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The density of the image of the exponential function and spacious locally compact groups

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A connected locally compact group G is called spacious if there exists a nonempty open set $U \subseteq G$ such that $U^n \cap U^{n+1} = \emptyset$ for every positive integer n. G is called weakly exponential if the union of its one-parameter subgroups is dense in G. We prove the following conjecture of Hofmann and Mukherjea [Math. Ann. **234**, 263-273 (1978)]: G is spacious iff it fails to be weakly exponential.

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Let G be a connected locally compact group. Following Hofmann and Mukherjea [3] we call G *spacious* if it admits a nonempty open subset U such that $U^n \cap U^{n+1} = \emptyset$ for every positive integer $n \in \mathbb{N}$. We call G *weakly exponential* if the union of its one-parameter subgroups is dense in G.

When $G = \mathrm{SL}(n, \mathbb{R})$ and $U \subseteq G$ is the set of matrices with all entries strictly negative, then obviously $U^n \cap U^{n+1} = \emptyset$ for all n. $\mathrm{SL}(n, \mathbb{R})$ is thus an elementary example of a spacious connected locally compact group. On the other hand, when $G = \mathbb{R}$, $U \subseteq \mathbb{R}$ is open, and $x \in U$, there will be a positive integer n such that $y = nx/(n+1) \in U$. So $nx = (n+1)y \in U^n \cap U^{n+1}$, and therefore $\mathbb{R}$ is not spacious. Using this result one immediately concludes that a weakly exponential locally compact group is not spacious. Hofmann and Mukherjea conjectured that the converse is also true, i.e., G is spacious iff it fails to be weakly exponential [3, Conjecture 2.13]. The purpose of the present paper is to give a proof of this conjecture.

We begin by quoting two lemmas from [3].

Lemma 1. *A connected locally compact group G is weakly exponential iff every neighbourhood of the identity $e \in G$ contains a compact normal subgroup K such that G/K is a weakly exponential Lie group.*

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Of course, a connected Lie group G is weakly exponential iff the image of the exponential map is dense in G.

Lemma 2. *Let R denote the radical of a connected locally compact group G. Then G is weakly exponential iff G/R is.*

It is easy to see that if N is a closed normal subgroup of G such that G/N is spacious then G is spacious. Therefore it is clear from Lemmas 1 and 2 that it suffices to prove the conjecture in the case that G is a connected semisimple Lie group (as pointed out in [3]).

Let G be a connected semisimple Lie group with Lie algebra $\mathfrak{g}$. Recall that a Cartan subalgebra of $\mathfrak{g}$ is a maximal abelian subalgebra $\mathfrak{h}$ such that $\mathrm{ad}X$ is semisimple for every $X \in \mathfrak{h}$. The Cartan subgroup $H(\mathfrak{h})$ associated with $\mathfrak{h}$ is defined as the centralizer of $\mathfrak{h}$ in G: $H(\mathfrak{h}) = Z_G(\mathfrak{h}) = \{g \in G;\ \mathrm{Ad}(g){\restriction}\mathfrak{h} = \mathrm{id}\}$. $H(\mathfrak{h})$ is a closed Lie subgroup with Lie algebra $\mathfrak{h}$. Thus if $H(\mathfrak{h})$ is connected then $H(\mathfrak{h}) = \exp(\mathfrak{h})$. Since the union of all Cartan subgroups of G is dense in G [5, Theorem 1.4.1.7], one obtains:

Lemma 3. *If all Cartan subgroups of a connected semisimple Lie group G are connected then G is weakly exponential.*

As shown in [3, Theorem 2.10] the converse is also true, but we do not need this in our argument. To prove the conjecture it suffices to show that if some Cartan subgroup is disconnected then G is spacious. We will achieve this goal in two steps. First, we will show that spaciousness occurs when the minimal parabolic subgroups of G are disconnected. Then we will show that if the minimal parabolic subgroups are connected then the Cartan subgroups are also connected.

