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Invariant orders in simply connected Lie groups

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In this paper we give a criterion for an invariant cone in a real Lie algebra to generate a bi-invariant order in the corresponding simply connected Lie group.

Introduction

A convex, closed, pointed, generating cone C in a real Lie algebra which is invariant under all inner automorphisms will be called an *invariant cone*. Let S be the closed semigroup generated by $\exp(C)$. If $S \cap S^{-1} = \{e\}$, where e is the identity of the group, then the invariant order is defined by $g \succcurlyeq h \Leftrightarrow gh^{-1} \in S$.

The assumption that a Lie algebra admits an invariant cone is rather restrictive. Among simple real Lie algebras, only hermitian ones have this property ([11],[10]). This class recently was described by K.-H. Neeb ([7]). A Lie algebra admitting an invariant cone is a semidirect product of a reductive Lie algebra which also admits an invariant cone and a two-step nilpotent ideal, and all possibilities for the representation which determines the product are known now. A reductive Lie algebra admits an invariant cone if and only if it is not compact semisimple and each it's simple ideal is either compact or hermitian.

In the original paper [11], E.B. Vinberg proved that each simply connected group corresponding to a simple hermitian Lie algebra admits an invariant order. Among others, he also formulated the problem of a description of invariant cones which generates an invariant order (these cones are called global).

For simple algebras, this problem was solved by G.I. Ol'shanskiĭ who discovered an obstruction to the globality: an invariant cone is global if and only if the dual cone contains some special element l. This l is nonzero only for nontubular simple hermitian algebras, so in tubular ones any invariant cone is global.

In the solvable case, each invariant cone is global ([5], [1]).

Various results for the general case were obtained by K.-H. Neeb. In particular, he proved that in a Lie algebra which has no simple nontubular ideals each invariant cone is global. Furthermore, an invariant cone is global if it's dual cone intersects the relative interior of the cone C_l generated by elements l in

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all nontubular simple ideals. It follows also from his results that in semisimple algebras the condition for the dual cone to intersect C_l is a necessary one.

Let $\mathcal{J}$ be a semisimple ideal in a Lie algebra $\mathcal{G}$ with nontubular simple summands only, $\mathcal{L}_{\mathcal{J}}$ be the linear span of these elements l in all summands, C be an invariant cone in $\mathcal{G}$, and π_J be the projection to $\mathcal{J}^*$ dual to the identical embedding. By the main result of this paper, the globality of C is equivalent to the condition

$$\pi_J C^* \cap \mathcal{L}_{\mathcal{J}} \neq \{0\}$$

for all such ideals $\mathcal{J}$. Since $(C \cap \mathcal{J})^* = \operatorname{clos} \pi_J C^*$, this condition is close to the evident necessary one that $C \cap \mathcal{J}$ is global in $\mathcal{J}$ but involves the degree of contact between C and $\mathcal{J}$.

A cone in a Lie algebra, by left shifts, naturally defines a left invariant cone field on the corresponding group; if the cone is invariant then the cone field is bi-invariant. A smooth curve is timelike if it's tangent vector at any point belongs to the cone at this point. It is possible to prove that the semigroup S above is the closure of the set of endpoints of timelike curves starting at e. Hence the existence of a closed timelike curve implies that the cone is not global. Another possibility for an invariant cone to be nonglobal is the existence of a sequence of timelike curves starting at e such that their lengths tend to the infinity but endpoints tend to e. In the setting of left invariant cone fields this effect occurs if the cone has a sufficiently high degree of contact with a Heisenberg subalgebra (Theorem 4.1). For bi-invariant cone fields, one may consider two different Heisenberg subgroups together with their actions, the first by left shifts and the second by right ones. Thus a global invariant cone cannot have a high degree of contact with a sum of two Heisenberg subalgebras. A construction of special closed curves in Heisenberg groups gives estimates for the degree of contact; gathered with some technical lemmas on the geometry of cones and a choice of two Heisenberg subalgebras in nontubular hermitian algebras this proves the necessity of the globality criterion.

If the interior of the invariant cone intersects a sum of two Heisenberg subalgebras then the mentioned construction gives an example of a closed timelike curve. This solves the problem of Ol'shanskiĭ formulated in [8] for $\mathrm{su}(2,1)$. In simple simply connected groups, such a curve exists if and only if the cone is not global, but even in the algebra $\mathrm{su}(2,1) \oplus \mathbb{R}$ there exists an invariant cone which is not global but admits no closed timelike curves.

The proof of the sufficiency is more complicated technically but more standard. It uses the techniques of increasing functions, in fact, a modification of the construction of the paper [11]. An essential tool is also Theorem 5.1 which deals with the delicate situation of an invariant cone whose image in a factor algebra is included to some global invariant one. The globality problem is reduced to the reductive case by the consideration of simplectic algebras $\mathrm{sp}(2n, \mathbb{R})$.

The material of the paper is organized as follows.

Section 1 contains basic definitions and notations as well as the statement of Main Theorem and the description of a geometrical procedure which allows to check the globality; there are also two examples.

In Section 2 the preparatory material on simple hermitian algebras is given in a form convenient for our purpose. The results of this section are known.

Mainly, we need a reformulation of the Ol'shanskiĭ criterion $l \in C^*$ (Proposition 2.1). A construction of an increasing function which, in fact, coincides with the function of Vinberg ([11]) shows that it is naturally connected with some finite dimensional representation.

In Section 3 it is shown that estimates of lengths of closed curves of a special kind can give estimates of the degree of contact (Proposition 3.1).

Section 4 is devoted to the consideration of Heisenberg groups. According to Theorem 4.1 proved in this section, the degree of contact of a global cone with a Heisenberg algebra at a point of it's center cannot exceed 2. The necessity of the criterion of the globality is also proved in this section.

The main result of Section 5 is Theorem 5.1 mentioned above. It is proved that an invariant cone is global if it's projection to a factor by an ideal is included to some global cone and the projection of the dual cone to the dual space of this ideal includes some cone dual to a global one. The additional hypothesis is that this ideal admits a comlementary subalgebra.

In Section 6, nonreductive algebras are considered; it is proved in Theorem 6.3 that the globality of an invariant cone in a nonreductive algebra is equivalent to the globality of it's intersection with some reductive subalgebra.

The results formulated in Section 1 are proved in Section 7.

The paper is rather selfcontained. Special results which are given without proofs concern the structure of Lie algebras admitting an invariant cone (see [3], [6], [2]; the last article could be a preliminary part to this paper).

The main result of the paper could be understood as follows: the Heisenberg group is the unique obstruction to the existence of an bi-invariant order which is not an obstruction to the existence of an invariant cone. It seems to be faithful that this obstruction (with general nilpotent groups instead of the Heisenberg group) is an essential one in a more general situation of homogeneous spaces.

1. Statement of the main theorem

The identity in a group is denoted by e, in an associative algebra by 1; $\mathbb{R}^+ = [0, \infty)$. Vector spaces everywhere in this paper are supposed to be finite dimensional and, if the contrary is not supposed in an explicit form, real. For a vector space V, V^* denotes the dual space. We consider only convex cones. A cone $C \subseteq V$ is called *pointed* if it is closed and $C \cap (-C) = \{0\}$.

The interior of a set S is denoted by $\mathrm{Int}(S)$, it's boundary by ∂S; $\mathrm{RelInt}(S)$ denotes the interior of a set $S \subseteq V$ in it's linear span.

A cone C is called *generating* if $\mathrm{Int}(C) \neq \varnothing$. A closed cone C is pointed (generating) if and only if the *dual cone*

$$C^* = \{\lambda \in V^* : \ \lambda(x) \geq 0 \quad \text{for all} \quad x \in C\}$$

is generating (pointed).

Let M be a smooth manifold ("smooth" everywhere in this paper means "infinitely differentiable"). A *cone field* on M is a subset $\mathfrak{C}$ of the total space of the tangent bundle TM such that $\mathfrak{C}(m) = \mathfrak{C} \cap T_m M$ is a cone in the tangent space $T_m M$ for all $m \in M$. A cone field on a domain in an Euclidean space $\mathcal{E}$ will be called *continuous* if the projections of cones to the projective space $P\mathcal{E}$ depends on the point continuously with respect to the Hausdorff distance between sets; the Hausdorff distance may correspond to any Riemannian metric in $P\mathcal{E}$. Clearly, diffeomorphisms keep this property, thus the definition of the continuity naturally extends to smooth manifolds.

A smooth curve $\gamma : [a, b] \to G$ is called *timelike* if

$$\gamma'(t) \in \mathfrak{C}(\gamma(t)) \quad \text{for all} \quad t \in [a, b] \tag{1.1}$$

For a piecewise smooth curve, one has to replace $\gamma'(t)$ in (1.1) with both one-side derivatives. By $\mathcal{T}(m)$ will be denoted the set of all timelike piecewise smooth curves in M which start at m.

The *future* $\mathcal{F}(m)$ of the point m is the closure of the set of endpoints of curves in $\mathcal{T}(m)$. The *past* $\mathcal{P}(m)$ of m is the future of m with respect to the cone field $-\mathfrak{C}$.

A cone field $\mathfrak{C}$ is called *global* if the tangent cone to the set $\mathcal{F}(m)$ coincides with $\mathfrak{C}(m)$ for each $x \in M$. If the cone field $\mathfrak{C}$ is global then the dual cone field $\mathfrak{C}^*$ in the cotangent bundle T^*M defined by $\mathfrak{C}^*(m) = \mathfrak{C}(m)^*$ will be called *exact*.

A smooth function f is called *increasing at a point* m if $d_g f \in \mathfrak{C}(m)^*$. If $d_g f \in \operatorname{Int}(\mathfrak{C}(m)^*)$ then f will be called *strictly increasing at* m. An *increasing* (*strictly increasing*) *function on* M is a function which is increasing (strictly increasing) at each point. Set

$$\mathfrak{C}^\star(m) = \{d_m f : f \quad \text{is an increasing function on} \quad M\} \tag{1.2}$$

Clearly, $\mathfrak{C}^\star(m) \subseteq \mathfrak{C}^*(m)$.

The length of a vector x in an euclidean space $\mathcal{E}$ is denoted by $|x|$; for the distance between a point and a set or between two sets we shall use the notation dist. The length of a curve γ is denoted by $l(\gamma)$.

For a subalgebra $\mathcal{X} \subseteq \mathcal{G}$, let $\operatorname{Ad}(\mathcal{X})$ be the subgroup of $\mathrm{GL}(\mathcal{G})$ generated by the set $\{e^{\operatorname{ad}(x)} : x \in \mathcal{X}\}$. If $\operatorname{clos} \operatorname{Ad}(\mathcal{X})$ is compact, $\mathcal{X}$ is called *compactly embedded.*

A pointed generating $\operatorname{Ad}(\mathcal{G})$-invariant cone in a real Lie algebra $\mathcal{G}$ will be called an *invariant cone.* Each cone $C \subseteq \mathcal{G}$ defines the unique left invariant cone field $\mathfrak{C}$ on a Lie group G with the tangent Lie algebra $\mathcal{G}$ such that $\mathfrak{C}(e) = C$. If C is an invariant cone then $\mathfrak{C}$ is also right invariant. More generally, a cone in a tangent (cotangent) space at a point p of a homogeneous space defines an invariant cone field $\mathfrak{C}$ if and only if it is invariant under the isotropy group. Such a cone will be called *global* (*exact*) if it generates the global (exact) invariant cone field. A cone dual to an invariant cone in a Lie algebra will be called an *exact invariant cone.*

A simple real Lie algebra is called *hermitian* if the center of it's maximal compactly embedded subalgebra is nontrivial. Simple hermitian Lie algebras are exactly Lie algebras of groups of holomorphic automorphisms of irreducible hermitian symmetric spaces. A simple hermitian Lie algebra $\mathcal{G}$ is called *tubular* if the corresponding symmetric space is equivalent to a *tube domain*, i.e. a domain in $\mathbb{C}^n$ of the type $\mathbb{R}^n + iC$, where iC is a pointed generating cone in $\mathbb{R}^n$. Otherwise, $\mathcal{G}$ is *nontubular*. An equivalent algebraic definition is: $\mathcal{G}$ is nontubular if and only if some special element l is nonzero (for the exact definition of l, see (2.10)).

Let $\mathcal{G}$ be a real Lie algebra, $\mathcal{J}_1, \dots, \mathcal{J}_n$ be all it's simple hermitian nontubular ideals, $l_k \neq 0$ be the element defined by (2.10) for $\mathcal{J}_k$, $\overline{n} = \{1, \dots, n\}$. For a set $I \subseteq \overline{n}$, let $\mathcal{J}_I$ be the sum of $\mathcal{J}_k$, $k \in I$, $\mathcal{L}_I$ be the linear subspace generated by l_k, $k \in I$, π_I be the projection $\mathcal{G}^* \to \mathcal{J}_I$ dual to the embedding $\mathcal{J}_I \to \mathcal{G}$, where $\mathcal{J}_I{}^*$ is identified with $\mathcal{J}_I$ by the Killing form. If $I = \overline{n}$, set $\pi_I = \pi$, $\mathcal{J}_I = \mathcal{J}$, and $\mathcal{L}_I = \mathcal{L}$. Note that l_k are linearly independent.

Theorem (*Main Theorem*). *An invariant cone C in a real Lie algebra $\mathcal{G}$ is global if and only if*

$$\text{(GlC)} \qquad \pi_I C^* \cap \mathcal{L}_I \neq \{0\}$$

for each subset $I \subseteq \overline{n}$. Moreover, the globality of C is equivalent to each of the following conditions:

(1) *the cone πC^* includes some exact cone in $\mathcal{J}$;*
(2) *$\pi_I C^*$ includes some exact cone in $\mathcal{J}_I$ for any $I \subseteq \overline{n}$.*

The *globality criterion* (GlC) may be checked by the intersection of the cone πC^* with a rather small subspace of $\mathcal{G}$. Let $\mathcal{V}_r$ be the two-dimensional subspace generated by l_r and m_r, where l_r and m_r are defined by (2.10) for the nontubular ideal $\mathcal{J}_r$, $\mathcal{V} = \mathcal{V}_1 + \dots + \mathcal{V}_n$. Note that $\mathcal{V}$ is invariant under all projections π_I, $I \subseteq \overline{n}$. Set $C' = \pi C^* \cap \mathcal{V}$.

The checking procedure deals with a set $I \subseteq \overline{n}$ and the cone $C_I = \pi_I C'$ (if $I = \varnothing$, put $\mathcal{L}_I = C_I = \{0\}$). Initially, $I = \overline{n}$. Each step consists of two operations:

(1) find the least subset $I' \subseteq I$ such that $C_I \cap \mathcal{L}_I \subseteq \mathcal{L}_{I'}$;
(2) replace I by $I \setminus I'$.

The procedure finishes if either $I' = \varnothing$ or $I = \varnothing$.

Proposition 1.1. *The invariant cone C is global if and only if the procedure above finishes with $I = \varnothing$.*

This section is concluded by two illustrating examples. We use the notation (2.10).

Among simple nontubular hermitian algebras, su$(2,1)$ has the least dimension. This is a real form of sl$(3,\mathbb{C})$ whose root system is A_2. Recall that A_2 consists of vertices of the regular hexagon. Let α be a compact root, β the simple noncompact one. Then $3ik = \alpha + 2\beta$, the automorphism ν transposes α and β, and (2.10) may be rewritten as follows

$$il = \frac{1}{3}(\beta - \alpha), \quad im = \frac{1}{3}(2\alpha + \beta), \quad ip = \alpha + \beta$$

Note that ip is the second positive noncompact root.

The essential part of possible geometrical effects obstructing to the globality occurs in algebras $\mathrm{su}(2,1)\oplus\mathbb{R}$ and $\mathrm{su}(2,1)\oplus\mathrm{su}(2,1)$. By a result of Hilgert and Hofmann, any closed convex generating Weyl invariant cone c in a Cartan subalgebra $\mathcal{C}$ of a reductive Lie algebra $\mathcal{G}$ such that $c_{\min}\subseteq c\subseteq c_{\max}$ is the intersection of the unique invariant cone C in $\mathcal{G}$ with $\mathcal{C}$. So we give examples of cones c. For $\mathrm{su}(2,1)$, $c_{\min}$ is generated by $-i\beta$ and p, $c_{\max}$ by l and m. The Weyl group of $\mathrm{su}(2,1)$ contains only one nontrivial element, namely, the reflection ρ with respect to the line orthogonal to α. Since k is a fixed point of it and $\rho\alpha=-\alpha$, any invariant cone in $\mathrm{su}(2,1)$ is uniquely determined by the condition that the vector $3k+it\alpha$ with $t\in[1,3]$ belongs to it's boundary. The space $\mathcal{V}$ coincides with $\mathcal{C}$.

Example 1.1. Let $\mathcal{G}=\mathrm{su}(2,1)\oplus\mathbb{R}$, $\mathcal{C}$ be a Cartan subalgebra of $\mathcal{G}$. Then $\mathcal{C}=\mathcal{C}'\oplus\mathbb{R}$, where $\mathcal{C}'$ is a Cartan subalgebra of $\mathrm{su}(2,1)$. Let f be a convex function on $(-3,3)$ such that

(1) f is even;
(2) $f(t)=0$ for $t\in[-1,1]$;
(3) $\lim\limits_{t\to 3} f(t)=+\infty$.

There exists the unique convex homogeneous of degree 1 function F on the interior of $c_{\max}$ in $\mathcal{C}'$ which coincides with $f(t)$ on the interval $k-it\alpha$, $t\in(-3,3)$. Set

$$c=\{(x,s):\, x\in\mathrm{Int}(c_{\max}),\, s\geq F(x)\}\cup\{(0,s):\, s\geq 0\}$$

Then c is a closed convex generating Weyl invariant cone in $\mathcal{C}$. It's projection to $\mathcal{C}'$ is equal to $\mathrm{Int}(c_{\max})\cup\{0\}$, so (GlC) is not satisfied for the corresponding invariant cone C. Hence the predual cone is not global while it's intersection with $\mathrm{su}(2,1)$ is global because the closure of the projection is $c_{\max}$. Note that the degree of contact of the predual cone with $\mathcal{C}'$ at the boundary of $c_{\min}$ depends on the growth of f as $t\to 3$ but always exceeds 1 (for the exact definition of the degree of contact, see Section 3).

Example 1.2. Set $\mathcal{G}=\mathcal{G}_1\oplus\mathcal{G}_2$, where $\mathcal{G}_1$ and $\mathcal{G}_2$ are isomorphic to $\mathrm{su}(2,1)$. Lets denote objects (vectors, cones, subspaces) in an ideal $\mathcal{G}_r$ introduced above for $\mathrm{su}(2,1)$, $r=1,2$, by the same letters with the index r.

