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On the Exponential Map of Almost Simple Real Algebraic Groups *

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Abstract. One of the basic problems for Lie groups is to describe the image of the exponential map. A Lie group H is called exponential resp. weakly exponential if its exponential map is surjective resp. has dense image in H. In this paper we classify all exponential groups which are identity components of almost simple real algebraic groups. The weakly exponential groups of this type were determined in our previous paper [15].

0. Introduction

The well-known notion of *exponential map* in the theory of Lie groups and related topics is as old as the theory itself. As one of key notions, it plays an important role in the theory as well as in other branches of mathematics, e.g. differential geometry, (transcedental) number theory, arithmetic algebraic geometry, functional analysis, dynamical systems, mathematical physics, etc. (see e.g. [2, 3, 4, 8, 22, 25]). However, until recently, the problem of describing the image of the exponential map, one of the basic and difficult problems in the theory, is still open, apart from partial results for some particular Lie groups (see e.g. [9, 10, 11, 12, 17, 19, 20, 21, 23, 24]). This problem gives rise to the following basic questions :

A) *Characterize the Lie groups for which the exponential map has dense image, and*

B) *Characterize the Lie groups, for which the exponential map is surjective.*

The Lie groups satisfying the condition mentioned in A (resp. B) are called *weakly exponential (w.e.)* (after [17, 18], where the question A was also raised) (resp. *exponential*).

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Dixmier has carried out in [9] a detailed investigation of the image of the exponential map for simply connected solvable Lie groups. Therefore the problem is focused on reductive Lie groups. One of the important classes of connected reductive Lie groups, which are frequently used in practice, is the class Σ consisting of real Lie groups $G(\mathbf{R})^\circ$, where G is an almost simple algebraic group defined over $\mathbf{R}$, $G(\mathbf{R})$ the group of real points of G, and $^\circ$ denotes the identity component.

In our recent paper [15] we have obtained a classification of groups in Σ which are w.e. (a solution of Question A for Σ). Here, using this result we give a solution of Question B for the class Σ.

It may be of interest to mention that among all almost simple (connected) complex Lie groups only the groups $\mathrm{PSL}_n(\mathbf{C})$ are exponential, see [19, 20, 21].

For connected real semisimple groups H, there is a simple criterion, due to Borel ([18, Theorem 2.10]), which can be used to decide whether H is w.e. or not. We have found an analogous criterion for exponentiality of groups belonging to the class Σ. In order to apply this criterion we need to know the structure of the centralizers of unipotent elements. In most cases the necessary facts can be found in the literature. In the case of the classical group $\mathrm{SO}^*(2n)$, the group which preserves a non-degenerate skew-hermitian form on a finite dimensional quaternionic vector space, the structure of these centralizers is not available in the sources that we have consulted. We have described these centralizers in detail in Section 5.

1. Statement of the main result

By Z_k we denote a cyclic group of order k, by $\mathbf{R}$ the real numbers, by $\mathbf{C}$ the complex numbers, and by $\mathbf{H}$ the quaternion algebra over $\mathbf{R}$ with standard involution. For a Lie group G with Lie algebra $\mathfrak{g}$ we denote by E_G the image of the exponential map $\exp : \mathfrak{g} \to G$. We use also standard notations for classical Lie groups. In particular, by $\mathrm{Spin}^*(2n)$ we denote the double cover of $\mathrm{SO}^*(2n)$, which is also the group of real points of $\mathrm{Spin}_{2n}(\mathbf{C})$ for appropriate real structure. If W is a non-degenerate subspace of V, we denote by $\mathrm{Spin}^*(W)$ the inverse image of $\mathrm{SO}^*(2n)$ in $\mathrm{Spin}^*(2n)$. If $\dim W = k$, then $\mathrm{Spin}^*(W) \simeq \mathrm{Spin}^*(2k)$. If $k = 1$, then $\mathrm{SO}^*(W)$ is the circle group $\mathrm{SO}(2)$, and the same is true for $\mathrm{Spin}^*(W)$. If $k = 2$, then $\mathrm{Spin}^*(W) \simeq \mathrm{SU}(2) \times \mathrm{SU}(1,1)$.

In our previous paper [15] we have obtained the list of all w.e. groups in Σ. Our main objective in this paper is to determine which of these groups are in fact exponential. It is well-known [16] that all compact groups from Σ are exponential. Moreover, if $G \to H$ is a surjective homomorphism of connected Lie groups with discrete kernel (we write $G \geq H$) and G is (weakly) exponential, then H is also (weakly) exponential. Thus we need only to describe non-compact exponential groups which are "maximal" with respect to $\geq$.

For convenience of the reader we list in Table 1 all non-compact almost simple w.e. groups in Σ which are maximal with respect to the above partial order.