Let G be a real connected semisimple Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{l}$ be a Cartan decomposition ($\mathfrak{k}$ the subalgebra) and Θ the associated Cartan involution . We shall denote by $\langle \cdot, \cdot \rangle$ the bilinear form $\langle X, Y \rangle = -B(X, \Theta Y)$, $X, Y \in \mathfrak{g}$, where $B(\cdot, \cdot)$ is the Killing form. $\langle \cdot, \cdot \rangle$ is a scalar product on $\mathfrak{g}$ and we shall write $\|X\|$ for the norm

$$\|X\| = (\langle X, X \rangle)^{1/2}. \tag{1}$$

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{l}$. For any linear functional λ let,

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g};\ (\mathrm{ad}Y)X = \lambda(Y)X \text{ for all } Y \in \mathfrak{a}\}. \tag{2}$$

When $\mathfrak{g}_\lambda \neq \{0\}$, λ is called a root of $\mathfrak{g}$ (relative to $\mathfrak{a}$). One proves that $\mathfrak{g} = \bigoplus_{\lambda \in \Delta} \mathfrak{g}_\lambda$, where the sum is over the set Δ of roots. Moreover, the $\mathfrak{g}_\lambda$'s are orthogonal with respect to the scalar product $\langle \cdot, \cdot \rangle$. Let $\geq$ be any total vector order in the dual space $\mathfrak{a}^*$ of $\mathfrak{a}$ and denote by Δ_+ set of positive (nonzero) roots. Set $\mathfrak{n} = \bigoplus_{\lambda \in \Delta_+} \mathfrak{g}_\lambda$. Then $\mathfrak{n}$ is a nilpotent subalgebra of $\mathfrak{g}$ and one obtains an Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Let K, A, and N denote the connected Lie subgroups of G with Lie algebras $\mathfrak{k}$, $\mathfrak{a}$, and $\mathfrak{n}$, respectively. Then the map $K \times A \times N \ni (k, a, n) \to kan \in G$ is a diffeomorphism onto G and one obtains an

Iwasawa decomposition $G = KAN$. We note that K contains the centre $C(G)$ and is compact iff $C(G)$ is finite.

We shall denote by M the centralizer of A in K. M is either discrete or a closed Lie subgroup with Lie algebra $\mathfrak{m} = \mathfrak{g}_0 \cap \mathfrak{k}$. The set MAN is a closed cocompact subgroup of G called the minimal parabolic subgroup associated with the Iwasawa decomposition $G = KAN$. All Iwasawa decompositions of G, and therefore also all minimal parabolic subgroups are conjugate.

Let $\tilde{\mathfrak{n}} = \bigoplus_{\lambda\in-\Delta_+} \mathfrak{g}_\lambda$. Then $\tilde{\mathfrak{n}}$ is a nilpotent subalgebra. We shall write $\tilde{N}$ for the connected Lie subgroup of G with Lie algebra $\tilde{\mathfrak{n}}$. Given a subgroup $H \subseteq G$ we shall denote by H_0 the connected component of the identity e. We note that $(MAN)_0 = M_0AN$. The following lemma is a version of Lemma III.6 in [1, p. 66].

Lemma 4. *There exists an open semigroup $S \subseteq G$, an open neighbourhood V of e in G, and an element $a \in A$ such that $VaM_0 \subseteq S \subseteq \tilde{N}M_0AN$.*