For $t\in[1,3]$, let c_t be the closed Weyl invariant cone in $\mathcal{C}$ generated by

$$p_1,\quad l_1+k_2,\quad k_1+l_2,\quad 3k_2+it\,\alpha_2 \tag{1.3}$$

and C_t be the corresponding invariant cone in $\mathcal{G}$. Then

$$c_{\min,1}\oplus c_{\min,2}\subseteq c_t\subseteq c_{\max,1}\oplus c_{\max,2}$$

The projection of c_t to $\mathcal{G}_r$ contains l_r. Let λ be the linear functional on $\mathcal{C}=\mathcal{C}_1\oplus\mathcal{C}_2$ such that $\lambda(l_1)=\lambda(l_2)=0$, $\lambda(m_1)=\lambda(m_2)=1$. If $t<3$ then λ is strictly positive on all generating elements (1.3), so c_t doesn't contain any nonzero element of the type $a\,l_1+b\,l_2$ with $a,b\in\mathbb{R}$. Hence C_t is not exact for $t<3$ by (GlC). If $t=3$ then $l_2\in c_t$ and the projection of c_t to $\mathcal{G}_1$ contains l_1. Thus C_3 is exact by Proposition 1.1.

2. Simple hermitian Lie algebras

Let $\mathcal{G}$ be a real simple Lie algebra, κ be a Cartan involution which defines the Cartan decomposition

$$\mathcal{G} = \mathcal{K} \oplus \mathcal{M}$$

where $\kappa\xi = \xi$ for $\xi \in \mathcal{K}$ and $\kappa\xi = -\xi$ if $\xi \in \mathcal{M}$, $\mathcal{K}$ is a maximal compactly embedded subalgebra. Then

$$\tilde{\mathcal{K}} = \mathcal{K} \oplus i\mathcal{M}$$

is a compact real form of the complexification $\mathcal{G}^{\mathbf{c}} = \mathcal{G} \oplus i\mathcal{G}$ of $\mathcal{G}$. The group $\mathrm{Ad}(\mathcal{K})$ is irreducible in $\mathcal{M}$.

If τ is the complex conjugation in $\mathcal{G}^{\mathbf{c}}$ with respect to $\tilde{\mathcal{K}}$ in $\mathcal{G}^{\mathbf{c}}$ then

$$\mu = \kappa\tau = \tau\kappa \tag{2.1}$$

is the complex conjugation with respect to $\mathcal{G}$.

The algebra $\mathcal{G}$ is called *hermitian* if $\mathcal{Z}(\mathcal{K}) \neq 0$. If this is the case then the center of $\mathcal{K}$ coincides with the centralizer $\mathcal{Z}(\mathcal{K})$, $\dim \mathcal{Z}(\mathcal{K}) = 1$ and $\mathcal{M}$ admits an $\mathrm{Ad}(\mathcal{K})$-invariant complex structure

$$\mathcal{J} = \mathrm{ad}(k) \tag{2.2}$$

for some $k \in \mathcal{Z}(\mathcal{K})$. This k is unique up to a sign. The automorphism κ admits an expression by $\mathrm{ad}(k)$

$$\kappa = e^{\pi\,\mathrm{ad}(k)} \tag{2.3}$$

A complex Lie algebra $\mathcal{G}^{\mathbf{c}}$ is simple because $\mathcal{G}$ admits no complex structure (otherwise, its maximal compactly embedded subalgebra would be simple).

Any Cartan subalgebra $\mathcal{C}$ of $\mathcal{K}$ is a Cartan subalgebra of $\mathcal{G}$. Let Δ denotes the root system for $\mathcal{C}$. Since $\mathcal{C}$ is compactly embedded, $\Delta \subset (i\mathcal{C})^*$.

Let B be any base of the root system Δ. Then there exists a Chevalley base $\mathfrak{B}$ for $\mathcal{G}^{\mathbf{c}}$

$$\mathfrak{B} = \{h_\alpha :\ \alpha \in B\} \cup \{x_\alpha :\ \alpha \in \Delta\} \tag{2.4}$$

where x_α is the eigenvector of $\mathcal{C}$ corresponding to a root α (so $[h, x_\alpha] = \alpha(h)x_\alpha$ for all $\alpha \in \Delta$ and $h \in \mathcal{C}$),

$$h_\alpha = [x_\alpha, x_{-\alpha}], \quad \alpha(h_\alpha) = 2 \quad \text{for all } \alpha \in \Delta$$

and structure constants of $\mathcal{G}^{\mathbf{c}}$ for the base $\mathfrak{B}$ are integer; in particular, they are real. Thus $\{h_\alpha, x_\alpha, x_{-\alpha}\}$ is a sl_2-triple.

We may also assume that

$$\tau x_\alpha = -x_{-\alpha} \tag{2.5}$$

The union of vectors h_α is a root system $\Delta^\vee$ dual to Δ.

Let q be an invariant symmetric bilinear form in $\mathcal{G}^c$ which is positive definite in $i\mathcal{C}$ (so q is a scalar multiplier of the Killing form), $|\;|$ be the corresponding length. Then for all $\alpha \in \Delta$ and $h \in i\mathcal{C}$ $\alpha(h) = 2q(h_\alpha, h)/q(h_\alpha, h_\alpha)$. Thus we may assume that Δ is embedded into $i\mathcal{C}$ in such a way that

$$\alpha = \frac{2}{q(h_\alpha, h_\alpha)} h_\alpha, \quad h_\alpha = \frac{2}{q(\alpha,\alpha)}\alpha \tag{2.6}$$

It will be convenient to normalize q by the condition

$$\max\{\,|\alpha| :\ \alpha \in \Delta\} = \sqrt{2} \tag{2.7}$$

If α is a root of the maximal length, we shall say that α is a *long* one; others will be called *short*; short roots are of length 1. The normalization (2.7) and (2.6) imply that

$$h_\alpha = \alpha,\ |\alpha| = \sqrt{2}, \quad \text{and} \quad h_\alpha = 2\alpha,\ |\alpha| = 1 \tag{2.8}$$

Let W be the Weyl group of Δ, V be a Weyl chamber in $i\mathcal{C}$ which contains ik. There exists the unique base B of Δ such that

$$V = \{x \in i\mathcal{C} :\ \alpha(x) \geq 0 \text{ for all } \alpha \in B\} \tag{2.9}$$

It will be supposed that B in (2.4) satisfies (2.9).

Since V is a fundamental domain for W, there exists the unique $\delta \in W$ such that $\delta V = -V$. This is an involution because $\delta^2 V = V$. Let ν be an automorphism of $\mathcal{G}^c$ which is equal to $-\delta$ in $\mathcal{C}$; then ν is an involution in $i\mathcal{C}$, $\nu V = V$, and ν is either outer or identical in $i\mathcal{C}$. Set

$$l = k + \delta k, \quad m = \nu k, \quad p = k + m \tag{2.10}$$

Clearly, $l \perp p$.

Let W_k and W_m denote subgroups of W generated by reflections in W which fix points ik and im respectively, $W_{km} = W_k \cap W_m$, W_p be the group generated by W_{km} and ν.

Lemma 2.1. *The set of fixed points in $i\mathcal{C}$ of a group W_x is equal to $i\mathbb{R}x$, $x = k, m, p$. The group W_{km} fixes points of the subspace generated by k and m, and only these points.*

Proof. Note that W_k includes the Weyl group of the reductive Lie algebra $\mathcal{K}$ which could be naturally identified with the Weyl group of the compact semisimple algebra $\mathcal{K}' = [\mathcal{K}, \mathcal{K}]$ corresponding to its Cartan subalgebra $\mathcal{K}' \cap \mathcal{C}$. The last group has no fixed points in $\mathcal{K}' \cap \mathcal{C}$. Therefore, $\mathcal{Z}(\mathcal{K}) = \mathbb{R}k$ is exactly the set of fixed points of W_k in $\mathcal{C}$.

For $x = m$, the assertion is clear because $m = \nu k = -\delta k$ and $\delta \in W$. If $x = p$ then an obvious case is $k = m$. Since $i\mathbb{R}k$ is the set of W_k-fixed points in $i\mathcal{C}$, $i\mathbb{R}^+k$ is an extreme ray of V. The same is true for m. The cone V is simplicial, so the subspace generated by ik and im is the intersection of those boundary hyperplanes of V which contains ik and im; since all reflections with respect to these hyperplanes are contained in W_{km}, the set of W_{km}-fixed points is exactly this subspace. It remains to note that if $k \neq m$ then $i\mathbb{R}p$ is the set of ν-fixed points in the plane generated by ik and im because ν transposes them.■

Remark 2.1. In fact, W_k is the Weyl group of the reductive Lie algebra $\mathcal{K}$ because any reflection in $\mathcal{C}$ with respect to a hyperplane orthogonal to a root $\alpha \in \Delta_0$ could be realized by the inner automorphism $\exp(\frac{\pi}{2}\operatorname{ad}(x_\alpha - x_{-\alpha}))$.

The following corollary was already used in the proof of the lemma; the proof is clear.

Corollary 2.1. *$i\mathbb{R}^+k$ and $i\mathbb{R}^+m$ are extreme rays of V.* ■

By the corollary, ik is proportional to a fundamental weight of the base B and the dual base $B^\vee$ of $\Delta^\vee$. Since eigenvalues of $\operatorname{ad}(k)$ are $0, \pm i$,

$$\alpha(ik) \in \{0, 1, -1\} \tag{2.11}$$

for all $\alpha \in \Delta$. Therefore, ik is a fundamental weight of $\Delta^\vee$. Moreover, ik is a *microweight* of $\Delta^\vee$ ((2.11) is a definition).

Set

$$\Delta_j = \{\alpha \in \Delta : \ \alpha(ik) = j\}$$

Then $\Delta = \Delta_{-1} \cup \Delta_0 \cup \Delta_1$. Roots in Δ_0 will be called *compact*, in $\Delta_{-1} \cup \Delta_1$ *noncompact*, in Δ_1 — *positive noncompact*. Note that there is only one root in B which is not orthogonal to ik; this is the *simple noncompact root*.

By (2.1), (2.3), and (2.5),

$$\mu x_\alpha = -x_{-\alpha}, \ \alpha \in \Delta_0 \quad \text{and} \quad \mu x_\alpha = x_{-\alpha}, \ \alpha \in \Delta_1 \cup \Delta_{-1} \tag{2.12}$$

So, there are natural bases for $\mathcal{K}$ and $\mathcal{M}$ in the decomposition $\mathcal{G} = \mathcal{K} \oplus \mathcal{M}$

$$\mathfrak{B}_{\mathcal{K}} = iB^\vee \cup \{x_\alpha - x_{-\alpha} : \ \alpha \in \Delta_0^+\} \cup \{i\,(x_\alpha + x_{-\alpha}) : \ \alpha \in \Delta_0^+\}$$

$$\mathfrak{B}_{\mathcal{M}} = \{x_\alpha + x_{-\alpha} : \ \alpha \in \Delta_1\} \cup \{i\,(x_\alpha - x_{-\alpha}) : \ \alpha \in \Delta_1\} \tag{2.13}$$

where Δ_0^+ is a subset of positive roots for some ordering (so $\Delta_0^+ \cup (-\Delta_0^+) = \Delta_0$ and $\Delta_0^+ \cap (-\Delta_0^+) = \varnothing$).

Since m is conjugated to k in $\mathcal{G}^{\mathbf{c}}$, $\operatorname{ad}(m)$ has the same eigenvalues as $\operatorname{ad}(k)$. Put

$$\Delta_{r,s} = \{\alpha \in \Delta : \ \alpha(ik) = r, \, \alpha(im) = s\}$$

and denote by $\mathcal{G}^{\mathbf{c}}_{r,s}$ the corresponding eigenspace

$$\mathcal{G}^{\mathbf{c}}_{r,s} = \{x \in \mathcal{G}^{\mathbf{c}} : \operatorname{ad}(ik)x = rx, \ \operatorname{ad}(im)x = sx\}$$

Then $\mathcal{G}^{\mathbf{c}}_{0,0}$ is the complex linear span of $\mathcal{C}$ and $\{x_\alpha : \ \alpha \in \Delta_{0,0}\}$, and other $\mathcal{G}^{\mathbf{c}}_{r,s}$ are generated by x_α, $\alpha \in \Delta_{r,s}$.

Since ik and im belong to the same Weyl chamber V, $\alpha(ik)$ and $\alpha(im)$ cannot have opposite signs for all $\alpha \in \Delta$. Therefore, $\Delta_{-1,1} = \Delta_{1,-1} = \varnothing$. Thus we obtain an A_2-type gradation

$$\mathcal{G}^{\mathbf{c}} = \mathcal{G}^{\mathbf{c}}_{-1,-1} \oplus \mathcal{G}^{\mathbf{c}}_{-1,0} \oplus \mathcal{G}^{\mathbf{c}}_{0,-1} \oplus \mathcal{G}^{\mathbf{c}}_{0,0} \oplus \mathcal{G}^{\mathbf{c}}_{1,0} \oplus \mathcal{G}^{\mathbf{c}}_{0,1} \oplus \mathcal{G}^{\mathbf{c}}_{1,1}$$

For the element p defined by (2.10), $\operatorname{ad}(ip)$ has the eigenvalue $r+s$ in the space $\mathcal{G}^{\mathbf{c}}_{r,s}$. So we have a gradation of $\mathcal{G}^{\mathbf{c}}$

$$\mathcal{G}^{\mathbf{c}} = \mathcal{G}^{\mathbf{c}}_{-2} \oplus \mathcal{G}^{\mathbf{c}}_{-1} \oplus \mathcal{G}^{\mathbf{c}}_{0} \oplus \mathcal{G}^{\mathbf{c}}_{1} \oplus \mathcal{G}^{\mathbf{c}}_{2}$$

where $\mathcal{G}^{\mathbf{c}}_{r+s} = \mathcal{G}^{\mathbf{c}}_{r,s} + \mathcal{G}^{\mathbf{c}}_{s,r}$.

Our aim is to receive the same gradation for $\mathcal{G}$. To do it, we'll find an element in $\mathcal{M}$ conjugated to ip in $\mathcal{G}^{\mathbf{c}}$.

Lemma 2.2. *Let $Q(\Delta)$ be the lattice in $i\mathcal{C}$ generated by Δ, $\lambda \in Q(\Delta)$, $\lambda \neq 0$. Then*

$$\lambda = \sum_{\alpha \in E} n_\alpha \alpha \tag{2.14}$$

for some positive integer n_α and $E \subset \Delta$ such that $q(\lambda, \alpha) > 0$, $q(\alpha, \beta) \geq 0$ for all $\alpha, \beta \in E$.

Proof. Consider the class of representations of λ of the type (2.14) with nonnegative n_α and $q(\lambda, \alpha)$. This class is not empty because these conditions are satisfied for the unique representation by the base corresponding to a Weyl chamber which contains λ. Suppose that for the data $n_\alpha, \alpha \in E$ the sum of n_α attains the minimal value over this class. Then $q(\alpha, \beta) \geq 0$ for all $\alpha, \beta \in E$ since otherwise $\alpha + \beta$ is a root and it is possible to decrease the sum. So $\alpha \in E$, $q(\lambda, \alpha) = 0$ implies $n_\alpha = 0$, and α could be excluded from E. ■

Lemma 2.3. *Let $\tilde{\Delta}$ be the subsystem of long roots in Δ. Then $ik - i\delta k \in Q(\tilde{\Delta})$.*

Proof. First, note that for any weight λ of a root system Δ (this means that $q(\lambda, \alpha^\vee)$ is integer for all $\alpha \in \Delta$) and $w \in W$, $\lambda - w\lambda \in Q(\Delta)$ (for reflections in W, this is a simple calculation; for any $w \in W$, this is true because W is generated by reflections).

So, the assertion is clear if roots in Δ are of equal length. We may exclude from the consideration G_2 and F_4 since they have no hermitian real forms. Thus we have to consider B_n and C_n.

Let $\tilde{W}$ be the Weyl group of $\tilde{\Delta}$. If $\Delta = C_n$ then $\delta = -1 \in \tilde{W}$ since $\tilde{\Delta} = nA_1$. For $\Delta = B_n$, $\delta = -1$ but $\delta \notin \tilde{W}$ for odd n; being a microweight of $\Delta^\vee = C_n$, ik is a short root ε of B_n, so $ik - i\delta k = 2\varepsilon$ could be represented as a sum of two long roots in any B_2-subsystem which contains ε. ■

Lemma 2.4. *There exists a subset $E \subseteq \Delta_{1,1}$ consisting of long roots such that*

$$ip = \sum_{\alpha \in E} h_\alpha \tag{2.15}$$

and the set E is a maximal in Δ_1 subset of pairwise orthogonal roots.

Proof. By Lemma 2.3, Lemma 2.2, and the definition of p, there exists a representation of ip of the type (2.14), and roots in E may be supposed to be long. For each $\xi \in \Delta$,

$$2 \geq q(ip, \xi) = \sum_{\alpha \in E} n_\alpha q(\alpha, \xi)$$

If $\xi \in E$ then $q(\xi, \xi) = 2$, so $n_\xi = 1$, $q(ip, \xi) = 2$, and $q(\alpha, \xi) = 0$ for $\alpha \neq \xi, \alpha \in E$. Therefore, roots in E are pairwise orthogonal and $E \subseteq \Delta_{1,1}$. By (2.11), $\alpha = h_\alpha$ for $\alpha \in E$. Thus (2.15) is proved. The maximality of E is a consequence of (2.15) and the obvious inequality $q(ip, \alpha) \geq 1$ for $\alpha \in \Delta_1$. ■

Lemma 2.5. *The linear span of all $x_\alpha, x_{-\alpha}, h_\alpha$, $\alpha \in E$, is a subalgebra of $\mathcal{G}^{\mathbb{C}}$ isomorphic to the direct sum of* sl_2*-subalgebras corresponding to roots in E.*

Proof. For all $\alpha \in E$, x_α and $x_{-\alpha}$, are annihilated by all summands in (2.15) which are not equal to h_α; if $\alpha, \beta \in E$, $\alpha \neq \beta$, then $\pm\alpha \pm \beta$ are not roots because α, β are orthogonal and long. ∎

Corollary 2.2. *The element ip can be embedded into a* sl_2*-triple in $\mathcal{G}^{\mathbb{C}}$.*

Proof. Put

$$\mathfrak{h} = ip, \quad \mathfrak{e} = \sum_{\alpha \in E} x_\alpha, \quad \mathfrak{f} = \sum_{\alpha \in E} x_{-\alpha} \tag{2.16}$$

Then $[\mathfrak{e}, \mathfrak{f}] = \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{e}] = 2\mathfrak{e}$, $[\mathfrak{h}, \mathfrak{f}] = -2\mathfrak{f}$ by the lemma and (2.15).