Table 1

Maximal non-compact almost simple w.e. groups in Σ

A I	$\mathrm{PSL}_2(\mathbf{R})$
A II	$\mathrm{SL}_n(\mathbf{H})$, $n \geq 2$
A III	$\mathrm{SU}(p,q)$, $p > q \geq 1$ $\mathrm{SU}(p,p)/Z_{2^m}$, $2p = 2^m r$, r odd
B I	$\mathrm{Spin}(2n,1)$, $n \geq 2$
C II	$\mathrm{Sp}(p,q)$, $p \geq q \geq 1$
D I	$\mathrm{Spin}(2n-1,1)$, $n \geq 3$ $\mathrm{PSO}(2n-2,2)^0$, $n \geq 3$, n odd $\mathrm{Spin}(2n-2,2)/\langle z \rangle$, $n \geq 4$, n even
D III	$\mathrm{Spin}^*(2n)$, $n \geq 3$, n odd $\mathrm{SO}^*(2n)$, $n \geq 4$, n even $\mathrm{Spin}^*(2n)/\langle z' \rangle$, $n \geq 4$, n even
E III, IV	G^*
E VII	G^*/Z_2
F II	G^*

In this table z and z' are elements of order two which generate the center $Z \simeq Z_2 \times Z_2$ of $\mathrm{Spin}_{2n}(\mathbf{C})$, such that $\langle zz' \rangle$ is the kernel of the canonical projection $\mathrm{Spin}_{2n}(\mathbf{C}) \to \mathrm{SO}_{2n}(\mathbf{C})$. It is well-known that in both cases, D I and D III, Z is contained in the group of real points $\mathrm{Spin}(2n-2,2)$ and $\mathrm{Spin}^*(2n)$, respectively. Moreover there is an automorphism of $\mathrm{Spin}(2n-2,2)$ which interchanges z and z', but there is no such automorphism of $\mathrm{Spin}^*(2n)$ if n is even. In the latter case we can distinguish z and z' by considering any compact torus T of dimension $n/2$ which is centralized by a maximal split torus of $\mathrm{Spin}^*(2n)$. Then $z \in T$ and $z' \notin T$ (see [15] for details).

We can now state our main result.

Main Theorem. *The non-compact exponential groups in Σ are the groups listed below and their quotients by finite central subgroups :*

1. $\mathrm{SL}_n(\mathbf{H})$, $n \geq 2$;
2. $\mathrm{PSU}(p,p) = \mathrm{SU}(p,p)/Z_{2p}$, $p \geq 1$;
3. $\mathrm{SU}(p,q)/Z_d$, *where* $p > q \geq 1$, d *is an odd divisor of* $n = p+q$, *and every odd prime dividing* d *resp.* n/d *is* $\leq n/(p-q)$ *resp.* $> n/(p-q)$;

4. $\mathrm{Spin}(2n,1)$, $n \geq 2$;
5. $\mathrm{Sp}(p,q)$, $p \geq q \geq 1$;
6. $\mathrm{Spin}(2n-1,1)$, $n \geq 3$;
7. $\mathrm{PSO}(2n-2,2)^0$, n *odd* ≥ 3 ;
8. $\mathrm{Spin}(2n-2,2)/\langle z \rangle$, n *even* ≥ 4 ;
9. $\mathrm{Spin}^*(2n)$, n *odd* ≥ 3 ;
10. $\mathrm{SO}^*(2n)$, n *even* ≥ 4 ;
11. $\mathrm{Spin}^*(2n)/\langle z' \rangle$, n *even* ≥ 4 ;
12. G^* *of type* E IV.

In the last three rows of Table 1 we denote by G^* the group of real points of the simply connected algebraic group of indicated type. It is known that $\mathrm{PSL}_2(\mathbf{R}) \simeq \mathrm{SU}(1,1)/Z_2$ is exponential and so are the groups listed in the cases 1, 5, and 10, see [10]. Hence we can omit from further considerations the groups of type A I, A II, and C II listed in Table 1. For all other types, listed in Table 1, the proof will be given separately in the remaining sections.

2. A criterion for exponentiality

In this section we give a general criterion for exponentiality of groups belonging to Σ. Theorem 2.2 below plays an important role in our study, just like Borel's theorem does in the study of weak exponentiality of Lie groups, see [15], [18, Theorem 2.10]. It is an analog, in the real case, of [12, Theorem 3.2]. Recall that a Cartan subgroup of a connected reductive real Lie group G is the centralizer in G of a Cartan subalgebra of the Lie algebra of G, see [28, p. 108]. First we need the following mild extension of Borel's theorem.

Theorem 2.1. *A connected reductive real Lie group G is w.e. if and only if all Cartan subgroups of G are connected.*

Proof. If G is w.e. the same holds true for G/R, where R is the radical of G. If C is a Cartan subgroup of G then $C \supset R$ and C/R is a Cartan subgroup of G/R. By Borel's theorem [18, Theorem 2.10], C/R is connected. As R is connected, C is also connected.

Conversely, assume that all Cartan subgroups of G are connected. Since Cartan subgroups of G/R are images of the Cartan subgroups of G, they are also connected. By Borel's theorem G/R is w.e. By [18, Corollary 2.1A, p. 273] G is w.e. ∎

In the sequel 1-PSG means "one-parameter subgroup".

Theorem 2.2. *Let G be the identity component of real points of an* $\mathbf{R}$*-group H. Then G is exponential if and only if $Z_G(u)$ is w.e. for every unipotent element $u \in G$.*

Proof. Assume that G is exponential and let $u \in G$ be unipotent. We have a Levi decomposition $Z := Z_G(u) = RC$, where R is the unipotent radical of Z

and C is reductive (not necessarily connected). Let $s \in C$ be semisimple and set $x = su$. Since $x \in E_G$, $x \in P$ where P is a 1-PSG of $Z_G(x)$. Since $Z_G(x) \subset Z$ and u is unipotent, it follows that $\overline{P\langle u\rangle}$ is a connected abelian Lie subgroup of Z containing $s = xu^{-1}$. Hence $s \in E_Z$, and $C \simeq Z/R$ implies that $s \in E_C$. It is known [28, Section 1.4] that semisimple elements are dense in C, and so C is w.e. By using the fact that R is exponential and [18, Corollary 2.1A], we conclude that Z is w.e.