Proof. Choose $Z \in \mathfrak{a}$ so that $\lambda(Z) > 0$ for all $\lambda \in \Delta_+$. Set $a = \exp(Z)$ and $\gamma = \exp\big(-\min\{\lambda(Z); \lambda \in \Delta_+\}\big) < 1$. Then for every $X \in \tilde{\mathfrak{n}}$ we have $\|\mathrm{Ad}(a)X\| \le \gamma\|X\|$. where $\|\cdot\|$ is the norm (1). Let Ω be an open ball in $\mathfrak{n}$ with centre 0. Put $E = \pi\big(\exp(\overline{\Omega})\big)$ and $E' = \pi\big(\exp(\Omega)\big)$ where $\pi\colon G \to G/M_0AN$ is the canonical map. E is clearly a compact subset and $E' \subseteq E$. Now, $\exp(\Omega)$ is open because $\tilde{N}$ is a simply connected nilpotent Lie group. As the map $\tilde{N} \times MAN \ni (x, y) \to xy$ is a diffeomorphism onto an open subset of G [2, pp. 406-407], it follows that $E' = \pi\big(\exp(\Omega)\big) = \pi\big(\exp(\Omega)M_0AN\big)$ is open. Furthermore, $aE = \pi\big(\exp\big(\mathrm{Ad(a)}\overline{\Omega}\big)\big) \subseteq \pi\big(\exp(\Omega)\big) = E'$.

Define $S = \{g \in G; gE \subseteq E'\}$. Then S is a semigroup containing a. S is open because E is compact while E' is open.

Let V be a neighbourhood of e in G such that $Va \subseteq S$. Let $v \in V$ and $g \in M_0$. Then $vagE = va\pi\big(g\exp(\overline{\Omega})\big) = va\pi\big(\exp\big(\mathrm{Ad}(g)\overline{\Omega}\big)\big)$. Since $\mathrm{Ad}(g) \in \mathrm{Ad}(K)$ is an isometry, $\mathrm{Ad}(g)\overline{\Omega} = \overline{\Omega}$ and therefore $vagE = vaE \subseteq E'$. So $VaM_0 \subseteq S$.

It remains to show that $S \subseteq \tilde{N}M_0AN$. But $M_0AN = \pi(e) \in E$. Hence, $\pi(S) = S\pi(e) \subseteq E' \subseteq \pi(\tilde{N})$. So $S \subseteq \tilde{N}M_0AN$. ∎

Lemma 5. *If M is disconnected then G is spacious.*

Proof. By Lemma 4 there exists an open semigroup S, an open neighbourhood V of e in G, and $a \in A$ such that $VaM_0 \subseteq S \subseteq \tilde{N}M_0AN$. We can assume that $V = \exp(\Omega)$ where $\Omega = \{X \in \mathfrak{g}; \|X\| < \varepsilon\}$ for sufficiently small $\varepsilon > 0$. Let $x \in M - M_0$. As $\mathrm{Ad}(x)$ preserves the norm $\|\cdot\|$, we have $xVx^{-1} = V$. Furthermore, $VaM_0 \subseteq S$ and $xVaM_0x^{-1} = VaM_0$ (because x centralizes a and normalizes M_0). Consequently, $S_1 = \bigcup_{i=1}^{\infty}\big(VaM_0\big)^i \subseteq S \subseteq \tilde{N}MAN$ is an open semigroup such that $xS_1x^{-1} = S_1$. Hence, setting $U = xS_1$ we have $U^n \cap U^{n+1} \subseteq x^nS_1 \cap x^{n+1}S_1 = x^n\big(S_1 \cap xS_1\big) \subseteq x^n\big(\tilde{N}M_0AN \cap \tilde{N}xM_0AN\big)$ (because x normalizes $\tilde{N}$). Using the fact that the map $\tilde{N} \times M \times AN \ni (g, h, k) \to ghk$ is one-to-one we conclude that $U^n \cap U^{n+1} = \emptyset$ for all $n \in \mathbb{N}$. So G is spacious. ∎