In $\mathrm{sl}(2, \mathbb{C})$, $\mathfrak{h}$ and $\mathfrak{e} + \mathfrak{f}$ are conjugated by an inner automorphism which may be written explicitly. Set

$$\theta = e^{\frac{1}{4}\pi\,\mathrm{ad}(\mathfrak{e}-\mathfrak{f})} \tag{2.17}$$

Then a calculation with 2-matrices shows that

$$\theta\mathfrak{h} = \mathfrak{e} + \mathfrak{f}, \quad \theta\mathfrak{e} = -\frac{1}{2}(\mathfrak{h} - \mathfrak{e} + \mathfrak{f}), \quad \theta\mathfrak{f} = -\frac{1}{2}(\mathfrak{h} + \mathfrak{e} - \mathfrak{f}) \tag{2.18}$$

By Lemma 2.3 and the same calculation,

$$\theta x_\alpha = -\frac{1}{2}(h_\alpha - x_\alpha + x_{-\alpha}), \quad \theta x_{-\alpha} = -\frac{1}{2}(h_\alpha + x_\alpha - x_{-\alpha}) \tag{2.19}$$

for all $\alpha \in E$. Thus

$$i\theta x_\alpha, i\theta x_{-\alpha} \in \mathcal{G} \quad \text{for all} \quad \alpha \in E \tag{2.20}$$

by (2.13). ∎

Lemma 2.6. *For θ defined by (2.17), $\theta E \subset \mathcal{M}$. The real linear span $\mathcal{A}$ of θE is a maximal abelian subspace of $\mathcal{M}$; for $\mathcal{B} = (E^\perp \cap \mathcal{C})$, $\mathcal{A} + \mathcal{B}$ is a Cartan subalgebra of $\mathcal{G}$.*

Proof. By Lemma 2.3 and (2.18), $\theta h_\alpha = x_\alpha + x_{-\alpha} \in \mathcal{M}$. This implies the first assertion.

The set of θ-fixed points in $\mathcal{C}$ includes $E^\perp$; therefore,

$$\theta\mathcal{C} = i\mathcal{A} \oplus \mathcal{B} \tag{2.21}$$

So $\mathcal{Z}(\mathcal{A} + \mathcal{B}) = \theta(\mathcal{C} + i\mathcal{C}) \cap \mathcal{G} = \mathcal{A} + \mathcal{B}$ and $\mathcal{A} + \mathcal{B}$ is a Cartan subalgebra of $\mathcal{G}$.

The remainder is the equality $\mathcal{Z}(\mathcal{A}) \cap \mathcal{M} = \mathcal{A}$ which is a consequence of the following lemma. ∎

Lemma 2.7. *Let $\mathcal{F}$ be a set of θ-fixed points in $\mathcal{Z}(\mathcal{A})$. Then $\mathcal{F} \subseteq \mathcal{K}$ and $\mathcal{Z}(\mathcal{A}) = \mathcal{A} \oplus \mathcal{F}$.*

Proof. If $\beta \in E$ and $\alpha \perp E$ then $\beta + \alpha$ cannot be a root because roots in E are long. Therefore, $[\mathsf{e}, x_\alpha] = [\mathsf{f}, x_\alpha] = 0$ and $\theta x_\alpha = x_\alpha$ for all $\alpha \in E^\perp$.

The centralizer of E in $\mathcal{G}^{\mathbf{c}}$ is generated by $\mathcal{C}$ and root vectors x_α, $\alpha \in E^\perp \cap \Delta$. So $\mathcal{Z}(\mathcal{A}) = \mathcal{Z}(\theta E) = \mathcal{A} \oplus \mathcal{F}$ by (2.21).

By Lemma 2.4, noncompact roots cannot be orthogonal to E. Thus $\mathcal{F} \subseteq \mathcal{K}$. ∎

Corollary 2.3. *A simple hermitian Lie algebra $\mathcal{G}$ admits a gradation*

$$\mathcal{G} = \mathcal{G}_{-2} \oplus \mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_1 \oplus \mathcal{G}_2 \tag{2.22}$$

Proof. This is the gradation by eigenvalues of $\mathrm{ad}(\theta \mathsf{h})$ (note that $\theta\mathsf{h} \in \mathcal{M}$ by (2.15) and Lemma 2.6). ∎

Let $\mathcal{L} = \mathcal{K} \cap \mathcal{G}_0$. Clearly, $\mathcal{B} \subseteq \mathcal{L}$. Lets denote by $P_\mathcal{X}$ the q-orthogonal projection in $\mathcal{G}$ onto the linear subspace $\mathcal{X} \subset \mathcal{G}$. Then

$$P_\mathcal{M} = \frac{1}{2}(1 - \kappa), \quad P_\mathcal{K} = \frac{1}{2}(1 + \kappa)$$

Set $\mathcal{H}^+ = \mathcal{G}_1 \oplus \mathcal{G}_2$, $\mathcal{H}^- = \mathcal{G}_{-1} \oplus \mathcal{G}_{-2}$, and

$$\mathcal{P} = \mathcal{G}_0 \oplus \mathcal{H}^+, \quad \mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-, \quad \mathcal{H}_0 = \mathcal{H} \cap \mathcal{K}$$

Lemma 2.8. *In the decomposition* (2.22),

(1) $\kappa \mathcal{G}_j = \mathcal{G}_{-j}$, $j = 0, \pm 1, \pm 2$; *in particular,* $\kappa \mathcal{H}^+ = \mathcal{H}^-$;
(2) $l = 2P_\mathcal{L} k$;
(3) $l \in \mathcal{Z}(\mathcal{L}) \cap \mathcal{L}$;
(4) $P_\mathcal{K}\mathcal{H} = P_\mathcal{K}\mathcal{H}^+ = P_\mathcal{K}\mathcal{H}^- = \mathcal{H}_0$;
(5) $\mathcal{K} = \mathcal{L} \oplus \mathcal{H}_0$, *and this decomposition is orthogonal*;
(6) $\mathcal{G} = \mathcal{K} + \mathcal{P}$;
(7) *if $\mathcal{G}$ is nontubular then* $[\mathcal{G}_1, \mathcal{G}_1] = \mathcal{G}_2$ *and* $[\mathcal{G}_{-1}, \mathcal{G}_{-1}] = \mathcal{G}_{-2}$.

Proof. By Lemma 2.6, $\theta\mathsf{h} \in \mathcal{M}$, so $\kappa\theta\mathsf{h} = -\theta\mathsf{h}$. This implies (1).

Since $l \perp \Delta_{1,1} \subseteq E$, $l \in \mathcal{B} \subseteq \mathcal{L}$. By (2.18), $\mathsf{h} = -\theta\mathsf{e} - \theta\mathsf{f}$. Hence

$$p = i\theta(\mathsf{e} + \mathsf{f}) \tag{2.23}$$

and

$$k = \frac{1}{2}(l + p) = \frac{1}{2}(l + i\theta\mathsf{e} + i\theta\mathsf{f}) \tag{2.24}$$

So (2) is true because $i\theta\mathsf{e} \in \mathcal{G}_2$, $i\theta\mathsf{f} \in \mathcal{G}_{-2}$ by (2.20), and $\mathcal{L} \perp \mathcal{G}_2 \oplus \mathcal{G}_{-2}$ since the decomposition (2.22) is orthogonal.

It follows from (2) that l is a fixed point of $\mathrm{Ad}(\mathcal{L})$. Since $l \in \mathcal{L}$, this proves (3).

Note that $\kappa\mathcal{H} = \mathcal{H}$ by (1), so $P_\kappa\mathcal{H} = \mathcal{H} \cap \mathcal{K} = \mathcal{H}_0$; $P_\kappa\mathcal{H}^+ = P_\kappa\mathcal{H}^-$ because $\kappa\mathcal{H}^+ = \mathcal{H}^-$. Clearly, $\mathcal{H} \perp \mathcal{L}$, so $\mathcal{H}_0 \perp \mathcal{L}$. This implies that $\mathcal{H}_0 \cap \mathcal{L} = \{0\}$ since $-q$ is positive definite in $\mathcal{K}$. By (1), $\mathcal{G}_0$ is κ-invariant, so $P_\kappa\mathcal{G}_0 = \mathcal{L}$; therefore, $\mathcal{K} = P_\kappa(\mathcal{G}_0 + \mathcal{H}) = \mathcal{L} + \mathcal{H}_0$. Thus (4) and (5) are proved.

By (4) and (5), $\mathcal{K} + \mathcal{H}^+ \supset \mathcal{H}^-$. This proves (6).

To prove (7), set $\mathcal{G}'_2 = [\mathcal{G}_1, \mathcal{G}_1]$, $\mathcal{G}'_{-2} = [\mathcal{G}_{-1}, \mathcal{G}_{-1}]$, $\mathcal{G}'_0 = [\mathcal{G}_1, \mathcal{G}_{-1}]$ and

$$\mathcal{G}' = \mathcal{G}'_{-2} \oplus \mathcal{G}_{-1} \oplus \mathcal{G}'_0 \oplus \mathcal{G}_1 \oplus \mathcal{G}'_2$$

Then $[\mathcal{G}_0, \mathcal{G}'_k] = \mathcal{G}'_k$, $k = 0, \pm 2$. Since $[\mathcal{G}_2, \mathcal{G}_{-1}] \subseteq \mathcal{G}_1$, $[\mathcal{G}_2, \mathcal{G}'_{-2}] \subseteq \mathcal{G}'_0$; similarly, $[\mathcal{G}'_2, \mathcal{G}_{-2}] \subseteq \mathcal{G}'_0$. Clearly, $[\mathcal{G}_{\pm 1}, \mathcal{G}'] \subset \mathcal{G}'$. Thus $\mathcal{G}'$ is an ideal in $\mathcal{G}$ and this ideal is proper if (7) is not true. ■

In the theorem below, we summarize known properties of invariant cones in simple hermitian Lie algebras in the form convenient for our purpose. We follow papers of Vinberg [11] and Ol'shanskiĭ [8], [9]; see also [10].

Recall that the Weyl group of $\mathcal{C}$ in $\mathcal{G}$ may be identified with the group W_k of Lemma 2.1 (see Remark 2.1).

Theorem 2.1. *Let $\mathcal{C}$ be a compactly embedded Cartan subalgebra of $\mathcal{G}$.*

(1) *Any invariant cone C in $\mathcal{G}$ is uniquely determined by $C \cap \mathcal{C} = c$, and c is a W_k-invariant generating cone in $\mathcal{C}$.*

(2) *There exist unique up to a sign maximal and minimal invariant cones $C_{\max}$ and $C_{\min}$. The cone $C_{\min}$ may be chosen in such a way that $k \in C_{\min}$. Then elements $x \in C_{\max} \cap \mathcal{K}$ are distinguished by the property $\mathcal{J}\mathrm{ad}(x) \geq 0$ in $\mathcal{M}$ (for $C_{\max} \supseteq C_{\min}$; $\mathcal{J}$ is defined by (2.2)).*

(3) *Let $c_{\max} = C_{\max} \cap \mathcal{C}$, $c_{\min} = C_{\min} \cap \mathcal{C}$. Then each W_k-invariant cone $c \subset \mathcal{C}$ such that*

$$c_{\min} \subseteq c \subseteq c_{\max}$$

is the intersection of some invariant cone with $\mathcal{C}$.

(4) *The cone $C_{\min}$ is generated by k; $C_{\max}$ is the cone dual to $C_{\min}$ with respect to the invariant form $-q$.*

(5) *The cone $ic_{\min}$ is the minimal cone which contains all positive noncompact roots; $c_{\max} = \{x \in \mathcal{C} : q(\alpha, ix) \geq 0$ for all $\alpha \in \Delta_1\}$.* ■

In other words, $ic_{\max}$ is the union of Weyl chambers which contain ik. Note that, by (2.10), (5), and (2.15),

$$l, m \in c_{\max}, \quad p \in c_{\min} \tag{2.25}$$

Ol'shanskiĭ ([8]) discovered a criterion of the globality of an invariant cone for simple simply connected Lie groups. For a cone C, let C^* be the cone dual to C with respect to $-q$.

Theorem 2.2. *An invariant cone $C \supseteq C_{\min}$ in a simple hermitian Lie algebra $\mathcal{G}$ is global if and only if $l \in C^*$.* ■

If $l = 0$ then a simple hermitian Lie algebra $\mathcal{G}$ is called *tubular*. Otherwise, it is called *nontubular*. By the theorem, in a tubular Lie algebra any invariant cone is global.

In fact, the sufficiency of the condition $l \in C^*$ was proved by Vinberg in the original paper [11] (he formulated the result for $C_{\min}$). Ol'shanskiĭ observed this and proved the necessity.

Since Vinberg defined the element l by another way, we shall outline a modified version of his proof.

By (2.11), ik is a weight of Δ; since $ik \in V$, this is a highest weight of a finite dimensional representation ρ of $\mathcal{G}^{\mathbf{c}}$. Weights of ρ lie in the W-invariant convex polytope M generated by ik. If $\alpha \in \Delta_0$ then ik belongs to the convex hull of $W_k(ik+\alpha) = ik + W_k\alpha$. Since ik is a vertex of M, $ik+\alpha$ is not a weight of ρ. Clearly, the same is true for $\alpha \in \Delta_1$. So, if w denotes the corresponding highest vector then

$$\rho(x_\alpha)w = 0 \quad \text{for all} \quad \alpha \in \Delta_0 \cup \Delta_1 \tag{2.26}$$

The dual representation ρ^* has the highest weight $im = i\nu k$. Therefore,

$$\rho^*(x_\alpha)\omega = 0 \quad \text{for all} \quad \alpha \in \Delta_{0,0} \cup \Delta_{-1,0} \cup \Delta_{0,1} \cup \Delta_1$$

where ω is a highest vector for ρ^*. Note that $\mathcal{P} + i\mathcal{P}$ is the complex linear span of θx_α, $\alpha \in \Delta_{0,0} \cup \Delta_{0,1} \cup \Delta_1$, and $\theta\mathcal{C}$. Hence

$$\rho^*(\theta^{-1}x)\omega = 0 \quad \text{for all} \quad x \in \mathcal{P}' = [\mathcal{P}, \mathcal{P}] \tag{2.27}$$

Let $G^{\mathbf{c}}$ be the simply connected complex Lie group with the Lie algebra $\mathcal{G}^{\mathbf{c}}$. Subgroups of $\mathcal{G}^{\mathbf{c}}$ with tangent algebras $\mathcal{G}$, $\mathcal{K}$, $\mathcal{P}$ will be denoted by G, K, P. Note that K is compact.

The representation of $G^{\mathbf{c}}$ corresponding to ρ will be denoted by the same letter. Set

$$\Theta = e^{\rho(\frac{\pi}{4}(\mathrm{e}-\mathrm{f}))} \quad \text{and} \quad f(g) = \omega(\Theta\rho(g)w) \tag{2.28}$$

Lemma 2.9. *For the function f, the following conclusions hold:*

(1) *there exist functions a, b on G such that $f(xgy) = a(x)b(y)f(g)$ for all $x \in P$, $y \in K$, $g \in G$;*
(2) *$f(g) \neq 0$ for all $g \in G$.*

Proof. By (2.26), w is a common eigenvector of $\rho(\eta)$, $\eta \in \mathcal{K}$; analogously, ω is a common eigenvector of $\rho^*(\theta^{-1}\xi)$, $\xi \in \mathcal{P}$, by (2.27).

So (1) may be verified by a calculation (note that $\rho^*(\theta^{-1}\xi) = \Theta^{-1}\rho^*(\xi)\Theta$ and $\rho^*(g) = \rho(g^{-1})^*$).

The function f cannot be identically zero on $G^{\mathbf{c}}$ since ρ is irreducible and on G because f is holomorphic. Thus $f(g) \neq 0$ for all $g \in G$ by (1) and the following lemma. ∎

Lemma 2.10. $G = KP$.

Proof. Let X be the homogeneous space G/P. Consider the action of the group K in X. By Lemma 2.8, (6), the orbit of the class P in X is open. Since K is compact, this is closed, so the action of K in X is transitive which proves the lemma. ∎

Let f be as in Lemma 2.9. We may assume that $f(e) = 1$. Since f has no zeroes in G, the definition

$$F(g) = \Im \log f(g), \quad F(e) = 0 \tag{2.29}$$

of the function F on the universal covering group $\tilde{G}$ is correct.

Lemma 2.11. *If $l \in C^*$ then F is an C-increasing function in $\tilde{G}$. Moreover,*

$$F(xgy) = F(x) + F(g) + F(y) \tag{2.30}$$

for all $x \in \tilde{K}$, $g \in \tilde{G}$, $y \in \tilde{P}$, where $\tilde{K}$ and $\tilde{P}$ are subgroups in the universal covering group $\tilde{G}$ corresponding to subalgebras $\mathcal{K}$ and $\mathcal{P}$. Furthermore, after the multiplication on a suitable constant and the identification of $\mathcal{G}$ and $\mathcal{G}^*$ by means of the invariant form q,

$$d_e F = \frac{1}{2} l + i\theta \mathsf{e} \tag{2.31}$$

Proof. First of all, note that (2.30) is a consequence of Lemma 2.9 and (2.29). Lemma 2.10 and (2.30) imply that the function F is increasing if $d_e F \in C^*$. Thus the lemma will be proved if we'll prove (2.31) and the inclusion $i\theta\mathsf{e} \in C_{\min}$.

It follows from (2.28) that $d_e f(\zeta) = \omega(\Theta\rho(\zeta)w)$ for all $\zeta \in \mathcal{G}^{\mathbf{c}}$. Set $\varphi = d_e F$. By (2.28) and (2.26), φ has the zero restriction to the semisimple part $\mathcal{K}' = [\mathcal{K}, \mathcal{K}]$ of $\mathcal{K}$; by (2.28) and (2.27), $\varphi = 0$ on $\mathcal{P}'$. So restrictions of φ to $\mathcal{K}$ and $\mathcal{P}$ are scalar multipliers of functionals $q(k, \cdot)$ and $q(\Im(i\theta m), \cdot)$ respectively, where $\Im$ is the imaginary part with respect to the algebra $\mathcal{G}$ in $\mathcal{G}^{\mathbf{c}}$. Since $\theta l = l$,

$$\Im(i\theta m) = \Re(\theta m) = \frac{1}{2}\Re\theta(p - l) = -\frac{1}{2} l$$

according to (2.23) and Lemma 2.6. Combining this with Lemma 2.8, (2), we receive the equality $k + \xi = \frac{1}{2} l + \eta$, where $\xi \in \mathcal{K}^{\perp} = \mathcal{M}$, $\eta \in \mathcal{P}^{\perp} = \mathcal{H}^{+}$. It follows from Lemma 2.8, (6), that this equality determines φ up to a scalar multiplier. Since $i\theta(\mathsf{e} - \mathsf{f}) \in \mathcal{M}$ and $i\theta\mathsf{e} \in \mathcal{G}_2$, by (2.24),

$$k + \frac{i}{2}\theta(\mathsf{e} - \mathsf{f}) = \frac{1}{2} l + i\theta\mathsf{e}$$

This proves (2.31).