We now prove the converse. Let $x \in G$ and let $x = su$ be its Jordan decomposition in $H(\mathbf{R})$, with s semisimple and u unipotent. Then necessarily $u \in G$, and so $s \in Z := Z_G(u)$. We can choose a Levi decomposition $Z = RC$ such that $s \in C$. By hypothesis Z is w.e. This entails that $C \simeq Z/R$ is w.e. As s is semisimple, it belongs to a Cartan subgroup P of C, which is connected by Theorem 2.1. Hence $x = su$ belongs to the connected abelian Lie group $\overline{P\langle u\rangle}$ and so $x \in E_G$. ∎

We need the following lemmas in the sequel.

Lemma 2.3. *Let $G = G_1 \times \cdots \times G_n$ where G_i are connected Lie groups, and let $z_i \in G_i$ be central elements of order 2. Assume that G_1 and $G_i/\langle z_i\rangle$, $i > 1$, are exponential. Then $\bar{G} := G/\langle z_1z_2, z_1z_3, \ldots, z_1z_n\rangle$ is exponential.*

Proof. Let $x = x_1 \cdots x_n \in G$, with $x_i \in G_i$, and let $\bar{x} \in \bar{G}$ be its image. Since each of the groups $G_i/\langle z_i\rangle$ is exponential, we can choose $k_i \in \{0,1\}$, $i > 1$, such that $x_i z_i^{k_i} \in E_{G_i}$. If $k_1 = k_2 + \cdots + k_n$ and $z = z_1^{k_1} \cdots z_n^{k_n}$ then $xz \in E_G$ because G_1 is exponential. As $\bar{z} = 1$, we have $\bar{x} \in E_{\bar{G}}$. ∎

The group $\bar{G}$ in the above lemma will be denoted by $\prod^* G_i$ and we refer to it as a *direct product of G_i's with identified subgroups $\langle z_i\rangle$.*

Lemma 2.4. *If $\phi : G \to H$ is a homomorphism of groups and G is a k-group, where k is a field, and $X \subset G$ is such that* Ker $\phi \subset X \cap G(k)$, *then*

$$\phi(X \cap G(k)) = \phi(G(k)) \cap \phi(X).$$

Proof. Let $y = \phi(g) = \phi(x)$, where $g \in G(k)$, $x \in X$. Then $x = gz$, where $z \in \text{Ker}\,\phi \subset G(k)$. Hence $x \in X \cap G(k)$, and $y \in \phi(X \cap G(k))$. ∎

3. Groups of type A III

In this section we investigate the exponentiality of groups of type A III listed in Table 1. In the proof of the following proposition, and also in subsequent sections, we shall use results and notations of [10]. The readers are advised to consult this paper before reading further.

Proposition 3.1. *Let $G = \mathrm{SU}(p,q)/Z_d$, where $d \mid n = p+q$, $n/d = 2^s t$, t odd, and $p \geq q \geq 1$. Then G is exponential if and only if one of the following holds :*

(a) $p = q$ and $d = n$,

(b) $p > q$ and every prime divisor of t is bigger than $n/(p-q)$.

In particular, if n is a power of 2, then G is always exponential.

Proof. Let Γ be a conjugacy class of $G^* = \mathrm{SU}(p,q)$ and $\hat{\Gamma}$ its image under the projection $\pi : G^* \to G$. Then

$$\pi^{-1}(\hat{\Gamma}) = \Gamma \cup \theta\Gamma \cup \cdots \cup \theta^{d-1}\Gamma,$$

where $\theta = e^{2\pi i/d}$ is a generator of Z_d.

So $\hat{\Gamma} \subset E_G$ if and only if at least one of the conjugacy classes $\theta^s\Gamma$ is contained in E_{G^*}. As in [10, p. 80] we may write

$$\Gamma = \sum_{i=1}^{r} \Gamma_{m_i-1}(\lambda_i, \bar{\lambda}_i^{-1}) + \sum_{i=r+1}^{k} \Gamma^{\pm}_{m_i-1}(\lambda_i), \tag{3.1}$$

where $|\lambda_i| > 1$ (resp. $|\lambda_i| = 1$) for $i \le r$ (resp. $i > r$). Let p_Γ be the greatest common divisor of the integers $2m_1, \dots, 2m_r, m_{r+1}, \dots, m_k$. The integer q_Γ is defined by :

$$2\sum_{i=1}^{r} m_i \ \arg\lambda_i + \sum_{i=r+1}^{k} m_i \ \arg\lambda_i = 2\pi q_\Gamma,$$

uniquely mod p_Γ. We claim that G is exponential if and only if $\mathrm{GCD}(p_\Gamma, n/d) = 1$ for all Γ.

For, we have the following basic relations :

$$2(m_1 + \cdots + m_r) + m_{r+1} + \cdots + m_k = n,$$

$$(m_1 + \cdots + m_r) + [m_{r+1}/2] + \cdots + [m_k/2] \le q.$$

We know that $\Gamma \subset E_{G^*}$ if and only if $p_\Gamma \mid q_\Gamma$ (*loc. cit.*). Since $p_{\theta^s\Gamma} = p_\Gamma$ and $q_{\theta^s\Gamma} = q_\Gamma + ns/d$, it follows that $\theta^s\Gamma \subset E_{G^*}$ if and only if $p_\Gamma \mid q_\Gamma + ns/d$. Thus

$$\hat{\Gamma} \subset E_G \iff \exists s \in \mathbf{Z},\ p_\Gamma \mid q_\Gamma + ns/d. \tag{3.2}$$

Hence if $\mathrm{GCD}(p_\Gamma, n/d) = 1$ for all Γ then G is exponential.