Given a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ we shall write $\mathfrak{h}_{\mathfrak{k}}$ for $\mathfrak{h} \cap \mathfrak{k}$ and $\mathfrak{h}_{\mathfrak{l}}$ for $\mathfrak{h} \cap \mathfrak{l}$. If $\mathfrak{h}$ is stable under Θ then $\mathfrak{h} = \mathfrak{h}_{\mathfrak{k}} \oplus \mathfrak{h}_{\mathfrak{l}}$. We shall denote by $\mathcal{H}_{\mathfrak{a}}$ the set of all Cartan subalgebras containing $\mathfrak{a}$. $\mathcal{H}_{\mathfrak{a}}$ coincides with the set of maximal abelian subspaces of $\mathfrak{g}$ containing $\mathfrak{a}$. If $\mathfrak{h}$ is a member of $\mathcal{H}_{\mathfrak{a}}$ then $\mathfrak{h}$ is Θ-stable, $\mathfrak{h}_{\mathfrak{l}} = \mathfrak{a}$ and $\mathfrak{h}_{\mathfrak{k}} \subseteq \mathfrak{m} = \mathfrak{g}_0 \cap \mathfrak{k}$.

Lemma 6. $\mathfrak{m} = \bigcup_{\mathfrak{h} \in \mathcal{H}_{\mathfrak{a}}} \mathfrak{h}_{\mathfrak{k}}$.

Proof. Let $X \in \mathfrak{m}$. If $\mathfrak{h}$ is a maximal abelian subspace containing both X and $\mathfrak{a}$, then $\mathfrak{h} \in \mathcal{H}_{\mathfrak{a}}$ and $X \in \mathfrak{h}_{\mathfrak{k}}$. ■

Lemma 7. *There exist:*

(*i*) *a compact connected semisimple Lie group $G_\#$ with Lie algebra $\mathfrak{g}_\#$ and trivial centre;*

(*ii*) *a linear isomorphism $F : \mathfrak{g} \to \mathfrak{g}_\#$ such that $F[X,Y] = [FX,FY]$ for $X \in \mathfrak{k}$, $Y \in \mathfrak{g}$, and $F[X,Y] = -[FX,FY]$ for $X,Y \in \mathfrak{l}$;*

(*iii*) *an analytic homomorphism $\varphi\colon K \to G_\#$ such that $F \restriction \mathfrak{k}$ is the differential of φ at e and $F\mathrm{Ad}(g) = \mathrm{Ad}(\varphi(g))F$ for all $g \in K$.*

Proof. Let $\tilde{\mathfrak{g}}$ be the complexification of $\mathfrak{g}$. Then $\mathfrak{k} \oplus i\mathfrak{l}$ is a compact real form of $\tilde{\mathfrak{g}}$. Set $G_\# = \mathrm{Int}(\mathfrak{k} \oplus i\mathfrak{l})$. Identify $\mathfrak{k} \oplus i\mathfrak{l}$ with the Lie algebra of $G_\#$ and define F by $F(X+Y) = X + iY$, $X \in \mathfrak{k}$, $Y \in \mathfrak{l}$. Given $g \in K$ define $\varphi(g)$ to be the complexification of $\mathrm{Ad}(g)$ restricted to $\mathfrak{k} \oplus i\mathfrak{l}$. ■

Remark 8. $\mathrm{Ker}\varphi = C(G)$. $[F\mathfrak{k}, F\mathfrak{l}] \subseteq F\mathfrak{l}$, $[F\mathfrak{l}, F\mathfrak{l}] \subseteq F\mathfrak{k}$.

Lemma 9. *Let $W \subseteq \mathfrak{g}$ be a subset. Denote by $Z_K(W)$ (resp., $Z_{G_\#}(FW)$) the centralizer of W (resp., FW) in K (resp., in $G_\#$). Then $Z_K(W) = \varphi^{-1}(Z_{G_\#}(FW))$.*

Proof. Obvious by Lemma 7(iii). ■

We shall write $K_\#$ for the connected Lie subgroup of $G_\#$ with Lie algebra $F\mathfrak{k}$. Clearly, $K_\# = \varphi(K)$.