Since $\operatorname{ad}^2(e)\mathsf{e} = 0$, $\operatorname{ad}^2(e)\mathsf{f} = 2e$, by (2.23),

$$\lim_{t\to\infty} \frac{1}{t^2} e^{t \operatorname{ad}(i\theta \mathsf{e})} p = 2i\theta\mathsf{e}$$

hence $i\theta\mathsf{e} \in C_{\min}$ by (2.25) and (2.20). ∎

The proof of the globality could be concluded by an application of Theorem 2.1 and Corollary 5.6 (cf. [11]).

If $\mathcal{G}$ is tubular then $l = 0$. Hence $l \in C^*$. So one may assume that $\mathcal{G}$ is nontubular proving the necessity. It will be done in Section 4 for a more general setting; in this subsection, some conditions equivalent to $l \in C^*$ are given.

Let V be the inverse image of the group W_{km} in the normalizer of $\mathrm{Ad}(\mathcal{C})$ in $\mathrm{Ad}(\mathcal{G})$ under the homomorphism of restriction. This is a compact group isomorphic to a semidirect product of the torus $\mathrm{Ad}(\mathcal{C})$ and the finite group W_{km}. The averaging operator over V will be denoted by A_V. As in Proposition 1.1, the two-dimensional subspace of $\mathcal{G}$ generated by k and m is denoted by $\mathcal{V}$.

Proposition 2.1. *Let $\mathcal{G}$ be a nontubular simple hermitian Lie algebra, C be an invariant cone in $\mathcal{G}$, $C \subseteq C_{\max}$. Then $A_V = P_{\mathcal{V}}$, where $P_{\mathcal{V}}$ is the q-orthogonal projection to $\mathcal{V}$. Moreover,*

$$A_V(C^* \cap p^{\perp}) = C^* \cap \mathbb{R}^+ l \tag{2.32}$$

and the condition $l \in C^*$ is equivalent to each of the following ones:

(1) $p \notin \mathrm{Int}(C)$;

(2) $\mathrm{Int}(C) \cap \mathcal{H} = \varnothing$;

(3) $p^{\perp} \cap C^* \neq \{0\}$;

Proof. Since $\mathcal{C}$ is the set of fixed points of $\mathrm{Ad}(\mathcal{C})$ and $\mathcal{V}$ is the set of fixed points of W_{km} in $\mathcal{C}$ by Lemma 2.1, $A_V = P_{\mathcal{V}}$ because the averaging operator over a compact group of orthogonal transformations is the orthogonal projection to the set of fixed points of this group. The group V fixes p, hence $p^{\perp}$ is V-invariant. Therefore, $A_V(C^* \cap p^{\perp}) = A_V\, C^* \cap p^{\perp}$ and (2.32) folows from the equality $\mathbb{R}l = p^{\perp} \cap \mathcal{V}$ and the inclusion $C \subseteq C_{\max}$.

It follows from the equality $A_V = P_{\mathcal{V}}$ that the dual to $C \cap \mathcal{V}$ cone in $\mathcal{V}$ is equal to $C^* \cap \mathcal{V}$. Note also that $\mathrm{RelInt}(C \cap \mathcal{V}) = \mathrm{Int}(C) \cap \mathcal{V}$ since $k \in \mathrm{Int}(C) \cap \mathcal{V}$. The cone $C \cap \mathcal{V}$ is an angle in the plane $\mathcal{V}$ which contains p by (2.25). Thus

$$p \in \mathrm{Int}(C) \quad \text{if and only if} \quad l \notin C^* \tag{2.33}$$

and the condition $l \in C^*$ is equivalent to (1) as well as to (3). Since $p \in \mathcal{H}$ by (2.23), (2) implies (1); the converse is true because $l \perp \mathcal{H}$ (recall that $l \in \mathcal{G}_0$). ■

3. A necessary condition for the globality

In this section, we show that the degree of contact of a cone with some special subspaces could be an obstruction to the globality (Proposition 3.4). We start with preparatory lemmas on the geometry of cones.

Let η be an increasing function defined on some interval in $\mathbb{R}$ with left endpoint 0, $\lim_{\varepsilon \to 0} \eta(\varepsilon) = 0$, $\mathcal{L}$ be a linear subspace of an Euclidean space $\mathcal{E}$. We shall say that a cone C has the *degree of contact with $\mathcal{L}$ at the point $x \in C$ greater than η* if

$$\mathrm{dist}(C, x + y) = o(\eta(|y|)) \quad \text{as} \quad y \to 0 \quad \text{in} \quad \mathcal{L}$$

The degree of contact of C with $\mathcal{L}$ at x is *greater or equal to* η if there exist a constant $Q > 0$ and a neighborhood U of zero in $\mathcal{L}$ such that

$$Q\eta(|y|) \geq \operatorname{dist}(C, x+y) \quad \text{for all} \quad y \in U$$

Suppose that $x \notin \mathcal{L}$; then the degree of contact is *equal* to η if it is greater or equal and the inverse inequality holds with some another constant. If $x \in \mathcal{L}$ then one has to replace $\mathcal{L}$ in this definition to any subspace $\mathcal{L}' \subset \mathcal{L}$ complementary to $\mathbb{R}x$ in $\mathcal{L}$; clearly, the definition doesn't depend on the choice of $\mathcal{L}'$.

If $\eta(\varepsilon) = \varepsilon^a$ then a will be called the degree of contact and denoted by $\operatorname{cont}(C, \mathcal{L}, x)$; $\operatorname{cont}(C, \mathcal{L}, x) > a$ ($\operatorname{cont}(C, \mathcal{L}, x) \geq a$) will mean that the degree of contact is greater than (greater or equal to) ε^a.

Lemma 2.12. *Suppose that* $\operatorname{cont}(C, \mathcal{L}, x) > 1$. *Then*

(1) *either* $x \in \operatorname{Int}(C)$ *or* $x \in \partial C$ *and* $(x + \mathcal{L}) \cap \operatorname{Int}(C) = \varnothing$;
(2) *the orthogonal projection of* C *to* $\mathbb{R}x + \mathcal{L}$ *includes a neighborhood of* x *in* $\mathbb{R}x + \mathcal{L}$;
(3) *if* $\xi \in C^*$ *and* $(\xi, x) = 0$ *then* $\xi \in \mathcal{L}^{\perp}$.

Proof. If $x \in \partial C$ then there exists a hyperplane H in $\mathcal{E}$ such that $x, 0 \in H$, $H \cap \operatorname{Int}(C) = \varnothing$. Suppose that $(x + \mathcal{L}) \cap \operatorname{Int}(C) \neq \varnothing$ and consider the ray R in $x + \mathcal{L}$ with the origin x which is orthogonal to $H \cap (x + \mathcal{L})$ and separated from C by H. Then $\operatorname{dist}(x + y, C) > K|y|$ for $y \in R$ and some $K > 0$ which is impossible because $\operatorname{cont}(C, \mathcal{L}, x) > 1$. The same construction for the projection of C to $\mathbb{R}x + \mathcal{L}$ shows that x cannot be a boundary point of it. To prove (3), note that $u \in \mathcal{L}$ and $(\xi, u) \leq 0$ imply

$$0 \geq (\xi, x + tu) \geq -|\xi| \operatorname{dist}(x + tu, C) = o(t) \quad \text{as} \quad t \to +0 \qquad \blacksquare$$

Lemma 3.1. *Let* $f \geq 0$ *be a convex function on a neighborhood of the origin of an euclidean space* $\mathcal{E}$, Gr *be it's graph in* $\mathbb{R} \times \mathcal{E}$ *endowed with the product Euclidean distance. Then there exist a neighborhood* $V \subset \mathcal{E}$ *of zero and a constant* $K > 0$ *such that* $f(x) \leq K \operatorname{dist}(x, Gr)$ *for all* $x \in V$.

Proof. We may assume that $f^{-1}(0)$ is compact since it is possible to add to f a convex function vanishing in some neighborhood of the origin and strictly positive outside it. Then, for sufficiently small $\varepsilon > 0$, the set

$$V_\varepsilon = \{x \in U : f(x) \leq \varepsilon\}$$

is compact. Let ε be such that both V_ε and $V_{2\varepsilon}$ have this property. Set $\delta = \inf\{|x - y| : \ x \in V_\varepsilon, y \notin V_{2\varepsilon}\}$. Then, for any $x, y \in V_\varepsilon, x \neq y$,

$$\frac{|f(x) - f(y)|}{|x - y|} \leq \frac{\varepsilon}{\delta} \tag{3.1}$$

(to prove it, consider the straight line passing through x and y and the restriction of f to it).

Let $d(x) = \operatorname{dist}(x, Gr)$. For any $x \in V_\varepsilon$, there exists the unique $x^* \in U$ such that $|x - x^*|^2 + f^2(x^*) = d^2(x)$. For this x^*,

$$f(x) \geq d(x) \geq f(x^*) \quad \text{and} \quad d(x) \geq |x - x^*|$$

and the equality in any of these inequalities holds if and only if $f(x) = d(x) = f(x^*) = 0$, $x = x^*$. By (3.1),

$$f(x) - f(x^*) \leq \frac{\varepsilon}{\delta}|x - x^*| \leq \frac{\varepsilon}{\delta}d(x)$$

Thus the assertion of the lemma holds for $V = V_\varepsilon$ and $K = \frac{\varepsilon}{\delta} + 1$. ■

For $x \in C$, $v \in \mathcal{E}$, $y \in \mathcal{E}$, set

$$\Phi_{x,v}(y) = \inf\{t > 0 : \ x + y + tv \in C\}$$

and $\Phi_{x,v}(y) = +\infty$ if $x + y + tv \notin C$ for all $t > 0$. For a subspace $\mathcal{L} \subseteq \mathcal{E}$ and $x \in \mathcal{E}$, set $B(\mathcal{L}, x, r) = \{x + y : \ y \in \mathcal{L}, |y| \leq r\}$; $B_r = \{y \in \mathcal{E} : \ |y| \leq r\}$..

Lemma 3.2. *Let C be a generating closed cone in $\mathcal{E}$, $x \in C$, $x \neq 0$, and C has the degree of contact with $\mathcal{L}$ at x greater than $\eta(\varepsilon) = O(\varepsilon)$. Suppose that $v \in \mathcal{E}$ and $x + v \in \operatorname{Int}(C)$. Then there exists a function φ on some interval $(0, \xi)$, $\xi > 0$, such that $\varphi(\varepsilon) = o(\eta(\varepsilon))$ as $\varepsilon \to 0$ and*

$$\varphi(\varepsilon)v + B(\mathcal{L}, x, \varepsilon) \subset \operatorname{Int}(C) \tag{3.2}$$

for all $\varepsilon \in (0, \xi)$.

Proof. The assertion is clear if $x \in \operatorname{Int}(C)$, so we may assume that $x \in \partial C$. Let $\mathcal{H}$ be a tangent hyperplane to C at x. It follows from the assumption of the lemma that $\operatorname{cont}(C, \mathcal{L}, x) > 1$, hence $\mathcal{H} \supseteq \mathcal{L} + \mathbb{R}x$ by Lemma 3.1, (3). Let δ be such that $x + v + B_\delta \subset \operatorname{Int}(C)$. Then $\Phi_{x,v}(y) < 1$ for all $y \in B(\mathcal{H}, x, \delta)$, $\partial C \cap (x + y + [0,1]v)$ is a single point, and $\partial C \cap (B(\mathcal{H}, x, \delta) + [0,1]v)$ may be considered as the graph of the convex function $\Phi_{x,v}(y)$ on $B(\mathcal{H}, x, \delta)$. It follows from Lemma 3.2 that

$$\Phi_{x,v}(y) \leq K \operatorname{dist}(x + y, C) = o(\eta(|y|)) \tag{3.3}$$

for all $y \in B(\mathcal{L}, x, \delta)$ and some $K > 0$. Note also that

$$x + y + tv \in \operatorname{Int}(C) \quad \text{if} \quad \Phi_{x,v}(y) < t < 1 \tag{3.4}$$

Let ψ be any positive increasing function on $(0, \delta)$ such that $\psi(\varepsilon) = o(\eta(\varepsilon))$. Set

$$\varphi(\varepsilon) = \psi(\varepsilon) + \sup\{\Phi_{x,v}(y) : \ y \in B(\mathcal{L}, x, \varepsilon)\}$$

By (3.3) and (3.4), φ satisfies the lemma for some ξ, $0 < \xi \leq \delta$. ■

Let G be a Lie group endowed with a left invariant Riemannian metric. For a subspace $\mathcal{L}$ of it's Lie algebra $\mathcal{G}$, any $x \in \mathcal{G}$, and $\varepsilon > o$, let $\Gamma(\mathcal{L}, x, \varepsilon)$ be the set of all piecewise smooth curves $\gamma : [a, b] \to G$ such that, for both one-side derivatives and all $t \in [a, b]$,

$$\gamma'(t) \in d_e\lambda_{\gamma(t)}(B(\mathcal{L}, x, \varepsilon))$$

where $\lambda_g(h) = gh$ is the left shift. It's subset of closed curves will be denoted by $\Gamma_{\rm cl}(\mathcal{L}, x, \varepsilon)$. Let $l(\gamma)$ be the length of a curve γ. Set

$$\Lambda(\mathcal{L}, x, \varepsilon) = \inf\{l(\gamma) : \ \gamma \in \Gamma_{\rm cl}(\mathcal{L}, x, \varepsilon)\}$$

and $\Lambda(\mathcal{L}, x, \varepsilon) = +\infty$ if $\Gamma_{\rm cl}(\mathcal{L}, x, \varepsilon) = \varnothing$.

Proposition 3.1. *Suppose that C is pointed and generating cone in $\mathcal{G}$, $x \in C$, and*

$$\frac{Q}{\varepsilon} < \Lambda(\mathcal{L}, x, \varepsilon) < +\infty \tag{3.5}$$

for some $Q > 0$ and all sufficiently small $\varepsilon > 0$. If there exists $v \in \mathcal{G}$ such that

$$[v, x] = 0, \quad [v, \mathcal{L}] \subseteq \mathcal{L}, \quad x + v \in \mathrm{Int}(C) \tag{3.6}$$

and the degree of contact of C with $\mathcal{L}$ at x is greater than $\eta(\varepsilon) = \Lambda(\mathcal{L}, x, \varepsilon)^{-1}$ then C is not global.

Proof. If $x \in \mathrm{Int}(C)$ then $C \supset B(\mathcal{L}, x, \varepsilon)$ for small ε and the assertion is true since $\Lambda(\mathcal{L}, x, \varepsilon) < +\infty$ by (3.5). So we may assume that $x \in \partial C$. Another inequality in (3.5) implies that $\mathrm{cont}(C, \mathcal{W}, x) > 1$. By Lemma 3.3, there exists a function $\varphi(\varepsilon) = o(\eta(\varepsilon))$ satisfying (3.2). For $\gamma \in \Gamma_{\mathrm{cl}}(\mathcal{L}, x, \varepsilon)$, $\gamma : [0, L] \to G$, set

$$\tilde{\gamma}(t) = \gamma(t) \exp(\varphi(2\varepsilon) t v), \quad 0 \le t \le L \tag{3.7}$$

Then

$$d_{\tilde{\gamma}(t)} \lambda_{\tilde{\gamma}(t)}^{-1}(\tilde{\gamma}'(t)) = \mathrm{Ad}(\exp(-\varphi(2\varepsilon) v)) d_{\gamma(t)} \lambda_{\gamma(t)}^{-1}(\gamma'(t)) + \varphi(2\varepsilon) v$$

By (3.6), $\mathrm{Ad}(\exp(\tau v))x = x$ and $\mathrm{Ad}(\exp(\tau v))\mathcal{L} \subseteq \mathcal{L}$ for all $\tau \in \mathbb{R}$. So there exists $\mu > 0$ such that $\mathrm{Ad}(\exp(\tau v))B(\mathcal{L}, x, \varepsilon) \subseteq B(\mathcal{L}, x, 2\varepsilon)$ if $0 \le \tau \le \mu$. Since $d_{\gamma(t)} \lambda_{\gamma(t)}^{-1}(\gamma'(t)) \in B(\mathcal{L}, x, \varepsilon)$, by the choice of φ, $\tilde{\gamma}$ is timelike if

$$\varphi(2\varepsilon) L \le \mu \tag{3.8}$$

and ε is sufficiently small. Note that

$$|\tilde{\gamma}'(t) - d_e \lambda_{\tilde{\gamma}(t)}(x)| \le 2\varepsilon + \varphi(2\varepsilon)|v| \tag{3.9}$$

Hence $|L|x| - l(\tilde{\gamma})| \le L(2\varepsilon + \varphi(2\varepsilon)|v|)$. Analogously, $|L|x| - l(\gamma)| \le L\varepsilon$. Since $l(\gamma)$ may be arbitrary close to $\Lambda(\mathcal{L}, x, \varepsilon)$, for all sufficiently small $\varepsilon > 0$ there exists a curve $\gamma \in \Gamma_{\mathrm{cl}}(\mathcal{L}, x, \varepsilon)$ such that (3.8) is true.

We may choose now a sequence $\{\varepsilon_n\}$, $\varepsilon_n \to 0$ as $n \to \infty$, and a sequence $\tilde{\gamma}_n : [0, L_n] \to G$ of timelike curves such that $\tilde{\gamma}_n(L_n) = \exp(\varphi(2\varepsilon_n) L_n v) \to e$ as $n \to \infty$. Since $\varepsilon_n \to 0$, by (3.7) and (3.9),

$$\lim_{n \to \infty} \tilde{\gamma}_n(L_n - t) = \exp(-tx)$$

for all sufficiently small $t > 0$. Therefore, the tangent to $\mathcal{F}(e)$ cone contains $-x$. Thus C is not global. ■

Remark 3.1. In general, C admits no timelike closed curves. This is true if L is a normal closed connected subgroup of G with the tangent Lie subalgebra $\mathcal{L}$, the projection of C to $\mathcal{G}/\mathcal{L}$ is global, and $C \cap \mathcal{L}$ is global in $\mathcal{L}$.

This is not clear if the converse to the Proposition 3.1 is true in the setting of left invariant cone fields. It follows from the results of this paper that for simply connected Lie groups and bi-invariant cone fields the converse is true.

We conclude this section with two geometrical lemmas which will be used in the proof of the necessity part of the main theorem.