Conversely, assume that G is exponential and let Γ be as in (3.1). Let $\Gamma' \subset G^*$ be obtained from Γ by replacing each λ_i by some λ'_i. As $p_{\Gamma'} = p_\Gamma$ and $\hat{\Gamma}' \subset E_G$, (3.2) implies that $p_\Gamma \mid q_{\Gamma'} + ns/d$ for some $s \in \mathbf{Z}$. Since $q_{\Gamma'}$ can take arbitrary integer values for appropriate choice of the λ'_i, it follows that $\mathrm{GCD}(p_\Gamma, n/d) = 1$. Hence our claim is proved.

Assume that $p = q$. We can choose Γ such that $p_\Gamma = 2$ if n/d is even, and $p_\Gamma = 2n/d$ if n/d is odd. Hence if $d \ne n$ then G is not exponential.

¿From now on we assume that $p > q$, which implies that p_Γ is odd for all Γ, see [7]. Hence if $t = 1$ then $\mathrm{GCD}(p_\Gamma, n/d) = 1$ and G is exponential. It remains to consider the case $t > 1$. Assume $p - q \le n/p_0$, where p_0 is the smallest prime divisor of t. Then $u = (p - q + n/p_0)/2$ and $v = n/p_0 - u$ are nonnegative integers. For

$$\Gamma = u\Gamma^{+}_{p_0-1}(1) + v\Gamma^{-}_{p_0-1}(1),$$

we have $p_\Gamma = p_0$ and $\mathrm{GCD}(n/d, p_\Gamma) = p_0 > 1$. Hence G is not exponential.

Conversely assume that G is not exponential. Then $\delta := \mathrm{GCD}(n/d, p_\Gamma) > 1$ for some Γ as in (3.1). We may assume that the m_i are odd for $r < i \le r+l$ and even for $i > r+l$. From the basic relations we have

$$q \ge \sum_{i=1}^{r} m_i + \sum_{i=r+1}^{r+l} (m_i - 1)/2 + \sum_{i=r+l+1}^{k} m_i/2 = (n-l)/2,$$

and so $p - q = n - 2q \le l \le k \le kp_\Gamma/\delta \le n/\delta \le n/p_0$. ■

The groups $\mathrm{SU}(p,q)/Z_d$ that satisfy condition (b) of this proposition are exactly the quotients of the group given in case 3 of the main theorem.

The following corollary is an immediate consequence of the proposition.

Corollary 3.2. *The group* $\mathrm{SU}(p,q)$, $p \ge q \ge 1$, *is exponential if and only if* $p > q$ *and every odd prime divisor of* $n = p + q$ *is bigger than* $n/(p-q)$.

4. Groups of types B I and D I

In this section we investigate the exponentiality of groups of types B I and D I listed in Table 1.

Proposition 4.1. *For* $n \ge 5$, $\mathrm{Spin}(n-1,1)$ *is exponential.*

Proof. Let $\phi : G^* = \mathrm{Spin}(n-1,1) \to G = \mathrm{SO}(n-1,1)$ be the canonical homomorphism. If $u^* \in G^*$ is unipotent then by Theorem 2.2 we have to show that $Z_{G^*}(u^*)$ is w.e. Let $u = \phi(u^*)$ and

$$Z_G(u) = RC, \quad Z_{G^*}(u^*) = R^*C^*,$$

where R and R^* are unipotent radicals and C and C^* the Levi factors. It is well-known that R^* is exponential, hence we have only to show that C^* is w.e. As G^* is w.e. (see Table 1), we may assume that $u^* \ne 1$. Then $u \ne 1$ and the conjugacy class of u in $\mathrm{O}(n-1,1)$ is $\Gamma_2^+(1) + (n-3)\Gamma_0^+(1)$, see [10, p. 83]. Let $\tilde{Z}$ be the centralizer of u in $\mathrm{O}(n-1,1)$. We have $\tilde{Z} = R\tilde{C}$ where $\tilde{C}$ is a Levi factor. By [26, Chapter 4]), $\tilde{C} \simeq \mathrm{O}(1) \times \mathrm{O}(n-3)$, and so $\tilde{C} \cap G^0 \simeq \mathrm{SO}(n-3)$. By Lemma 2.4, $\phi(C^*)$ is a Levi factor of $Z_{G^0}(u) = \tilde{Z} \cap G^0$ and is isomorphic to $\mathrm{SO}(n-3)$. Hence $C^* = \phi^{-1}(\tilde{C} \cap G^0) \simeq \mathrm{Spin}(n-3)$ is exponential. ■

Proposition 4.2. *For even* $n \ge 4$, $\mathrm{Spin}(2n-2,2)/\langle z \rangle$ *is exponential.*