Lemma 10. *If M is connected then $K_\# \cap \exp(F\mathfrak{a}) \subseteq \exp(F\mathfrak{m})$.*

Proof. Let $g \in K_\# \cap \exp(F\mathfrak{a})$. Clearly, $g \in Z_{G_\#}(F\mathfrak{a})$. But $g = \varphi(g')$ for some $g' \in K$. By Lemma 9, $g' \in Z_K(\mathfrak{a}) = M$. As M is connected and has compact Lie algebra $\mathfrak{m}$, $M = \exp(\mathfrak{m})$. So $g = \varphi(g') \in \varphi(\exp(\mathfrak{m})) = \exp(F\mathfrak{m})$ by Lemma 7(iii). ■

For the next lemma see [5, Proposition 1.4.1.2].

Lemma 11. *Let $\mathfrak{h}$ be a Θ-stable Cartan subalgebra. Set $H_K(\mathfrak{h}) = H(\mathfrak{h}) \cap K$. Then $H(\mathfrak{h}) = H_K(\mathfrak{h})\exp(\mathfrak{h}_{\mathfrak{l}})$ and $H(\mathfrak{h})$ is connected iff $H_K(\mathfrak{h})$ is connected.*

Theorem 12. *The following conditions are equivalent for a connected semisimple Lie group G:*

(*a*) *the minimal parabolic subgroups of G are connected;*

(*b*) *the Cartan subgroups of G are connected;*

(c) *G is weakly exponential;*
(d) *G is not spacious.*

Proof. (a) $\Rightarrow$ (b): Every Cartan subalgebra $\mathfrak{h}$ is conjugate in $\mathrm{Int}(\mathfrak{g})$ to a Θ-stable Cartan subalgebra $\mathfrak{h}'$ such that $\mathfrak{h}'_{\mathfrak{l}} \subseteq \mathfrak{a}$ [5, Proposition 1.3.1.1]. Therefore it suffices to consider Cartan subgroups $H = H(\mathfrak{h})$ where $\mathfrak{h}$ is Θ-stable and $\mathfrak{h}_{\mathfrak{l}} \subseteq \mathfrak{a}$. Due to Lemma 11 it suffices to show that $H_K = \exp(\mathfrak{h}_{\mathfrak{k}})$.

Let $g \in H_K$. Then $\hat{g} = \varphi(g) \in Z_{G_\#}(F\mathfrak{h})$ (Lemma 9). It is easy to see that $F\mathfrak{h}$ is a maximal abelian subalgebra of $\mathfrak{g}_\#$. As $\mathfrak{g}_\#$ is compact, this means that $F\mathfrak{h}$ is a Cartan subalgebra [4, Lemma 4.12.1]. As $G_\#$ is compact and connected, $H(F\mathfrak{h}) = \exp(F\mathfrak{h})$ [4, Theorems 4.12.3 and 4.12.5]. Therefore $\hat{g} = \exp(FX)$ for some $X \in \mathfrak{h}$. Write $X = X_1 + X_2$, $X_1 \in \mathfrak{h}_{\mathfrak{k}}$, $X_2 \in \mathfrak{h}_{\mathfrak{l}}$. Then $\hat{g} = \exp(FX_1)\exp(FX_2)$, and $FX_2 \in F\mathfrak{h}_{\mathfrak{l}} \subseteq F\mathfrak{a}$. As $\hat{g} \in K_\#$, $\hat{g}_2 = \exp(FX_2) = \exp(-FX_1)\hat{g} \in K_\# \cap \exp(F\mathfrak{a})$. By Lemma 10 $\exp(FX_2) = \exp(FX_2')$ where $X_2' \in \mathfrak{m}$. As $\mathfrak{m} = \bigcup_{\mathfrak{h}' \in \mathcal{H}_{\mathfrak{a}}} \mathfrak{h}'_{\mathfrak{k}}$ (Lemma 6), there exists $\mathfrak{h}' \in \mathcal{H}_{\mathfrak{a}}$ with $X_2' \in \mathfrak{h}'_{\mathfrak{k}}$. Then $\mathfrak{h}$ and $\mathfrak{h}'$ are two Θ-stable Cartan subalgebras such that $\mathfrak{h}_{\mathfrak{l}} \subseteq \mathfrak{h}'_{\mathfrak{l}} = \mathfrak{a}$. By [5, Proposition 1.3.1.3] there exists $k \in K$ such that $\mathrm{Ad}(k)\restriction \mathfrak{h}_{\mathfrak{l}} = \mathrm{id}$ and $\mathrm{Ad}(k)\mathfrak{h}'_{\mathfrak{k}} \subseteq \mathfrak{h}_{\mathfrak{k}}$. Now, $\varphi(k)\hat{g}_2\varphi(k)^{-1} = \varphi(k)\exp(FX_2)\varphi(k)^{-1} = \exp\big(\mathrm{Ad}(\varphi(k))FX_2\big) = \exp\big(F\mathrm{Ad}(k)X_2\big) = \exp(FX_2) = \hat{g}_2$ (by Lemma 7(iii)). Hence, $\hat{g}_2 = \varphi(k)\hat{g}_2\varphi(k)^{-1} = \varphi(k)\exp(FX_2')\varphi(k)^{-1} = \exp\big(\mathrm{Ad}(\varphi(k))FX_2'\big) = \exp\big(F\mathrm{Ad}(k)X_2'\big) = \exp(FX_2'')$, where $X_2'' \in \mathfrak{h}_{\mathfrak{k}}$. So $\varphi(g) = \hat{g} = \exp(FX_1)\exp(FX_2'') = \exp(FY) = \varphi\big(\exp(Y)\big)$, where $Y \in \mathfrak{h}_{\mathfrak{k}}$.