Lemma 3.3. *Let $\mathcal{U}$, $\mathcal{V}$ be subspaces of the Euclidean space $\mathcal{E}$, $\mathcal{U} \cap \mathcal{V} = \{0\}$, C be a pointed generating cone, $u, v \in C$, $\mathrm{cont}(C, \mathcal{U}, u) \geq a \geq 1$ and $\mathrm{cont}(C, \mathcal{V}, v) \geq a$. Then $\mathrm{cont}(C, \mathcal{U} + \mathcal{V}, u + v) \geq a$. If $\mathrm{cont}(C, \mathcal{U}, u) > a$ and $\mathrm{cont}(C, \mathcal{V}, v) > a$ then $\mathrm{cont}(C, \mathcal{U} + \mathcal{V}, u + v) > a$.*

Proof. This is a consequence of inequalities

$$|x + y| \geq K(|x| + |y|) \quad \text{for some} \quad K > 0$$

which is true for all $x \in \mathcal{U}$, $y \in \mathcal{V}$ since $\mathcal{U} \cap \mathcal{V} = \{0\}$,

$$\mathrm{dist}(u + v + x + y, C) \leq \mathrm{dist}(u + x, C) + \mathrm{dist}(v + y, C)$$

and $t^a + s^a \leq (t + s)^a$ for $a \geq 1, t, s \geq 0$. ∎

Lemma 3.4. *Let $\mathcal{W}$, $\mathcal{L}$ be subspaces of $\mathcal{E}$, $\mathcal{W} \subset \mathcal{L}$, C be a pointed generating cone in $\mathcal{E}$, $x \in \mathcal{W}$, and the degree of contact of the cone $C \cap \mathcal{L}$ with $\mathcal{W}$ at x in $\mathcal{L}$ is greater or equal to η, where $0 < \eta(\varepsilon) = o(\varepsilon)$ near zero. If $P_{\mathcal{L}} C^* \cap \mathcal{W}^\perp = \{0\}$ then the degree of contact of C with $\mathcal{W}$ at x is greater than η.*

Proof. Suppose that the degree of contact of C with $\mathcal{W}$ at x is not greater than η. Then there exists a sequence $\{x_n\}$ in $\mathcal{W}$ and $\delta > 0$ such that

$$\mathrm{dist}(x + x_n, C) \geq \delta \, \mathrm{dist}(x + x_n, C \cap \mathcal{L}) > 0, \quad x_n \to 0 \quad \text{as} \quad n \to \infty \tag{3.10}$$

For $u \in \mathcal{W} + x$, let u^* ($u^\star$) be the nearest to u point in C ($C \cap \mathcal{L}$). Then

$$u^* - u \in C^*, \quad u^* - u \perp u^*, \quad \text{and} \quad u^\star - u \in \mathcal{L} \tag{3.11}$$

In the triangle with vertices $u, u^*, u^\star$, the angle at the vertex u^* cannot be acute since there is no points which are closer to u than u^* in the interval $[u^*, u^\star]$. Hence

$$|u^\star - u| \cos\alpha \geq |u^* - u| \tag{3.12}$$

where α is the angle at the vertex u. Set

$$y_n = (x + x_n)^* - (x + x_n)$$

Then (3.12) and (3.10) imply that $|P_{\mathcal{L}} y_n| \geq \delta |y_n|$. Thus the normalized sequence $|y_n|^{-1} y_n$ has a limit point y with the nonzero projection to $\mathcal{L}$. By (3.11), $y \in C^*$ and $y \perp x$, so $P_{\mathcal{L}} y \in (C \cap \mathcal{L})^*$ and $P_{\mathcal{L}} y \in \mathcal{W}^\perp$; the last inclusion (which gives a contradiction) follows from Lemma 3.1, (3), since $\mathrm{cont}(C \cap \mathcal{L}, \mathcal{W}, x) > 1$ by the assumption of the lemma. ∎

4. Heisenberg group as an obstruction to the globality

A Lie group will be called a *Heisenberg group* if it is simply connected, two-step nilpotent, and has no abelian direct factors. At the level of Lie algebras, it means that the center of the algebra coincides with it's commutator algebra. We shall receive estimates for lengths of closed curves of the type considered in Proposition 3.1.

A cone field on a manifold M is called *controllable* if the future $\mathcal{F}(m)$ of any point in M coincides with M. For simply connected nilpotent Lie groups, a criterion of controllability for a left invariant cone field defined by a generating cone C in it's Lie algebra $\mathcal{N}$ is as follows (see [3])

$$[\mathcal{N},\mathcal{N}]\cap \operatorname{Int}(C)\neq\varnothing \tag{4.1}$$

The results of this section could be considered as a more precise version of (4.1) for Heisenberg groups; certainly, similar estimates for lengths of timelike closed curves could be done for all nilpotent groups.

Let $\mathcal{H}$ be a *Heisenberg algebra*, i.e. the Lie algebra of a Heisenberg group, $\mathcal{Z}$ be it's center; by the definition above, $[\mathcal{H},\mathcal{H}]=\mathcal{Z}$. If $\mathcal{W}$ is any complementary to $\mathcal{Z}$ subspace of $\mathcal{H}$ then the multiplication in $\mathcal{H}$ is uniquely determined by the bilinear skew-symmetric form $[\;,\;]:\ \mathcal{W}\times\mathcal{W}\to\mathcal{Z}$. Conversly, any bilinear skew-symmetric form $\omega:\ \mathcal{W}\times\mathcal{W}\to\mathcal{Z}$ defines the Heisenberg algebra by the setting $[\mathcal{H},\mathcal{Z}]=0$ and $[u,v]=\omega(u,v)$ for all $u,v\in\mathcal{W}$ if the linear span of $\omega(\mathcal{W},\mathcal{W})$ is equal to $\mathcal{Z}$. For a linear space $\mathcal{W}$, there exists a *free Heisenberg algebra* $\mathcal{H}_{\mathcal{W}}=\mathcal{W}\oplus\mathcal{Z}_{\mathcal{W}}$ with the center $\mathcal{Z}_{\mathcal{W}}=\mathcal{W}\wedge\mathcal{W}$ and $\omega(u,v)=u\wedge v$. Clearly, every Heisenberg algebra is isomorphic to a factor algebra of a free Heisenberg algebra by a subspace of it's center.

Lemma 4.1. *In a free Heisenberg algebra $\mathcal{H}_{\mathcal{W}}$, there exists an Euclidean norm $|\;|$ such that $\mathcal{Z}_{\mathcal{W}}\perp\mathcal{W}$ and every $z\in\mathcal{Z}_{\mathcal{W}}$ admits a representation*

$$z=u_1\wedge v_1+\cdots+u_n\wedge v_n \tag{4.2}$$

where $2n\le\dim\mathcal{W}$, $u_k,v_k\in\mathcal{W}$, $k=1,\dots,n$, and

$$|z|^2=|u_1|^4+|v_1|^4+\cdots+|u_n|^4+|v_n|^4 \tag{4.3}$$

Proof. Let $|\;|$ be any Euclidean norm in $\mathcal{W}$ and $(\;,\;)$ be the corresponding scalar product. Elements of $\mathcal{W}\wedge\mathcal{W}$ could be identified with skew-symmetric bilinear forms on $\mathcal{W}$ by

$$(u\wedge v)(\xi,\eta)=(\xi,u)(\eta,v)-(\xi,v)(\eta,u)\quad\text{for all}\quad \xi,\eta,u,v\in\mathcal{W} \tag{4.4}$$

So, there is the natural isomorphism $z\to A_z$ between $\mathcal{Z}_{\mathcal{W}}$ and the space of all skew-symmetric linear operators in $\mathcal{W}$ defined by the formula $(A_z\xi,\eta)=z(\xi,\eta)$. Lets extend $|\;|$ to $\mathcal{H}_{\mathcal{W}}$ by the setting $|z|^2=-\operatorname{Tr}A_z^2$. Then $|z|^2=\lambda_1^2+\cdots+\lambda_n^2$, where $\lambda_k>0$ and $\pm i\lambda_1,\dots,\pm i\lambda_n$ are nonzero eigenvalues of A_z (so $2n\le\dim\mathcal{W}$). For every $k=1,\dots,n$, there exists a pair of vectors $u_k,v_k\in\mathcal{W}$ such that

$$A_zu_k=\lambda_kv_k,\quad A_zv_k=-\lambda_ku_k,\quad |u_k|=|v_k|=\sqrt{\lambda_k},\quad u_k\perp v_k$$

and $u_k,v_k\perp u_j,v_j$ if $k\ne j$. Since $A_{u\wedge v}\xi=(\xi,u)v-(\xi,v)u$ by (4.4),

$$A_{u_k\wedge v_k}u_k=\lambda_kv_k,\quad A_{u_k\wedge v_k}v_k=-\lambda_ku_k,\quad k=1,\dots,n$$

This data satisfies the assertion of the lemma because $A_z=\sum\limits_{k=1}^{n}A_{u_k\wedge v_k}$. ∎

Corollary 4.1. *In any Heisenberg algebra $\mathcal{H} = \mathcal{W} \oplus \mathcal{Z}$, where $\mathcal{W}$ is a complementary to the center $\mathcal{Z}$ of $\mathcal{H}$ subspace, and every Euclidean norm $|\ |$ in it, there exists a constant $K > 0$ such that each $z \in \mathcal{Z}$ admits a representation of the type*

$$z = [u_1, v_1] + \cdots + [u_n, v_n] \tag{4.5}$$

where $u_k, v_k \in \mathcal{W}$, $k = 1, \ldots, n$, $2n \leq \dim \mathcal{W}$, and

$$\max\{|u_k|^2, |v_k|^2 : \ k = 1, \ldots, n\} \leq K|z| \tag{4.6}$$

Proof. Clear. ∎

The Heisenberg group H may be identified with $\mathcal{H}$ endowed with a multiplication given by the Campbell-Hausdorff formula

$$xy = x + y + \frac{1}{2}[x, y] \tag{4.7}$$

Then the mapping exp is identical, $x^{-1} = -x$, and the multiplicative commutator coincides with the Lie product

$$\{x, y\} = xyx^{-1}y^{-1} = [x, y] \tag{4.8}$$

Let $|\ |$ be an Euclidean norm in $\mathcal{H}$, $z \in \mathcal{Z}$, $r > 0$. We keep the notation of Proposition 3.1 with $\mathcal{G} = \mathcal{H}$.

Lemma 4.2. *There exists a constant $K > 0$ such that for any $z \in \mathcal{Z} \setminus \{0\}$ and some $x_1, \ldots, x_{4n} \in B(\mathcal{W}, z, \sqrt{K|z|})$*

$$z^{-1} = x_1 \ldots x_{4n} \tag{4.9}$$

where $2n \leq \dim \mathcal{W}$.

Proof. Let u_k, v_k be as in Corollary 4.1. For $k = 1, \ldots, n$, set

$$x_{4k-3} = z + v_k, \ x_{4k-2} = z + u_k, \ x_{4k-1} = z - v_k, \ x_{4k} = z - u_k$$

Then $x_{4k-3}\, x_{4k-2}\, x_{4k-1}\, x_{4k} = \{v_k, u_k\} = -[u_k, v_k]$ by (4.7) and (4.8). Since the group multiplication in $\mathcal{Z}$ is the addition, the assertion follows from Corollary 4.1 (with the same K). ∎

Lemma 4.3. *For $x \in \mathcal{H}$, lets consider the parametrized interval $\overline{x} : [0, 1] \to \mathcal{H}$, $\overline{x}(t) = tx$, as a curve in H. If $x \in B(\mathcal{W}, z, r)$ for $z \in \mathcal{Z}$ and $r > 0$ then $\overline{x} \in \Gamma(\mathcal{W}, z, r)$ and $l(\overline{x}) = |x|$.*

Proof. Since exp is identical, $\mathbb{R}x$ with the natural parametization is an one-parametrical group. So the left invariant Riemannian and the Euclidean metrics coincide on it, $l(\overline{x}) = |x|$, and the restriction of the left invariant vector field on H with the vector x at the indentity to $\mathbb{R}x$ is the constant vector field x. ∎

Suppose that $\gamma_1, \gamma_2 \in \Gamma(\mathcal{W}, z, r)$, $\gamma_k : [0, L_k] \to H$, $\gamma_k(0) = 0$, $k = 1, 2$. Then the curve $\gamma_1 \cdot \gamma_2 : [0, L_1 + L_2] \to H$ which coincides with γ_1 on $[0, L_1]$ and with $\gamma_1(L_1)\, \gamma_2(t - L_1)$ on $[L_1, L_1 + L_2]$ also belongs to $\Gamma(\mathcal{W}, z, r)$. Note that

$$l(\gamma_1 \cdot \gamma_2) = l(\gamma_1) + l(\gamma_2) \tag{4.10}$$

because the metric is left invariant.

Proposition 4.1. *For any left invariant Riemannian metric on a Heisenberg group $\mathcal{H}$ and a subspace $\mathcal{W}$ complementary to the center $\mathcal{Z}$ of it's Lie algebra $\mathcal{H}$, there exists $Q > 0$ such that for every $\zeta \in \mathcal{Z} \setminus \{0\}$ and $\varepsilon \in (0, |\zeta|)$*

$$\Lambda(\mathcal{W}, \zeta, \varepsilon) \leq \frac{Q|\zeta|^2}{\varepsilon^2} \tag{4.11}$$

Proof. Let $z \in \mathcal{Z} \setminus \{0\}$ and $x_1 \ldots x_{4n}$ be as in Lemma 4.4. Then, by Lemma 4.5,

$$\gamma = \overline{z} \cdot \overline{x}_1 \cdot \ldots \cdot \overline{x}_{4n} \in \Gamma_{\mathrm{cl}}(\mathcal{W}, z, \sqrt{K|z|})$$

and (4.10) together with (4.6) give the estimate

$$l(\gamma) = |z| + |x_1| + \cdots + |x_{4n}| \leq (4n+1)|z| + 4n\sqrt{K|z|} \tag{4.12}$$

Note that the change $t \to rt$ of the parameter defines one-to-one correspondence between $\Gamma_{\mathrm{cl}}(W, r\zeta, r\varepsilon)$ and $\Gamma_{\mathrm{cl}}(W, \zeta, \varepsilon)$ for any $r > 0$. Set $z = r\zeta$, where $r = K\varepsilon^{-2}|\zeta|$. Then $r\varepsilon = \sqrt{Kr|\zeta|}$, so (4.12) implies that

$$\Lambda(\mathcal{W}, \zeta, \varepsilon) \leq (4n+1)K\frac{|\zeta|^2}{\varepsilon^2} + 4nK\frac{|\zeta|}{\varepsilon} \leq \frac{Q|\zeta|^2}{\varepsilon^2}$$

for $Q = (8n+1)K$. ∎

Theorem 4.1. *Let C be a pointed generating cone in a Lie algebra $\mathcal{F}$, $\mathcal{H} \subseteq \mathcal{F}$ be a Heizenberg subalgebra, $\mathcal{H} = \mathcal{W} \oplus \mathcal{Z}$, where $\mathcal{Z}$ is the center of $\mathcal{H}$ and $\mathcal{W}$ is a complementary to $\mathcal{Z}$ subspace of $\mathcal{H}$. Suppose that $z \in C \cap \mathcal{Z}$, $z \neq 0$, and set*

$$\mathcal{N} = \{x \in \mathcal{F} \colon [x, \mathcal{W}] \subseteq \mathcal{W},\ [x, z] = 0\}$$

If $(z + \mathcal{N}) \cap \mathrm{Int}(C) \neq \varnothing$ and $\mathrm{cont}(C, \mathcal{W}, z) > 2$ then C is not global.

Proof. Combine Proposition 3.1 and Proposition 4.1. ∎

We are ready now to prove the necessity of (GlC) and this is convenient to do it here because the proof of the sufficiency uses other methods.

The simply connected Lie group with the Lie algebra $\mathcal{G}$ is denoted by G, it's connected subgroup corresponding to a subalgebra is denoted by the same Latin letter with the same indices as the subalgebra. We keep the notation of Section 2, the Main Theorem, and Proposition 1.1 up to the end of this section. For any subspace or vector in a simple hermitian nontubular Lie algebra which was introduced in Section 2 the sum of these subspaces or vectors for all simple summands of an ideal $\mathcal{J}_I$ will be denoted by the same letter with the index I. We need also some preparatory lemmas.

Lemma 4.4. *Let $\mathcal{G}$ be a simple hermitian nontubular Lie algebra, C be any invariant cone in $\mathcal{G}$ which includes $c_{\min}$, $\mathcal{W} = \mathcal{G}_1 + \mathcal{G}_{-1}$, where p is defined by (2.10). Then $\mathrm{cont}(C, \mathcal{W}, p) \geq 2$.*

Proof. By (2.23), Lemma 3.3, and the equality $\kappa\theta\mathsf{e} = \theta\mathsf{f}$, this is sufficient to prove that $\mathrm{cont}(C, \mathcal{G}_1, i\theta\mathsf{e}) \geq 2$. For any $x \in C$

$$t^{-2} e^{\mathrm{ad}(ti\theta\mathsf{e})} x = t^{-2} x + t^{-1} \mathrm{ad}(i\theta\mathsf{e}) x + \frac{1}{2} \mathrm{ad}^2(i\theta\mathsf{e}) x \in C \tag{4.13}$$

If $x = k + u$, $u \in \mathcal{G}_{-1}$ then, by (2.22), (2.23), and (2.24),

$$t^{-2} e^{\mathrm{ad}(ti\theta\mathsf{e})} x = t^{-2} x + t^{-1} \mathrm{ad}(i\theta\mathsf{e}) u + t^{-1} \mathrm{ad}(i\theta\mathsf{e}) k + i\theta\mathsf{e}$$

Replacing e with f and $u \in \mathcal{G}_{-1}$ with $v \in \mathcal{G}_1$ and summarizing these two formulas we obtain

$$t^{-2}(2k + u + v) + t^{-1}(\mathrm{ad}(i\theta\mathsf{e})u + \mathrm{ad}(i\theta\mathsf{f})v) + p \in C$$

for all sufficiently small $t > 0$. Note that $\mathrm{ad}(i\theta\mathsf{e})\mathcal{G}_{-1} = \mathcal{G}_1$ and $\mathrm{ad}(i\theta\mathsf{e})$ is invertible on $\mathcal{G}_{-1}$ because

$$\mathrm{ad}(i\theta\mathsf{h})y = [\mathrm{ad}(i\theta\mathsf{e}), \mathrm{ad}(i\theta\mathsf{f})]y = \mathrm{ad}(i\theta\mathsf{f})\,\mathrm{ad}(i\theta\mathsf{e})y$$

for $y \in \mathcal{G}_{-1}$. An analogous assertion is true for f and $\mathcal{G}_1$. Thus, for any compact neighborhood U of zero in $\mathcal{W}$, $\mathrm{dist}(C, e + tU) = O(t^2)$. ■

Lemma 4.5. *For any simple hermitian nontubular Lie algebra $\mathcal{G}$ with the gradation* (2.22)*, l, p, $\mathcal{H}^+$, $\mathcal{H}^-$, and $c_{\min}$ defined in Section* 2*, the following conclusions hold:*

1. $\mathrm{ad}(l)$ *keeps the gradation* (2.22);
2. $p + l \in \mathrm{Int}(C_{\min})$;
3. $\mathcal{H}^+$ *and* $\mathcal{H}^-$ *are Heizenberg subalgebras of* $\mathcal{G}$;
4. $\mathrm{ad}(l)\mathcal{G}_{\pm 2} = \{0\}$.