Proof. Let $\phi : G^* = \mathrm{Spin}(2n-2,2) \to G = \mathrm{SO}(2n-2,2)$ and $\psi : G^* \to G' := G^*/\langle z \rangle$ be the canonical maps. For a unipotent $u' \in G'$, there is a unique unipotent $u^* \in G^*$ such that $\psi(u^*) = u'$, and we set $u = \phi(u^*)$. We have Levi decompositions

$$Z_{G'}(u') = R'C', \quad Z_{G^*}(u^*) = R^*C^*, \quad Z_G(u) = RC,$$

where $R' \simeq R^* \simeq R$ are unipotent, and so exponential. As above, it suffices to show that C' is w.e. and we may assume that $u' \ne 1$. From [10, p. 83] it follows that there are exactly four non-trivial unipotent conjugacy classes Γ in $\mathrm{O}(2n-2,2)$. For each of these classes we list in Table 2 the Levi factors $\tilde{C}$ of the centralizer of $u \in \Gamma$ in $\mathrm{O}(2n-2,2)$, and also $\tilde{C} \cap G^0$. For the description of Γ see [10, p. 83] and for $\tilde{C}$ see [26, Chapter 4].

Table 2

Levi factors of centralizers of unipotent elements in $\mathrm{SO}(2n-2,2)^{\circ}$

	Γ	$\tilde{C}$	$\tilde{C} \cap G^0$
1	$\Gamma_4^+(1) + (2n-5)\Gamma_0^+(1)$	$\mathrm{O}(1) \times \mathrm{O}(2n-5)$	$Z_2 \times \mathrm{SO}(2n-5)$
2	$\Gamma_2^-(1) + (2n-3)\Gamma_0^+(1)$	$\mathrm{O}(1) \times \mathrm{O}(2n-3)$	$Z_2 \times \mathrm{SO}(2n-3)$
3	$2\Gamma_2^+(1) + (2n-6)\Gamma_0^+(1)$	$\mathrm{O}(2) \times \mathrm{O}(2n-6)$	$\mathrm{SO}(2) \times \mathrm{SO}(2n-6)$
4	$\Gamma_1(1,1) + (2n-4)\Gamma_0^+(1)$	$\mathrm{Sp}_2(\mathbf{R}) \times \mathrm{O}(2n-4)$	$\mathrm{Sp}_2(\mathbf{R}) \times \mathrm{SO}(2n-4)$

In case 1 resp. 2 we have $C^* = \phi^{-1}(C) = \langle z \rangle \times H$, where $H \simeq \mathrm{Spin}(2n-5)$ resp. $\mathrm{Spin}(2n-3)$. Hence $C' = \psi(C^*)$ is connected and compact. The same is true in case 3, because then $C^* \simeq \mathrm{SO}(2)\mathrm{Spin}(2n-6)$.

In the last case we have $C^* \simeq \mathrm{Sp}_2(\mathbf{R}) \times \mathrm{Spin}(2n-4)$. Since z is embedded diagonally in C^*, $C' = \phi(C^*) = C^*/\langle z \rangle$ is exponential by Lemma 2.3. ∎

Proposition 4.3. *For $n \geq 3$, $\mathrm{PSO}(2n-2,2)^0$ is exponential.*

Proof. If $a \in \mathrm{SO}(2n-2,2)^0$ then by [10, Theorem 6.3] a or $-a$ is an exponential in this group. Consequently every element of $\mathrm{PSO}(2n-2,2)^0$ is an exponential. ∎

5. Centralizers of unipotent elements in $\mathrm{SO}^*(2n)$

Springer and Steinberg [26, Chapter 4] have described the structure of the centralizers of unipotent elements in unitary, symplectic and orthogonal groups over any field of characteristic $\neq 2$. In order to apply our exponentiality criterion, we also need a description of these centralizers for the group $G = \mathrm{SO}^*(2n)$, consisting of all $\mathbf{R}$-linear automorphisms preserving a nondegenerate skew-hermitian form on a right $\mathbf{H}$-vector space V of dimension n. We shall use the same method as in [26, Chapter IV].

For an unipotent element $u \in G$, there exists a unique nilpotent element X in the Lie algebra $\mathfrak{g}$ of G such that $u = \exp(X)$. It is well-known that the centralizers of X and u in G coincide. We can choose a basis of V in which X has the canonical form as follows, see [7] and [11, pp. 224-225].

There are vectors e_i, f_j, g_k in V ($i \in I$, $j \in J$, $k \in K$) and nonnegative even (resp. odd) integers α_i (resp. β_j, γ_k) such that $X^{\alpha_i+1}e_i = X^{\beta_j+1}f_j = X^{\gamma_k+1}g_k = 0$, and the vectors

$$e_i, Xe_i, X^2e_i, \ldots, X^{\alpha_i}e_i,$$

$$f_j, Xf_j, X^2 f_j, \ldots, X^{\beta_j} f_j, \tag{5.1}$$
$$g_k, Xg_k, X^2 g_k, \ldots, X^{\gamma_k} g_k$$

form a basis of V. Moreover their scalar products are

$$\langle X^a e_i, X^{\alpha_i - a} e_i \rangle = (-1)^a \sqrt{-1}, \quad 0 \le a \le \alpha_i,$$

$$\langle X^b f_j, X^{\beta_j - b} f_j \rangle = (-1)^b, \quad 0 \le b \le \beta_j,$$

$$\langle X^c g_k, X^{\gamma_k - c} g_k \rangle = (-1)^c, \quad 0 \le c \le \gamma_k,$$

and all other products are zero.