Since $\mathrm{Ker}\varphi = C(G)$, $g = \exp(Y)c$ where $c \in C(G)$. But $C(G) \subseteq M = \exp(\mathfrak{m})$ because M is connected and $\mathfrak{m}$ is compact. So $c = \exp(Z)$ for some $Z \in \mathfrak{m}$. Then by Lemma 6 there exists $\mathfrak{h}' \in \mathcal{H}_{\mathfrak{a}}$ such that $Z \in \mathfrak{h}'_{\mathfrak{k}}$. Again, $\mathfrak{h}$ and $\mathfrak{h}'$ are two Θ-stable Cartan subalgebras such that $\mathfrak{h}_{\mathfrak{l}} \subseteq \mathfrak{h}'_{\mathfrak{l}} = \mathfrak{a}$. So there exists $k \in K$ such that $\mathrm{Ad}(k)\restriction\mathfrak{h}_{\mathfrak{l}} = \mathrm{id}$ and $\mathrm{Ad}(k)\mathfrak{h}'_{\mathfrak{k}} \subseteq \mathfrak{h}_{\mathfrak{k}}$ [5, Proposition 1.3.1.3]. We then have $c = kck^{-1} = \exp(\mathrm{Ad}(k)Z) = \exp(Z')$ with $Z' \in \mathfrak{h}_{\mathfrak{k}}$. Consequently, $g = \exp(Y + Z')$, and $Y + Z' \in \mathfrak{h}_{\mathfrak{k}}$. Thus $H_K = \exp(\mathfrak{h}_{\mathfrak{k}})$.

(b) $\Rightarrow$ (c): Lemma 3.

(c) $\Rightarrow$ (d): explained at the beginning of the paper.

(d) $\Rightarrow$ (a): Lemma 5. ∎

Corollary 13. *A connected locally compact group fails to be weakly exponential iff it is spacious.*

Remark 14. Using Lemma 7 one can prove an analog of Lemma 6 for the group M. Namely, $M = \bigcup_{\mathfrak{h} \in \mathcal{H}_{\mathfrak{a}}} H_K(\mathfrak{h})$. This, together with Lemma 11 allows to obtain the implication (b) $\Rightarrow$ (a) of Theorem 12 directly, without involving (c) and (d).

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