Proof. The first assertion is true because $l \in \mathcal{G}_0$ by Lemma 2.8. To prove the second, note that $p + l = 2k$ by (2.10) and that the averaging procedure over $\mathrm{Ad}(\mathcal{K})$ shows that $\mathbb{R}k \cap \mathrm{Int}(C) \neq \varnothing$ for any invariant cone C in $\mathcal{G}$. The third is a consequence of Lemma 2.8, (7). The forth is true because l is orthogonal to all roots in $\Delta_{1,1}$ and $\Delta_{-1,-1}$ by the definition of these sets and is a fixed point of the automorphism θ. ■

Lemma 4.6. *If $\mathcal{I}$ is a simple noncompact ideal of a Lie algebra with an invariant cone C then $C \cap \mathcal{I} \neq \{0\}$; moreover, $C \cap \mathcal{I}$ is an invariant cone in $\mathcal{I}$.*

Proof. Note that in (4.13) $x \in C$ need not be an element of $\mathcal{I}$. Thus $C \cap \mathcal{I} \neq \{0\}$. Since the linear span of $C \cap \mathcal{I}$ is an ideal in $\mathcal{I}$, $C \cap \mathcal{I}$ is generating; clearly, this cone is pointed and $\mathrm{Ad}(\mathcal{I})$-invariant. ■

Proposition 4.2. *Suppose that for some nontubular semisimple ideal $\mathcal{J}_I$ of a real Lie algebra $\mathcal{G}$ with an invariant cone C $\pi_I C^* \cap \mathcal{L}_I = \{0\}$. Then C is not global.*

Proof. Set $\tilde{\mathcal{W}} = \mathcal{W} + \mathbb{R}p$ where $\mathcal{W}$ is as in Lemma 4.5. By Lemma 4.6 and (2.25), $p \in C$. So, by Lemma 4.5 and Lemma 3.3, for any nontubular ideal $\mathcal{J}_I$ of $\mathcal{G}$ $\operatorname{cont}(C, \tilde{\mathcal{W}}_I, p_I) \geq 2$. Our aim is to prove that

$$\operatorname{cont}(C, \tilde{\mathcal{W}}_I, p_I) > 2 \tag{4.14}$$

Suppose (4.14) is not true. Then, by Lemma 3.4 with $\mathcal{L} = \mathcal{J}_I$, $\mathcal{W} = \tilde{\mathcal{W}}_I$, there exists $x \in C^*$ such that $\pi_I x \neq 0$ and $\pi_I x \perp \tilde{\mathcal{W}}_I$. Let V_I be the group generated by groups V of Proposition 2.1 for all simple summands of $\mathcal{J}_I$. The averaging procedure A_{V_I} over this group which clearly commutes with π_I, by Proposition 2.1, shows that

$$0 \neq \pi_I A_{V_I} x \in \mathcal{L}_I = \mathcal{W}_I^{\perp} \cap \mathcal{V}_I$$

Since $A_{V_I} x \in C^*$, this contradicts with the assumption of the proposition. Thus (4.14) is proved.

Let $\mathcal{N} = \mathcal{Z}(\mathcal{J}_I)$. Set $F = H_I^- \times P_I \times N$ (the direct product of groups). There is the unique action of F on G such that H_I^- and N act by lefts shifts and P_I acts by right ones. Since $\mathcal{G} = \mathcal{H}_I^- \oplus \mathcal{P}_I \oplus \mathcal{N}$, the orbit of e is open and the stabilizer is discrete. The restriction of the bi-invariant cone field corresponding C to this orbit is an invariant cone field on it. The universal covering space of the orbit coincides with the universal covering group $\tilde{F}$ of F. This is sufficient to prove that the left invariant cone field on $\tilde{F}$ defined by the cone C is not global. ■

By Lemma 4.5, (3), $\mathcal{H}_I = \mathcal{H}_I^+ \oplus \mathcal{H}_I^-$ is a Heizenberg subalgebra of $\mathcal{F}$. By the same lemma, (1), $[l_I, \mathcal{W}_I] \subseteq \mathcal{W}_I$, where the Lie product is taken in $\mathcal{F}$. By (2.23), p_I belongs to the center of $\mathcal{H}_I$; by (2.23) and Lemma 4.5, (4), $[l_I, p_I] = 0$. Since $\mathcal{W}_I \subset \tilde{\mathcal{W}}_I$, (4.14) implies that $\operatorname{cont}(C, \mathcal{W}_I, p_I) > 2$. Thus it remains to prove that

$$(p_I + l_I + \mathcal{N}) \cap \operatorname{Int}(C) \neq \varnothing$$

to satisfy the assumption of the Theorem 4.1.

If $(p_I + l_I + \mathcal{N}) \cap \operatorname{Int}(C) = \varnothing$ then $p_I + l_I$ is a boundary point of the projection of the cone C to $\mathcal{J}_I$ along $\mathcal{N}$. This is a contradiction with Lemma 4.5, (2), since C includes the sum of cones $C_{\min}$ for all simple summands by Lemma 4.6.

5. A sufficient condition for the globality

Let G_1 be a closed connected normal subgroup of a Lie group G, $G_2 = G/G_1$, $\iota : \mathcal{G}_1 \to \mathcal{G}$ and $\pi : \mathcal{G} \to \mathcal{G}_2$ be the corresponding homomorphisms of Lie algebras.

Theorem 5.1. *Suppose that $\mathcal{G}_1$ admits a complementary as a vector space subalgebra in $\mathcal{G}$. If C is an invariant cone in $\mathcal{G}$ such that πC is included to some global invariant cone in $\mathcal{G}_2$ and $\iota^* C^*$ includes some exact invariant cone in $\mathcal{G}_1^*$ then C is global.*

Note that neither $\iota^* C^*$ is closed nor $\operatorname{clos}(\iota^* C^*)$ is pointed in general. We need some lemmas to prove this theorem.

We shall say that a smooth function f on a Lie group G has a *left bounded differential* if the set $\{d_e(f \circ \lambda_g) : g \in G\}$, where $\lambda_g(h) = gh$ is a left shift, is bounded in $\mathcal{G}^*$.

Lemma 5.1. *For any global invariant cone $C \subset \mathcal{G}$ there exists an increasing strictly increasing at e smooth function f on G with the left bounded differential.*

Proof. Let χ be the characteristic function of the set $\mathcal{F}(e)$, φ be a nonnegative nonzero smooth function with a compact support, U be a bounded neighborhood of zero in $\mathcal{G}$, $\succcurlyeq$ denotes the invariant order on G generated by C, and $*$ denotes the convolution. For $\xi \in \mathcal{G}$, let L_ξ be the left invariant vector field on G such that $L_\xi(e) = \xi$. Clearly, f has the left bounded differential if and only if

$$\sup\{|L_\xi f(g)| : g \in G, \xi \in U\} < \infty \tag{5.1}$$

Set $f = \chi * \varphi$. Then $L_\xi f = \chi * L_\xi \varphi$ and (5.1) implies that the differential of f is left bounded. Since χ has the property that $\chi(g) \geq \chi(h)$ if $g \succcurlyeq h$, f is increasing. If $\operatorname{supp} \varphi = \exp(V)$, where V is a sufficiently small neighborhood of zero in $\mathcal{G}$, then the convolution is close the abelian one, hence f is strictly increasing at e if $\operatorname{supp} \varphi$ is a ball in a suitable Euclidean metric. ∎

Lemma 5.2. *Let $\mathcal{L}$, $\mathcal{M}$ be linear spaces, $\alpha : \mathcal{L} \to \mathcal{M}$ a linear mapping, and C be a closed cone in $\mathcal{L}$. The cone αC is closed if and only if there exist compact neighborhoods of zero Q in $\mathcal{M}$ and K in $\mathcal{L}$ such that*

$$\alpha(C \cap K) \supseteq (\alpha C) \cap Q$$

Proof. If $x \in \operatorname{clos}(\alpha C) \setminus \alpha C$ then the same is true for any point in the ray $\mathbb{R}^+ x$. Hence αC is closed if and only if $(\alpha C) \cap Q$ is compact. Standard compactness arguments show that this condition is equivalent to the following one: for any compact convex neighborhood K' of zero in $\mathcal{L}$ and all sufficiently great integer $n > 0$, $\alpha(C \cap (nK')) \supseteq \alpha C \cap Q$. ∎

Lemma 5.3. *Let C be a pointed generating cone in a linear space $\mathcal{L} = \mathcal{M} \oplus \mathcal{N}$, μ be the projection to $\mathcal{M}$ in $\mathcal{L}$ along $\mathcal{N}$. Suppose that μC is closed and $C \cap \mathcal{N}$ is generating in $\mathcal{N}$. Then for any bounded subset $Q \subset \mu C$ and every compact $Q' \subset \operatorname{RelInt}(C \cap \mathcal{N})$ there exists $t > 0$ such that $Q + tQ' \subset C$. Moreover, for this t, $Q \cap \operatorname{RelInt}(\mu C) + tQ' \subset \operatorname{Int}(C)$.*

Proof. It follows from Lemma 5.2 that, for some compact $K \subset C$, $\mu K \supseteq Q$. Let $\nu = 1 - \mu$. Then $\nu K \subset \mathcal{N}$ and $Q \subseteq K - \nu K$. For some sufficiently great $t > 0$, $tQ' - \nu K \subset \operatorname{RelInt}(C \cap \mathcal{N})$ since νK is compact; therefore, $Q + tQ' \subseteq K - \nu K + tQ' \subset C$. By a suitable choice of K, we may assume that for any $x \in \operatorname{RelInt}(\mu C) \cap Q$ there exists $y \in K \cap \operatorname{Int}(C)$ with $\mu y = x$; for this y, $x + tQ' \subset y + tQ' - \nu K \subset \operatorname{Int}(C)$. ∎

Lemma 5.4. *Let $\mathfrak{C}$ be a continuous pointed cone field on a smooth manifold M. Suppose that $\mathfrak{C}^\star(m)\cap\mathrm{Int}(\mathfrak{C}(m)^*)\neq\varnothing$ for all $m\in M$, where $\mathfrak{C}^\star$ is defined by* (1.2). *Then the cone field $\mathfrak{C}$ is global.*

Proof. The assumption of the lemma could be reformulated as follows: for each $m\in M$ there exists an increasing function on M which is strictly increasing at m. Let f be such a function for $m\in M$. We may assume that $f(m)=0$. Set $F_\varepsilon=\{x\in M:\ f(x)\geq\varepsilon\}$.

Choose a local Euclidean coordinate system near m. Let

$$\lambda\in\mathrm{Int}(\mathfrak{C}^*(m))$$

Then λ considered as a function on the chosen neighborhood of m is strictly increasing in some smaller neighborhood because $\mathfrak{C}$ is continuous. Hence, for any $\lambda_1,\dots,\lambda_n\in\mathrm{Int}(\mathfrak{C}^*(m))$, it is possible to find a neighborhood U of m, and $\varepsilon>0$ such that

$$\mathcal{F}(m)\subseteq F_\varepsilon\cup\{x\in U:\ \lambda_k(x)\geq 0,\ k=1,\dots,n\}$$

and the tangent to $\mathcal{F}(m)$ cone cannot be greater than $\mathfrak{C}(m)$ by the Hahn-Banach theorem. ∎

Corollary 5.1. *If C is an invariant cone in a Lie algebra $\mathcal{G}$ such that $C^\star$ is generating then C is global.*

Corollary 5.2. *If $\iota:\ \mathcal{G}_1\to\mathcal{G}_2$ is an isomorphic embedding of a Lie algebra $\mathcal{G}_1$ to $\mathcal{G}_2$ such that $\iota C_1\subset C_2$, where C_k is an invariant cone in $\mathcal{G}_k$, $k=1,2$, and $C_2^\star$ is generating then C_1 is global.*

Corollary 5.3. *An invariant cone C is global if and only if $C^\star\cap\mathrm{Int}(C^*)\neq\varnothing$*

Proof. The first corollary is a consequence of the inclusion $C^\star\subseteq C^*$, the second is true because the operation dual to the intersection is the projection which clearly keeps the property of a cone to be generating, the third follows from Lemma 5.1. ∎

Proof of Theorem 5.1. Let $C_k\subset\mathcal{G}_k$, $k=1,2$, be global invariant cones such that $\iota^*C^*\supseteq C_1^*$, $C_2\supseteq\pi C$. We may assume that $\mathcal{G}$ is equipped by a scalar product in such a way that $\mathcal{V}=\mathcal{G}_1^\perp$ is a subalgebra of $\mathcal{G}$. Let V be an open neighborhood of zero in $\mathcal{V}$ such that the mapping $(g,v)\to g\exp(v)$ is a diffeomorphism between the inverse image E of $\exp(\pi V)$ under the homomorphism $G\to G/G_1$ and $G_1\times V$. Note that it is possible to replace V by a smaller neighborhood of zero if it would be necessary. Since the cone C_2 is global, we may assume that V has the property that any C_2-timelike curve starting at e which leave V cannot come back (the construction of Lemma 5.4 implies the existence of such V). Hence any C-timelike curve in G which leaves E cannot come back and it is sufficient to prove the globality of the bi-invariant cone field $\mathfrak{C}$ corresponding to C on the set E.

Since C_2 is pointed, $C^*\cap\mathcal{V}$ is generating in $\mathcal{V}$. Choose $\varphi\in\mathrm{RelInt}(C_2^*)$ and put $\tilde\varphi(g\exp(v))=\varphi(v)$. Then $\tilde\varphi$ is a C_2-strictly increasing near e function on V. Let $\rho_g(h)=hg$ be a right shift. Replacing V by a smaller neighborhood, we may assume that

$$Q'=\operatorname{clos}\{d_e(\tilde{\varphi}\circ\rho_{\exp(-v)}): v\in V\}$$

is a compact subset of $\operatorname{RelInt}(C_2^*)\subset C^*\cap\mathcal{V}$. Then $\tilde{\varphi}$ is a C-increasing function on E since $\tilde{\varphi}\circ\lambda_g=\tilde{\varphi}$ for all $g\in G_1$.

Let η be a C_1-increasing strictly increasing at e smooth function on G_1 with the left bounded differential. Set $\tilde{\eta}(g\exp(v))=\eta(g)$. Then

$$\tilde{\eta}\circ\rho_{\exp(u)}(g\exp(v))=\eta(g)$$

for u and v sufficiently close to zero in V because $\mathcal{V}$ is a subalgebra. Put

$$Q=\operatorname{clos}\{d_e(\eta\circ\lambda_g\circ\rho_{\exp(-v)}): g\in G_1,\, v\in V\}$$

Since functions $\eta\circ\lambda_g\circ\rho_{\exp(-v)}$ are constant in $\exp(V)$ near e, $Q\subset\mathcal{V}^{\perp}=\mathcal{G}_1$, so $Q\subset C_1^*$.

Let $C'=(\iota^*)^{-1}(C_1^*)\cap C^*$; note that $C'\cap\mathcal{V}=C^*\cap\mathcal{V}$ since $\ker\iota^*=\mathcal{V}$. By Lemma 5.3, $Q+tQ'\subset C'\subseteq C^*$ for some $t>0$. The cone field $\mathfrak{C}$ is left and right invariant, so $f=\tilde{\eta}+t\tilde{\varphi}$ is an increasing function on E. By the same lemma and the choice of η and φ, f is strictly increasing at e, hence near e in V; replacing once more V by a smaller neighborhood, we may assume that this is true in V. Thus for any $p=g\exp(v)\in E$ the function $f\circ\lambda_g$ is increasing on E and strictly increasing at p. By Lemma 5.4, $\mathfrak{C}$ is global in E. ■

6. Globality of invariant cones in nonreductive algebras

This section contains the reduction of the problem to the reductive case. We use the known material on the structure of Lie algebras admitting an invariant cone formulated in Theorem 6.1. The reduction is based on the consideration of special cases of algebras $\mathrm{sp}(2n,\mathbb{R})$ and their semidirect products with Heizenberg algebras.

Recall that the simplectic Lie algebra $\mathrm{sp}(2n,\mathbb{R})$ is a hermitian tubular one and that an invariant cone in it is unique up to a sign. A homomorphism ρ of two algebras $\mathcal{G}_1,\mathcal{G}_2$ with invariant cones C_1,C_2 will be called positive if $\rho(C_1)\subseteq C_2$.

Theorem 6.1. *Let $\mathcal{G}$ be a Lie algebra which admits an invariant cone C. Then*

(1) *$\mathcal{G}=\mathcal{R}_0\oplus\mathcal{W}\oplus\mathcal{Z}$, where $\mathcal{Z}$ is the center of $\mathcal{G}$, $\mathcal{R}_0$ is a reductive subalgebra of $\mathcal{G}$, $\mathcal{N}=\mathcal{W}\oplus\mathcal{Z}$ is the maximal nilpotent ideal, $[\mathcal{W},\mathcal{W}]\subseteq\mathcal{Z}$, and $\mathcal{W}$ is the unique $\mathrm{Ad}(\mathcal{R}_0)$-invariant subspace of $\mathcal{N}$ complementary to $\mathcal{Z}$ which contains no nontrivial fixed points*;

(2) *if $\mathcal{W}\neq\{0\}$ then $C\cap\mathcal{Z}\neq\{0\}$ and, for any $w\in\mathcal{W}$, $\mathrm{ad}^2(w)C\subseteq C\cap\mathcal{Z}$*;

(3) *if* $\lambda \in \mathcal{Z}^*$ *is strictly positive on* $C \cap \mathcal{Z}$ *then the skew-symmetric bilinear form* $\omega_\lambda(u,v) = \lambda([u,v])$ *on* $\mathcal{W}$ *is nondegenerate; moreover,* $\omega_\lambda(\mathrm{ad}(h)w, w) > 0$ *for every* $h \in \mathrm{Int}(C)$ *and* $w \in \mathcal{W}$, $w \neq 0$;

(4) *the representation of* $\mathcal{G}$ *in* $\mathcal{W} \cong \mathcal{N}/\mathcal{Z}$ *with the form* ω_λ *induced by* ad *is a positive simplectic representation with the kernel* $\mathcal{I}_s \oplus \mathcal{N}$, *where* $\mathcal{I}_s$ *is the maximal semisimple ideal of* $\mathcal{G}$;

(5) *for any simple noncompact ideal* $\mathcal{J}$ *of* $\mathcal{G}$ *and the natural projection* π *to* $\mathcal{J}$ *along it's centralizer* $\mathcal{Z}(\mathcal{J})$, $\operatorname{clos} \pi C$ *and* $C \cap \mathcal{J}$ *are invariant cones in* $\mathcal{J}$;

(6) *for any simple compact ideal* $\mathcal{J}$ *of* $\mathcal{G}$ *and the natural projection* π *to the centralizer* $\mathcal{Z}(\mathcal{G})$ *along* $\mathcal{J}$, $\pi C = C \cap \mathcal{Z}(\mathcal{J})$; *moreover,* $\mathcal{J} \cap C = \{0\}$;

(7) *any* $h \in \mathrm{Int}(C)$ *is compact (i.e.* $\mathrm{ad}(x)$ *has purely imaginary eigenvalues and is semisimple).* ∎

The statement of the theorem is close to [2]; these results are known (see [6] or [3]). We keep the notation of Theorem 6.1 with an addition

$$\mathcal{R} = \mathcal{R}_0 \oplus \mathcal{Z} \tag{6.1}$$

up to the end of this section. The subalgebra $\mathcal{R}$ defined by (6.1) will be called *the main reductive subalgebra of* $\mathcal{G}$.