Let i be an even resp. odd integer and let $V_i(0)$ be the $\mathbf{R}$-span of

$$\{e_r : \alpha_r = i\}$$

resp.

$$\{f_s : \beta_s = i\} \cup \{g_t : \gamma_t = i\}.$$

We set also

$$V_i(s) = X^s V_i(0) \quad \text{for } s \le i,$$

and

$$V_i = V_i(0) + \cdots + V_i(i).$$

By X_i we denote the restriction of X to V_i, and by Z_i the centralizer of X_i in $\mathrm{SO}^*(V_i)$. Define a one-parameter subgroup $\lambda : \mathbf{R}^* \to G$ by

$$\lambda(t) X^a e_i = t^{2a - \alpha_i} X^a e_i,$$

$$\lambda(t) X^b f_j = t^{2b - \beta_j} X^b f_j,$$

$$\lambda(t) X^c g_k = t^{2c - \gamma_k} X^c g_k.$$

¿From above it follows immediately that V_i are X-invariant and $V = \oplus_i V_i$.

Lemma 5.1. *$\lambda(t)$ normalizes $Z_G(X)$ and all weights of λ in the centralizer $\mathfrak{z}$ of X in $\mathfrak{g}$ are nonnegative.*

Proof. We have $\lambda(t)X = t^2 X \lambda(t)$ by direct computation on basis vectors. Consequently $\lambda(t)$ normalizes $Z_G(X)$. We prove that the weights of λ in $\mathfrak{z}$ are nonnegative. Assume that $Y \in \mathfrak{z}$ has weight p, i.e., $Y \ne 0$ and $\lambda(t) Y \lambda(t)^{-1} = t^p Y$. Then

$$\lambda(t) Y e_i = t^p Y \lambda(t) e_i = t^p Y t^{-\alpha_i} e_i = t^{p - \alpha_i} Y e_i$$

and similarly $\lambda(t) Y f_j = t^{p - \beta_j} Y f_j$ and $\lambda(t) Y g_k = t^{p - \gamma_k} Y g_k$. As $Y \ne 0$, at least one of the vectors Ye_i, Yf_j, Yg_k is nonzero. Say $Ye_i \ne 0$. Assume that $X^b f_j$ occurs in Ye_i. Then $X^{\alpha_i + 1} Y e_i = 0$ implies that $X^{\alpha_i + 1} X^b f_j = 0$, i.e. $\alpha_i + b \ge \beta_j$. As Ye_i has weight $p - \alpha_i$, the vector $X^b f_j$ must have the same weight, i.e. $2b - \beta_j = p - \alpha_i$. Hence $p = 2b + \alpha_i - \beta_j \ge b \ge 0$. The other cases are similar. ∎

Now we come to the main result of this section.

Proposition 5.2. *Let $\Gamma \subset G = \mathrm{SO}^*(2n)$ be a unipotent conjugacy class,*

$$u \in \Gamma = \sum_{i \text{ even}} k_i \Gamma_i(1,1) + \sum_{i \text{ odd}} (p_i \Gamma_i^+(1,1) + q_i \Gamma_i^-(1,1)),$$

and let $Z := Z_G(u) = RC$, where R is the unipotent radical and C a Levi factor of Z. Then $C = \prod_{i \geq 0} C_i$ where

$$C_i \simeq \mathrm{SO}^*(2k_i), \ i \text{ even}; \quad C_i \simeq \mathrm{Sp}(p_i, q_i), \ i \text{ odd}.$$

Proof. Let $X \in \mathfrak{g}$ be the nilpotent element such that $u = \exp(X)$. Then it is well-known that Z is also the centralizer of X in G. By Lemma 5.1, $\mathfrak{z} = \mathfrak{z}^0 \oplus \mathfrak{z}^+$ where $\mathfrak{z}^0$ is the centralizer of λ in $\mathfrak{z}$, i.e., the zero weight space of λ, and $\mathfrak{z}^+$ is the nilpotent ideal which is the sum of all other weight spaces. Since the centralizer of λ in Z is reductive (see [26, Section 2.23]), it follows that $\mathfrak{z}^+$ is the Lie algebra of R and we may assume that C is the centralizer of λ in Z.

Let r be the largest α_i. Then it is not hard to see that $V_r(0)$ is a weight space since all basis vectors of V are also weight vectors. Hence $V_r(0)$ is also C-invariant. (Indeed, if $x \in C$ and $\alpha_i = r$ then $\lambda(t)xe_i = x\lambda(t)e_i = t^{-r}xe_i$.) Therefore all the spaces $X^t(V_r(0))$, $t \leq r$ are also C-invariant. In particular, the weight space corresponding to the weight $-r$ is the direct sum of C-invariant subspaces.

Assume that there exists $\alpha_i \neq r$ and let s be the largest such α_i. Since r and s are even, $r - s = 2k$. By considering the basis vectors as above, one can see that the space $W := V_r(k) + V_s(0)$ is the weight space of weight $-s$, and so W is C-invariant. As C is reductive and $V_r(k)$ is C-invariant, there exists a C-invariant complement W_s of $V_r(k)$ in W. By replacing $V_s(0)$ by W_s, $X^tV_s(0)$ by X^tW_s, $t \leq s$ we may assume that the weight space W corresponding to the weight $-s$ is the direct sum of C-invariant subspaces. Meanwhile one should note that the subspace W_s consists of vector of the same weight $-s$. Thus we may change the original basis of $V_s(0)$ by a new one via the change $V_s(0)$ by W_s. Therefore we may assume that $V_s(0)$ is C-invariant. By repeating this argument, we may assume that all the spaces $V_i(0)$ (and therefore also the spaces $X^tV_i(0)$, $t \leq i$) are C-invariant subspaces. Consequently $C = \prod C_i$, where C_i is the centralizer of λ in Z_i.