Lemma 6.1. *Let* π *be the projection onto* $\mathcal{R}$ *along* $\mathcal{W}$, $h \in \mathrm{Int}(C)$. *Then*

(1) *there exists the unique* $u \in \mathcal{W}$ *such that* $e^{\mathrm{ad}(u)} h \in \mathrm{RelInt}(C \cap \mathcal{R})$;

(2) $\pi h \in \mathrm{RelInt}(C \cap \mathcal{R})$.

Proof. Let $h = r_0 + w + z$ be the decomposition of $h \in \mathrm{Int}(C)$ as in Theorem 6.1, (1). Then, for any $u \in \mathcal{W}$,

$$e^{\mathrm{ad}(u)} h = r_0 + (w + [u, r_0]) + (z + [u, w] + \frac{1}{2}[u,[u,h]])$$

Since $\mathrm{ad}(h)$ is nondegenerate on the factor space $\mathcal{W}/\mathcal{Z}$ by Theorem 6.1, (3), $\mathrm{ad}(r_0)$ is nondegenerate on $\mathcal{W}$, hence there exists the unique $u \in \mathcal{W}$ such that $w = -[u, r_0]$. So h is conjugated with $h' = r_0 + z - \frac{1}{2}[u,[u,h]]$; clearly, $h' \in \mathrm{Int}(C) \cap \mathcal{R} = \mathrm{RelInt}(C \cap \mathcal{R})$. By Theorem 6.1, (2), $[u,[u,h]] \in C \cap \mathcal{Z}$, hence

$$\pi h = h' + \frac{1}{2}[u,[u,h]] \in \mathrm{RelInt}(C \cap \mathcal{R})$$ ∎

Let $\mathcal{U}$ be a real linear space with a nondegenerate skew-symmetric bilinear form ω. The Lie algebra $\mathrm{sp}(\mathcal{U},\omega)$ of the Lie group $\mathrm{Sp}(\mathcal{U},\omega)$ of linear transformations keeping ω could be realized as the Lie algebra $\mathcal{Q}_2(\mathcal{U})$ of quadratic forms on $\mathcal{U}$ with the Poisson brackets as a Lie product. Recall that ω defines the Poisson brackets of functions f and h by $\{f,h\}(w) = \omega(Jd_w f, Jd_w h)$, where $J : \mathcal{U}^* \to \mathcal{U}$ is defined by $\omega(J\xi, x) = -\xi(x)$ for all $x \in \mathcal{U}$. The natural isomorphism between $\mathcal{Q}_2(\mathcal{U})$ and $sp(\mathcal{U},\omega)$ corresponds to q the linear vector field Jdq.

Fix $u \in \mathcal{U}$ and consider the isotropy subgroup St_u of u. This group keeps the hyperplane $u^\perp$ which contains u; it's Lie algebra st_u is the centralizer of the form $\omega(u,x)^2$. Choose $v \in \mathcal{U}$, $v \notin u^\perp$, such that $\omega(u,v) = 1$ and set $\mathcal{W} = u^\perp \cap v^\perp$. Then ω is nondegenerate in $\mathcal{W}$. The space $\mathcal{Q}_2(\mathcal{U})$ restricted to $\mathcal{W} + v$ may be identified with the space $\mathcal{P}_2(\mathcal{W})$ of polynomials of degree ≤ 2 which is also a Lie algebra with respect to the Poisson brackets in $\mathcal{W}$. Set $\tilde{p}(w+v) = p(w)$. Then, for any $p \in \mathcal{P}_2(\mathcal{W})$, $\tilde{p}$ admits the unique extension $q \in \mathcal{Q}_2(\mathcal{U})$ from $\mathcal{W} + v$ to $\mathcal{U}$ such that $d_u q = 0$. This construction defines an isomorphic embedding $\iota : \mathcal{P}_2(\mathcal{W}) \to \mathcal{Q}_2(\mathcal{U})$ with the image st_u. This construction may be realized up to an isomorphism in suitable coordinates as follows:

$$\{f,h\} = \sum_{k=1}^{n+1} \left(\frac{\partial f}{\partial x_k}\frac{\partial h}{\partial y_k} - \frac{\partial h}{\partial x_k}\frac{\partial f}{\partial y_k} \right)$$

and ι corresponds to the polynomial $p_2(w) + p_1(w) + p_0$ the quadratic form $p_2(w) + x_{n+1}\, p_1(w) + p_0\, x_{n+1}^2$ which doesn't depend on y_{n+1}.

Each of these algebras admits an invariant cone — the cone of nonnegative functions. In $\mathcal{Q}_2(\mathcal{U})$, this is the unique up to a sign invariant cone, in $\mathcal{P}_2(\mathcal{W})$ it is not unique.

The natural decomposition

$$\mathcal{P}_2(\mathcal{W}) = \mathcal{Q}_2(\mathcal{W}) \oplus \mathcal{Q}_1(\mathcal{W}) \oplus \mathcal{Q}_0(\mathcal{W})$$

where $\mathcal{Q}_k(\mathcal{W})$ is the space of homogenious polynomials of degree $k, k = 0,1,2$, is the decomposition of Theorem 6.1, (1), for $\mathcal{P}_2(\mathcal{W})$.

The algebra $\mathcal{P}_2(\mathcal{W})$ is the basic example of a nonreductive Lie algebra with an invariant cone. The subspace $\mathcal{P}_1(\mathcal{W})$ of polynomials of degree ≤ 1 is the Heisenberg algebra with the center $\mathcal{P}_0(\mathcal{W})$ consisting of constant functions.

Let C_q^+ and C_p^+ be cones of nonnegative functions in $\mathcal{Q}_2(\mathcal{U})$ and $\mathcal{P}_2(\mathcal{W})$ respectively.

The multiplication in $\mathcal{N}$ could be defined by the Campbell-Hausdorff formula (4.7). Let α be the representation of the simply connected group R_0 in $\mathcal{W}$ with the tangent representation ad. Since $\mathcal{G}$ is the semidirect product of $\mathcal{N}$ and $\mathcal{R}_0$ via ad, a calculation leads to the following multiplication low in the simply connected group $G = R_0 \times \mathcal{W} \times \mathcal{Z}$

$$(r_1, w_1, z_1)(r_2, w_2, z_2) = (r_1 r_2, w_1 + \alpha(r_1)\, w_2, z_1 + z_2 + \frac{1}{2}[w_1, \alpha(r_1)\, w_2]) \tag{6.2}$$

Set $g_0 = (e, w_0, 0)$, $g_k = (r_k, 0, z_k)$, $k = 1,2, g = (r,w,z)$. Then $g_0^{-1} = (e, -w_0, 0)$. Hence

$$g_0^{-1} g\, g_0 = (r, (1-\alpha(r))\, w_0 + w, z + \frac{1}{2}[w_0, w] - \frac{1}{2}[w_0 + w, \alpha(r)\, w_0]) \tag{6.3}$$

Put $\beta(r) = (1-\alpha(r))^{-1}$; then

$$\alpha(r)\beta(r) = \beta(r) - 1 = -\beta(r^{-1}) \tag{6.4}$$

Lemma 6.2. *Let g_k, $k = 0,1,2$ be as above. Suppose that $1-\alpha(r)$ is invertible. Then*

$$g_0\, g_1\, g_0^{-1}\, g_2 \sim (r_1 r_2, 0, z_1 + z_2 + \frac{1}{2}[(\beta(r_2) - \beta(r_1 r_2))u, u]) \tag{6.5}$$

where $\sim$ *means that left and right parts of* (6.5) *are conjugated by an element of* G *of the type* $(e,\xi,0)$.

Proof. Put $w_0=-\beta(r)w$ in (6.3). Then

$$g_0^{-1}g\,g_0=(r,0,z-\frac{1}{2}[\alpha(r)\,w_0,w_0])=(r,0,z-\frac{1}{2}[\beta(r)w,w]) \tag{6.6}$$

The setting $w=0$ and $u=(1-\alpha(r))\,w_0$ in (6.3) gives the equality

$$g_0^{-1}g\,g_0=(r,u,z+\frac{1}{2}[\beta(r)u,u]) \tag{6.7}$$

By (6.2), (6.3), and (6.7),

$$g_0\,g_1\,g_0^{-1}\,g_2=(r_1\,r_2,u,z_1+z_2+\frac{1}{2}[\beta(r_2)u,u])$$

Thus (6.6) implies (6.5). ■

Lemma 6.3. *Suppose that* $1-\alpha(r)$ *is invertible. Then for any* $w\in\mathcal{W}$ *and* $x\in\mathcal{R}_0$ *such that* $(x+\mathcal{I}_s\oplus\mathcal{N})\cap C\neq\varnothing$

$$-[\beta(r)\,\mathrm{ad}(x)\alpha(r)\beta(r)w,w]\in C\cap\mathcal{Z} \tag{6.8}$$

Proof. By the definition of α, $[\alpha(r)u,\alpha(r)v]=[u,v]$ for all $u,v\in\mathcal{W}$. Hence $[\alpha(r)u,v]=[u,\alpha(r^{-1})v]$ and $[(1-\alpha(r))u,v]=[u,(1-\alpha(r^{-1}))v]$ which implies that $[\beta(r)u,v]=[u,\beta(r^{-1})v]$. By (6.4),

$$[\beta(r)\,\mathrm{ad}(x)\alpha(r)\beta(r)w,w]=-[\mathrm{ad}(x)\beta(r^{-1})w,\beta(r^{-1})w]=-\,\mathrm{ad}^2(u)x$$

where $u=\beta(r^{-1})w\in\mathcal{W}$. If $\xi\in\mathcal{I}_s\oplus\mathcal{N}$ then $\mathrm{ad}^2(u)\xi=0$. Thus Theorem 6.1, (2), and the assumption of the lemma implies (6.8). ■

For a linear operator X, let $\sigma(X)$ denotes the set of eigenvalues of X, $\rho(X)=\max\{|\lambda|:\lambda\in\sigma(X)\}$ be the *spectral radius* of X. We identify $\mathrm{sp}(\mathcal{U},\omega)$ and $\mathcal{Q}_2(\mathcal{U})$ by

$$q_X(u)=\frac{1}{2}\omega(Xu,u),\quad X\in\mathrm{sp}(\mathcal{U},\omega)$$

Lemma 6.4. *Let* U *be a real linear space equipped by a skew-symmetric bilinear nondegenerate form* ω *and an Euclidean norm* $|\ |$. *Then there exists a neighborhood* Q *of zero in* $\mathcal{U}$ *and a norm* $\|\ \|$ *in* $\mathrm{sp}(\mathcal{U},\omega)$ *such that for every* $X\in Q\cap\mathrm{Int}(C_q^+)$ *and* $u\in\mathcal{U}$

$$\omega(\beta(e^X)u,u)\geq\frac{|u|^2}{\|X\|} \tag{6.9}$$

where $\beta(t)=(1-t)^{-1}$.

Proof. We may assume that $\omega(u,v)=(Ju,v)$, where $(\ ,\ )$ is a scalar product in $\mathcal{U}$ such that $|u|^2=(u,u)$, $J:\mathcal{U}\to\mathcal{U}$, $J^2=-1$, $J^*=-J$, and $X^*J+JX=0$ for all $X\in \mathrm{sp}(\mathcal{U},\omega)$. If $X\in C_q^+$ then JX is a nonnegative symmetric linear operator in $\mathcal{U}$. Note that $X\in \mathrm{Int}(C_q^+)$ implies $((JX)^{-1})^*=-JX^{-1}\in \mathrm{Int}(C_q^+)$. Hence

$$(-JX^{-1}u,u)\geq \frac{|u|^2}{\rho(JX)} \tag{6.10}$$

for all $u\in\mathcal{U}$. Since JX is symmetric, $\rho(JX)$ coincides with the operator norm of JX. The equality $\beta(e^t)=-t^{-1}+\frac{1}{2}+o(1)$ implies that, for a suitable choice of a sufficiently small bounded neibourhood Q and a norm $\|\ \|$ as a scalar multiplier of the operator norm, (6.9) is a consequence of (6.10). ∎

Lemma 6.5. *Let π be the natural homomorphism $\mathcal{G}\to\mathcal{G}'=\mathcal{G}/(\mathcal{I}_s\oplus\mathcal{N})$. Then the cone $\operatorname{clos}\pi(C)$ is global in $\mathcal{G}'$.*

Proof. Theorem 6.1, (4), defines an embedding of $\pi\mathcal{G}$ to $\mathcal{Q}_2(\mathcal{W})$ such that $\pi\,\mathrm{Int}(C)\subseteq \mathrm{Int}(C_q^+)$. Recall that $\mathcal{Q}_2(\mathcal{W})$ is tubular. Hence, by Corollary 5.8, $(C_q^+)^\star\neq\{0\}$; since $\mathcal{Q}_2(\mathcal{W})$ is simple, $(C_q^+)^\star\neq\{0\}$ is generating. Thus Corollary 5.7 implies the globality of $\operatorname{clos}\pi(C)$. ∎

We need once more a known result; for a proof, see [6], Chapter 6.

Theorem 6.2. *Let C be an invariant cone in a Lie algebra $\mathcal{G}$, S be the semigroup topologically generated by $\exp(C)$. If C is global then there exists a neighborhood V of zero in $\mathcal{G}$ such that $\exp(C\cap V)=S\cap\exp(V)$.* ∎

Lemma 6.6. *Let π be as in Lemma 6.5. There exists a neighborhood U of zero in $\mathcal{R}_0$ such that $r_1=\exp(\xi_1)$, $r_2=\exp(\xi_2)$, $\xi_1,\xi_2\in U$, and $\pi\,\xi_1,\pi\,\xi_2\in\mathrm{Int}(\pi C)$ implies*

$$[(\beta(r_2)-\beta(r_1r_2))w,w])\in C\cap\mathcal{Z} \tag{6.11}$$

for all $w\in\mathcal{W}$.

Proof. It follows from the matrix equality

$$(1-BA)^{-1}-(1-A)^{-1}=(1-A)^{-1}(B-1)A(1-BA)^{-1}$$

that

$$\frac{d}{dt}(1-e^{tX}A)^{-1}_{t=0}=(1-A)^{-1}XA(1-A)^{-1}$$

Hence for any $x\in\mathcal{R}_0$ and $r\in R_0$ such that $1-\alpha(r)$ is invertible in $\mathcal{W}$,

$$\frac{d}{dt}\beta(\exp(tx)r)_{t=0}=\beta(r)\,\mathrm{ad}(x)\alpha(r)\beta(r)$$

Put $x=\xi_1$ and $r=r_2$. Since $\pi\,\xi_k\in\pi C$, $k=1,2$, by Lemma 6.3,

$$-\frac{d}{dt}[\beta(\exp(t\,\xi_1)\,r_2)w,w]\in C\cap\mathcal{Z}$$

for all $w \in \mathcal{W}$. Note also that $\pi\,\xi_k \in \pi\,\mathrm{Int}(C)$, $k = 1, 2$, implies invertibility of $1 - \alpha(r_k)$ by Theorem 6.1, (3), and Lemma 6.4 if U is sufficiently small.

Set $r_t = \exp(t\,\xi_1)\,r_2$. The integration by t and the same consideration show that (6.11) is true if $r_t = \exp(\xi_t)$ with $\xi_t \in U$ such that $\pi\,\xi_t \in \mathrm{Int}(\pi C)$ for all $t \in [0,1]$.

By Lemma 6.5, the cone $\mathrm{clos}(\pi C)$ is global. Let V be as in Theorem 6.2 for the algebra $\mathcal{G}'$ in this lemma and chose a convex neighborhood U of zero in $\mathcal{R}_0$ with the property $\exp(\pi U)\exp(\pi U) \subseteq \exp(V)$. It follows from Theorem 6.2 that $\exp(\mathrm{Int}(\pi C) \cap V) = \mathrm{Int}(\exp(\pi C \cap V))$. Therefore,

$$\exp(\mathrm{Int}(\pi C) \cap U)\exp(\mathrm{Int}(\pi C) \cap U) \subseteq \exp(\mathrm{Int}(\pi C) \cap V)$$

Thus ξ_t has the desired property if $\xi_1, \xi_2 \in U$, $\pi\,\xi_1, \pi\,\xi_2 \in \mathrm{Int}(\pi C)$, and U is sufficiently small. ∎

Lets denote by S_R° the semigroup in R algebraically generated by the set $\exp(\mathrm{RelInt}(C \cap \mathcal{R}))$, $S_R = \mathrm{clos}(S_R^\circ)$,

$$S^\circ = \{g_0\, s g_0^{-1} : s \in S_R^\circ,\ g_0 = (e, w_0, 0),\ w_0 \in \mathcal{W}\} \tag{6.12}$$

We keep the notation of the Lemma 6.5. The semigroup topologically generated by $\exp(\pi C)$ in the simply connected group G' corresponding to the Lie algebra $\mathcal{G}'$ will be denoted by S'. Let p denotes the homomorphism of simply connected groups with the tangent homorphism π.

Lemma 6.7. *There exists a neighborhood V of zero in $\mathcal{G}'$ such that*

$$S_0 = S^\circ \cup p^{-1}(S' \setminus \exp(V)) \tag{6.13}$$

is a semigroup in G.

Proof. Set $g_k = (r_k, 0, z_k)$, $k = 1, 2$, and suppose that $g_1, g_2 \in S_R^\circ$. Since S_0 is invariant under all inner automorphisms and $p(S_0) \subseteq S'$, this is sufficient to find U such that

$$g_0\, g_1\, g_0^{-1} g_2 \in S_0 \tag{6.14}$$

if $p(g_1), p(g_2) \in U$ and g_0 is as in (6.12).