Now we define a sesquilinear form $\phi_i : V_i(0) \times V_i(0) \to \mathbf{H}$ by

$$\phi_i(v, w) = \langle v, X^i w \rangle.$$

Then

$$\begin{aligned} \phi_i(w, v) &= \langle w, X^i v \rangle = \langle (-X)^i w, v \rangle \\ &= (-1)^{i+1} \overline{\langle v, X^i w \rangle} = (-1)^{i+1} \overline{\phi_i(v, w)}, \end{aligned}$$

and so ϕ_i is skew-hermitian for even i and hermitian for odd i. It is also non-degenerate. Indeed, if i is even, we have $\phi_i(e_{i_r}, e_{i_s}) = 0$ if $r \neq s$ and $\phi_i(e_{i_r}, e_{i_r}) = -\sqrt{-1}$. If i is odd then the basis $f_{j_1}, \ldots, f_{j_p}, g_{l_1}, \ldots, g_{l_q}$ of $V_i(0)$ is orthogonal with respect to ϕ_i and $\phi_i(f_{j_r}, f_{j_r}) = 1$, $\phi_i(g_{l_s}, g_{l_s}) = -1$. Hence ϕ_i has signature (p, q).

It is clear that C_i preserves ϕ_i. Indeed for $x \in C_i$ and $v, w \in V_i(0)$ we have

$$\begin{aligned}\phi_i(x(v), x(w)) &= \langle x(v), X^i x(w)\rangle = \langle x(v), x(X^i w)\rangle \\ &= \langle v, X^i w\rangle = \phi_i(v, w).\end{aligned}$$

We claim that $C_i \simeq \mathrm{SO}^*(\phi_i)$ for i even, and $C_i \simeq \mathrm{U}(\phi_i)$, the unitary group of ϕ_i, for i odd.

For, every x in $\mathrm{SO}^*(\phi_i)$ (or $\mathrm{U}(\phi_i)$) extends uniquely to an automorphism $\tilde{x}$ of V_i commuting with X. Say $x \in \mathrm{SO}^*(\phi_i)$, and let

$$xe_r = \sum_s e_s b_s, \quad \alpha_r = i,\ b_s \in \mathbf{H},\ e_s \in V_i(0).$$

Then

$$\begin{aligned}\langle \tilde{x}X^a e_r, \tilde{x}X^{i-a}e_r\rangle &= \langle X^a \tilde{x}e_r, X^{i-a}\tilde{x}e_r\rangle \\ &= \langle X^a \sum_s e_s b_s, X^{i-a}\sum_t e_t b_t\rangle \\ &= \sum_{s,t} \bar{b}_s\langle X^a e_s, X^{i-a}e_t\rangle b_t \\ &= \sum_s \bar{b}_s\langle X^a e_s, X^{i-a}e_s\rangle b_s \\ &= (-1)^a \sum_s \bar{b}_s\sqrt{-1}b_s \\ &= (-1)^a\phi_i(xe_r, xe_r) = (-1)^a\phi_i(e_r, e_r) \\ &= (-1)^a\langle e_r, X^i e_r\rangle \\ &= (-1)^a\sqrt{-1} \\ &= \langle X^a e_r, X^{i-a}e_r\rangle.\end{aligned}$$

The same holds for the scalar products of other basis vectors. This means that this extension $\tilde{x}$ preserves the form $\langle .,.\rangle$ on V_i, i.e., $\tilde{x} \in C_i$. Hence our claim (and the proposition) is proved. ∎

We shall need later a certain overgroup of C_i in $\mathrm{SO}^*(V_i)$ when i is even. We shall now describe this overgroup (see (5.2) below). We know that C_i leaves invariant each of the subspaces $V_i(s)$ and commutes with the action of X. The space V_i is an orthogonal direct sum of non-degenerate subspaces $V_i(s)+V_i(i-s)$, for $0 \le s < i/2$, and $V_i(i/2)$. Hence

$$\mathrm{SO}^*(V_i(i/2)) \times \prod_{0\le s<i/2} \mathrm{SO}^*(V_i(s)+V_i(i-s))$$

is a subgroup of $\mathrm{SO}^*(V_i)$.

It is well-known that if V, W are totally isotropic spaces (with respect to some non-degenerate quadratic or (skew-) hermitian form) of the same dimension such that $V+W$ is non-degenerate then the intersection

$$(\mathrm{GL}(V) \times \mathrm{GL}(W)) \cap \mathrm{SO}(V+W)$$

is isomorphic to $\mathrm{GL}(V)$. In particular, since $V_i(s)$ and $V_i(i-s)$ are totally isotropic, the subgroup

$$H_i(s) = (\mathrm{GL}(V_i(s)) \times \mathrm{GL}(V_i(i-s))) \cap \mathrm{SO}^*(V_i(s) + V_i(i-s))$$

is isomorphic to $\mathrm{GL}(V_i(s))$. The group C_i is a subgroup of

$$\mathrm{SO}^*(V_i(i/2)) \times \prod_{0 \le s < i/2} H_i(s) \tag{5.2}$$

and the projection of C_i in each factor of (5.2) is isomorphic to C_i. Thus C_i is "diagonally" embedded in the above direct product.