Note that $S_R \supset C \cap \mathcal{Z}$ since exp is identical in $\mathcal{N}$. Let U be as in Lemma 6.6. Then, by (6.5) and (6.11),

$$g_0\, g_1\, g_0^{-1}\, g_2 \sim (r_1\, r_2, 0, z_1 + z_2 + \frac{1}{2}[(\beta(r_2) - \beta(r_1\, r_2))u, u]) \in g_1\, g_2 \exp(C \cap \mathcal{Z})$$

if $r_1\, r_2 \in \exp(U)$. Note that $g_2 \exp(C \cap \mathcal{Z}) \subset S_R^\circ$ since $g_2 \in \exp(\mathrm{Int}(C) \cap U)$ by the choice of U and Theorem 6.8, (1). Hence $g_1\, g_2 \exp(C \cap \mathcal{Z}) \subset S_R^\circ$. It remains to set $V = \pi U$ to satisfy (6.14). ∎

Theorem 6.3. *Let $\mathcal{G}$ be a nonreductive Lie algebra with an invariant cone C, $\mathcal{R}$ be the main reductive subalgebra of $\mathcal{G}$. Then C is global in $\mathcal{G}$ if and only if $C \cap \mathcal{R}$ is global in $\mathcal{R}$.*

Proof. If C is global in $\mathcal{G}$ then $C \cap \mathcal{R}$ is global in $\mathcal{R}$ by Lemma 6.1 and Corollary 5.2.

Suppose that $C \cap \mathcal{R}$ is global. Let S be the closed semigroup in G generated by $\exp(C)$, $\tilde{S} = \mathrm{clos}(S_0)$, where S_0 is defined by (6.13). By Lemma 6.1, (1), $\tilde{S} \supseteq \exp(\mathrm{Int}(C))$, so Lemma 6.7 implies that $\tilde{S} \supseteq S$. Clearly, $S \supset S^\circ$, hence

$$\tilde{S} \cap p^{-1}(\exp(V)) = S \cap p^{-1}(\exp(V)) \tag{6.15}$$

where p and V are as in Lemma 6.7. Furthermore, by Theorem 6.2 there exists a neighborhood U of zero in $\mathcal{R}$ such that

$$\exp(U) \cap S_R = \exp(C \cap U) \tag{6.16}$$

We may assume that U is convex, open in $\mathcal{R}$, $\exp$ is diffeomorphic on U, and $\pi U \subseteq V$, where $\pi = d_e p$. Moreover, the existence of an increasing strictly increasing at e function (Lemma 5.3) implies the existence of a neighborhood U' of zero in $\mathcal{R}$ such that

$$g_1, g_2 \in S_R, g_1 g_2 \in \exp(U') \Longrightarrow g_1, g_2 \in \exp(U) \cap S_R \tag{6.17}$$

We may also assume that U' has the same properties as U. Set

$$\tilde{U} = \{(r, w, z) : (r, 0, z) \in \exp(U'),\ w \in \mathcal{W}\}$$

Let $g = (r, w, z) \in S^\circ \cap \exp(\tilde{U})$. The neighborhood V in Lemma 6.7 could be supposed to be a diffeomorphic image of a sufficiently small neighborhood of zero in $\mathcal{G}'$ by $\exp$. By Theorem 6.1, (3), we may assume that $1 - \alpha(r)$ is invertible for all $r \in \mathrm{Int}(\exp(C \cap U'))$. Then, by (6.6) and the definition of S°,

$$g \sim h = (r, 0, z - \frac{1}{2}[\beta(r)w, w]) \in S_R^\circ \tag{6.18}$$

and there exists the unique $u \in \mathcal{W}$ such that $g = (e, u, 0)^{-1} h (e, u, 0)$. It follows from Lemma 6.4 and Theorem 6.1, (3), that $\lambda([\beta(r)w, w]) \geq 0$ for any strictly positive on $C \cap \mathcal{Z}$ linear functional $\lambda \in \mathcal{Z}^*$ if V is sufficiently small. Hence

$$\zeta = \frac{1}{2}[\beta(r)w, w] \in C \cap \mathcal{Z}$$

Recall that $\exp$ is identical on $\mathcal{N}$. Thus

$$\exp(\zeta) \in S_R \tag{6.19}$$

and

$$(r, 0, z) = h \exp(\zeta) \in S_R^\circ \cap \exp(U') \tag{6.20}$$

By (6.20), (6.19), (6.18), and (6.17), $h \in \exp(U) \cap S_R^\circ$. Hence $h \in \exp(C \cap \mathcal{R})$ by (6.16). Therefore,

$$g \in \exp(\mathrm{Int}(C)) \tag{6.21}$$

Lets fix $\lambda \in \mathcal{Z}^*$ strictly positive on $C \cap \mathcal{Z}$. It follows from Lemma 6.4 that

$$|w|^2 \leq K\lambda(\zeta) \tag{6.22}$$

for some $K > 0$. Hence (6.21) together with (6.22) imply that $S^\circ \cap \tilde{U}$ can be included into a neighborhood Q of e in G which is a diffeomorphic image of a neighborhood of zero in $\mathcal{G}$ if U' is sufficiently small. By (6.21),

$$Q \cap \mathrm{clos}(S^\circ) \subseteq \exp(C)$$

Thus (6.15) implies the globality of C. ■

7. Proof of the Main Theorem

In this final section we deal with the remainder of the proof of the Main Theorem. Recall that the necessity of (GlC) was proved in Section 4. The sufficiency was partially reduced to the reductive case in Section 6; for the complete reduction, we need two more lemmas. We keep the notation of the Main Theorem and Theorem 6.1. So $\mathcal{I}_s$ is the maximal semisimple ideal of $\mathcal{G}$. Let $\mathcal{I} = \mathcal{I}_s \oplus \mathcal{Z}$, π be the projection in $\mathcal{G}$ to $\mathcal{I}$ dual to the embedding. The dual space $\mathcal{R}^*$ to the main reductive subalgebra $\mathcal{R}$ is identified with $\mathcal{R}$ by an invariant bilinear form which is positive definite on it's maximal compactly embedded subalgebra, hence on $\mathcal{Z}$. Then the projection π restricted to $\mathcal{R}$ is an orthogonal one.

Lemma 7.1. *Suppose that πC^* includes some exact invariant cone. Then C is global.*

Proof. By Lemma 6.5, the projection of the cone C to $\mathcal{G}' = \mathcal{G}/(\mathcal{I}_s \oplus \mathcal{N})$ is included to some global invariant cone in $\mathcal{G}'$. Lemma 6.1, (2), implies that the same is true for the cone $C \cap \mathcal{R}$. Note that $(\mathcal{I}_s \oplus \mathcal{N}) \cap \mathcal{R} = \mathcal{I}$. Thus Theorem 5.1 and the assumption of the lemma implies the globality of $C \cap \mathcal{R}$. By Theorem 6.3, C is global in $\mathcal{G}$. ■

Clearly, a convex cone is exact if and only if it is closed and the dual cone is global. Thus the following lemma shows that if (GlC) implies the globality for reductive algebras then the assumption of Lemma 7.1 is satisfied.

Lemma 7.2. *Let $\mathcal{G} = \mathcal{I} \oplus \mathcal{J}$, where $\mathcal{I}$, $\mathcal{J}$ are ideals of $\mathcal{G}$, π be the projection to $\mathcal{I}$ along $\mathcal{J}$, C be an invariant cone in $\mathcal{G}$. Suppose that $\mathcal{I}$ is reductive and has no compact simple ideals. Let $\Lambda \subset \pi C$ be a finite set consisting of compact nonzero elements which could be included into some pointed cone. Then πC includes the closed $\mathrm{Ad}(\mathcal{I})$-invariant cone generated by Λ; moreover, this cone is pointed.*

Proof. Suppose first that $\mathcal{I}$ is semisimple. It follows from Theorem 6.1, (5), that $\operatorname{clos} \pi C$ is a pointed generating cone which includes the sum of minimal cones and is included to the sum of maximal ones. Therefore, for all $x \in \Lambda$ $\rho(\operatorname{ad}(x)) > 0$ (the spectral radius ρ is defined in Section 6). Hence the set $\operatorname{Ad}(\mathcal{I})\Lambda$ could be separated from zero by a neighborhood U which may be choosen in such a way that $\pi C \setminus U$ is convex since $\operatorname{clos} \pi C$ is pointed. Thus the closure K of the convex hull of $\operatorname{Ad}(\mathcal{G})\Lambda$ doesn't contain zero.

Let Λ' be a finite subset of C such that $\pi\Lambda' = \Lambda$, C' be the closed $\operatorname{Ad}(\mathcal{J})$-invariant cone generated by Λ', Q be a compact convex neiborhood of zero in $\mathcal{J}$ such that $P = Q \oplus K \supset \Lambda'$. Then $\operatorname{clos}(\mathbb{R}^+P)$ is a pointed cone with the trivial intersection with $\mathcal{J}$ and $C' \subseteq \operatorname{clos}(\mathbb{R}^+P)$, so $C' \cap \mathcal{J} = \{0\}$. Thus $\pi C'$ is pointed and the closed $\operatorname{Ad}(\mathcal{I})$-invariant cone generating by Λ coincides with $\pi C'$.

For a reductive $\mathcal{I}$, let Λ_0 be the intersection of Λ with the center of $\mathcal{I}$, $\Lambda_1 = \Lambda \setminus \Lambda_0$. By the assumption of the lemma, Λ_0 is included into some pointed cone in the center. By the consideration above, the $\operatorname{Ad}(\mathcal{I})$-invariant closed cone generated by the projection of Λ_1 to the semisimple part of $\mathcal{I}$ is pointed. Hence Λ_1 is included to the sum of the two cones above, $\operatorname{Ad}(\mathcal{I})$-invariant cone generated by Λ is pointed, and the closed convex hull of the orbit of Λ_1 can be separated from zero. It remains to repeat once more the construction of the previous paragraph. ■

In the last lemma we consider a special case of a reductive algebra $\mathcal{R}$. Let $\mathcal{R} = \mathcal{S} \oplus \mathcal{A}$, where $\mathcal{S}$ is semisimple and $\mathcal{A}$ abelian. We identify $\mathcal{R}$ and $\mathcal{R}^*$ as above. Let σ be the orhogonal projection $\mathcal{R} \to \mathcal{S}$.

Lemma 7.3. *Suppose that each simple ideal of $\mathcal{S}$ is nontubular. Let $\mathcal{L}$ be the same as in* (GlC) *for S. If $\dim \mathcal{A} \leq 1$ and $\sigma C \cap \mathcal{L}$ has nonempty interior in $\mathcal{L}$ then C is exact.*

Proof. Let $l \in \sigma C \cap \mathcal{L}$. Then $l = s_1 l_1 + \cdots + s_n l_n$, where l_k are defined by (2.10) for the simple summand $\mathcal{J}_k$. Since $\sigma C \cap \mathcal{L}$ has a nonempty interior in $\mathcal{L}$, by a choice of a sign of l_k, we may assume that $s_k > 0$, $k = 1, \ldots, n$. Let f_k be the function (2.29) for $\mathcal{J}_k$, $f = 2(s_1 f_1 + \cdots + s_n f_n)$. It follows from the construction of f that for all $x \in S$, $u \in K$, and $v \in P'$, where K, P' are products of subgroups K, P in Lemma 2.10 for all simple summands,

$$f(uxv) = f(x) + f(u) + f(v) \quad \text{and} \quad d_e f = l + \xi \tag{7.1}$$

where $\xi \in C_{\min}$, $C_{\min}$ is the sum of minimal cones in all simple summands. Since any invariant cone includes the minimal one, Theorem 6.1, (5), (7), and Lemma 7.2 imply that $\xi \in \sigma C$. Set $P = P' \times Z$. There exists $l_0 \in C$ such that $\sigma l_0 = l$. Put $l_1 = l_0 - l$. Then $l_1 \in \mathcal{A}$ and could be considered as a homomorphism $R \to \mathbb{R}$.

If $l_1 = 0$ then $l \in C$. By Theorem 6.1, (5), $\operatorname{Ad}(\mathcal{R})$-invariant closed cone generated by l is generating in $\mathcal{S}$. Hence $C \cap \mathcal{S}$ is exact by Corollary 5.7. This implies that the projection of the predual cone to $\mathcal{S}$ is included to the global cone $(C \cap \mathcal{S})^*$. Since the projection of C to $\mathcal{A}$ clearly includes some exact cone, the assumption of the Theorem 5.1 is satisfied for the predual cone, and C is exact.

Suppose that $l_1 \neq 0$. The function $F(x) = f(x) + l_1(x)$ satisfies (7.1) for all $u \in K$, $v \in P$, and $d_e F = l_0 + \xi$. By Lemma 2.11, F is an increasing function on R. By the definition of l, l_0, l_1, $d_e F$ has nonzero projections to each simple summand of $\mathcal{S}$. By Theorem 6.1, (5), the intersection of the $\mathrm{Ad}(\mathcal{G})$-invariant cone C_0 generated by $d_e F$ with $\mathcal{S}$ is generating in $\mathcal{S}$. Since $\operatorname{codim} \mathcal{S} = 1$, C_0 is generating in $\mathcal{R}$. Therefore, the cone generated by the orbit of $d_e F$ has a nonempty interior in $\mathcal{R}$ (although it need not be closed). By Corollary 5.6, C is exact. ∎

Proof of Main Theorem. The necessity of (GlC) was already proved (Proposition 4.2). If (GlC) is true for $\mathcal{G}$ then for any $I \subseteq \overline{n}$ there exists $l_I \in \pi_I C \cap \mathcal{L}_I$. Set $\Lambda = \{l_I : I \subseteq \overline{n}\}$. By Lemma 7.2, there exists a closed invariant cone in $\mathcal{I} = \mathcal{I}_s + \mathcal{Z}$ which includes Λ. For this cone (GlC) is satisfied. Hence, by Lemma 7.1, this is sufficient to prove that (GlC) implies the globality in $\mathcal{I}$. Thus we may assume that $\mathcal{G}$ is reductive proving the necessity.

If $\mathcal{J}$ is a compact simple ideal of $\mathcal{G}$ then the averaging procedure over the corresponding compact normal subgroup and Theorem 6.1, (6), show that the projection of $C^\star$ to the centralizer $\mathcal{Z}(\mathcal{J})$ coincides with $C^\star \cap \mathcal{Z}(\mathcal{J})$. So C is global in $\mathcal{G}$ if and only if $C \cap \mathcal{Z}(\mathcal{J})$ is global in $\mathcal{Z}(\mathcal{J})$ and it is possible to assume that $\mathcal{G}$ has no simple compact ideals.

Suppose that $\mathcal{G}$ is semisimple and each simple ideal of $\mathcal{G}$ is noncompact tubular. Then Theorem 6.1, (5), Theorem 5.1, and induction arguments show that each invariant cone in $\mathcal{G}$ is global.

Since in abelian algebras any pointed generating cone is exact, Theorem 5.1 implies that the same is true for reductive algebras with the semisimple part which contains no nontubular ideals.

We prove the sufficiency of (GlC) by the induction on the number of simple nontubular summands in $\mathcal{G}$; it's base is provided by the consideration above.

Recall that $J_1, \dots, J_n$ denote all nontubular ideals of $\mathcal{G}$. We keep the notation of Main Theorem.

Suppose that (GlC) is satisfied for $\mathcal{G}$. Let π be the projection to the semisimple part of $\mathcal{G}$ and $l \in \pi C \cap \mathcal{L}_{\overline{n}}$, $l \neq 0$, $\overline{n} = \{1, \dots, n\}$, I be the set of indices k such that l has the nonzero projection to J_k, $K = \overline{n} \setminus I$. Note that, by Theorem 6.1, (5), l generates a nontrivial invariant cone in J_I, hence $\pi_I C$ has the nonempty interior in J_I.

Choose $l_0 \in C$ such that $\pi_I l_0 = l$. Then the assumption of Lemma 7.3 is satisfied for the cone $C \cap \mathcal{R}_0$ in the linear span $\mathcal{R}_0$ of J_I and l_0. Hence $C \cap \mathcal{R}_0$ is exact. This means that the projection of the predual cone to $\mathcal{R}_0$ is included to a global one. Let π_1 be the orthogonal projection to $\mathcal{R}_1 = \mathcal{R}_0{}^{\perp}$. By Theorem 5.1, this is sufficient to prove that $C_1 = \pi_1 C$ includes some exact cone in $\mathcal{R}_1$.

Note that the element l defined by (2.10) is compact. Using the averaging over the maximal compact subgroup of a complementary semisimple summand and Theorem 6.1, (7), one may find a compact inverse image for each $l_I \in \mathcal{L}_I$, $I \subset \overline{n}$. So there exists a finite set $\Lambda \subset C_1$ consisting of compact elements such that for any $J \subset K$ $\pi_J \Lambda \cap \mathcal{L}_J \neq \{0\}$. Since it is possible to add to Λ any finite number of points with compact projections to $\mathcal{R}_1$, we may also assume that the convex closed generated by Λ $\mathrm{Ad}(\mathcal{R}_1)$-invariant cone in $\mathcal{R}_1$ is generating and has trivial intersection with the center of $\mathcal{R}_1$ if $\mathcal{R}_1$ is not abelian. By Lemma 7.2, this cone is pointed. Since (GlC) is satisfied for it, this cone is exact by the induction hypothesis.

The nontrivial part of assertions (1) and (2) of the theorem is that $\pi_I C$ includes some closed invariant cone satisfying (GlC) for any $I \subseteq \overline{n}$. This is a consequence of Lemma 7.2 and could be proved as above. ∎

Proof of Proposition 1.1. Note that the procedure analogous to the procedure of the proposition for global invariant cones in $\mathcal{G}$ finishes with $I = \varnothing$ if (GlC) is true since $\pi_I C^* \cap \mathcal{L}_I \neq \varnothing$ in each step. Conversly, if $\pi_K C^* \cap \mathcal{K}_I = \varnothing$ for some $K \subseteq \overline{n}$ then $K \subseteq I$ in each step because $K \cap I' = \varnothing$, hence the procedure finishes with $I' = \varnothing$.

The averaging over $\mathrm{Ad}(\mathcal{C})$ for a compactly embedded Cartan subalgebra $\mathcal{C}$ and, after that, over groups W_{km} for each simple summand as in the proof of Proposition 2.1 show that the projection of C to the sum of any family of spaces $\mathcal{V}_k$ coincides with the intersection with this sum. Therefore, $\pi_I C = \pi_I C'$ and the procedure of the proposition must finish with the same result as for invariant cones. ∎

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