Lemma 5.3. *Let $\phi : \mathrm{Spin}^*(2n) \to \mathrm{SO}^*(2n)$ be the canonical projection. If i is even and $V_i \neq 0$ then $\phi^{-1}(C_i) \simeq \mathrm{Spin}^*(2k_i)$.*

Proof. Set

$$V_i' = \sum_{0 \le s < i/2} V_i(s), \quad V_i'' = \sum_{i/2 < s \le i} V_i(s). \tag{5.3}$$

By the above remark, the group

$$H_i = (\mathrm{GL}(V_i') \times \mathrm{GL}(V_i'')) \cap \mathrm{SO}^*(V_i' + V_i'')$$

is isomorphic to $\mathrm{GL}(V_i')$. Put $V_i^0 = V_i(i/2)$. Since $C_i \subset \mathrm{SO}^*(V_i^0) \times H_i$ and H_i is simply connected (see [16]), it follows that

$$\phi^{-1}(C_i) \subset \phi^{-1}(\mathrm{SO}^*(V_i^0) \times H_i) = \mathrm{Spin}^*(V_i^0) \times H_i^*,$$

where $H_i^* \simeq H_i$. Hence $\phi^{-1}(C_i) \simeq \mathrm{Spin}^*(V_i^0)$. ∎

6. Special tori in groups of type D III

Let G be a semisimple $\mathbf{R}$-group. An anisotropic $\mathbf{R}$-torus $T \subset G$ is called *special* if $Z_G(T)$ contains a maximal $\mathbf{R}$-split torus S of G such that ST is a maximal $\mathbf{R}$-torus of G. Since the maximal $\mathbf{R}$-split tori are $G(\mathbf{R})$-conjugate (see [5, Section 5], the same is true for special tori.

Let n be even, and $\tilde{G} = \mathrm{Spin}_{2n}(\mathbf{C})$ with $\tilde{G}(\mathbf{R}) = G^* = \mathrm{Spin}^*(2n)$. We denote by $\{\pm 1\}$ the kernel of the canonical projection $\mathrm{Spin}_{2n}(\mathbf{C}) \to \mathrm{SO}_{2n}(\mathbf{C})$. If T is special in $\tilde{G}$, then precisely one of the elements of the center of G^*, distinct from $\{\pm 1\}$, belongs to T, see [15, Proof of Theorem 4.5]. In order to conform to the notation of that paper, let $\tilde{Z} = \{1, z, z', zz'\}$ be the center of G^*, with $zz' = -1$, $z \in T$, and $z' \notin T$. We also note that for any maximal $\mathbf{R}$-split torus S we have $S \cap \tilde{Z} = \{1\}$ (*loc.cit.*). In the next three lemmas, we refer to the real points of special (resp. split) tori also as special (resp. split) tori, which should not lead to any confusion. We shall use the notations $V, V_i, V_i(s), C_i$ etc. from the previous section.

Lemma 6.1. *If $V = U_1 \oplus U_2$ and $U_1 \perp U_2$, then the product of special tori of $\mathrm{SO}^*(U_1)$ and $\mathrm{SO}^*(U_2)$ is contained in a special torus of $\mathrm{SO}^*(V)$.*

Proof. Let T_i be a special torus of $\mathrm{SO}^*(U_i)$, $i = 1, 2$. Then there is a maximal split torus S_i of $\mathrm{SO}^*(U_i)$ centralizing T_i. Since $S_1 S_2$ is a maximal split torus of $\mathrm{SO}^*(V)$ which centralizes $T_1 T_2$, the lemma follows. ∎

Lemma 6.2. *Every special torus of C_i, i even, is contained in a special torus of* $\mathrm{SO}^*(V_i)$.

Proof. We define V_i' and V_i'' by (5.3) and set $V_i^0 = V_i(i/2)$. The spaces V_i' and V_i'' are totally isotropic. Their sum and V_i^0 are non-degenerate. As above, the group

$$H_i = (\mathrm{GL}(V_i') \times \mathrm{GL}(V_i'')) \cap \mathrm{SO}^*(V_i' + V_i'')$$

is isomorphic to $\mathrm{GL}(V_i')$. By (5.2) we have

$$C_i \subset \mathrm{SO}^*(V_i^0) \times H_i \subset \mathrm{SO}^*(V_i^0) \times \mathrm{SO}^*(V_i' + V_i'') \subset \mathrm{SO}^*(V_i).$$

Let P be a special torus of C_i. Since the projection of C_i to $\mathrm{SO}^*(V_i^0)$ is an isomorphism (see (5.2)), the image P' of P under this projection is also a special torus of $\mathrm{SO}^*(V_i^0)$. Since P is compact, its projection P'' to H_i is contained in a maximal compact torus Q of H_i. Since Q is special in $\mathrm{SO}^*(V_i' + V_i'')$, the assertion follows from Lemma 6.1. ∎

Lemma 6.3. *Let P_i be a torus of C_i such that P_i is special in C_i for i even, and P_i is maximal compact in C_i for i odd. Then the product P of all P_i's is contained in a special torus of* $\mathrm{SO}^*(V)$.

Proof. By Lemma 6.1 it suffices to show that each P_i is contained in a special torus of $\mathrm{SO}^*(V_i)$. If i is even this was proved in Lemma 6.2. If i is odd, it follows from the fact that we have the inclusions $C_i \subset H_i \subset \mathrm{SO}^*(V_i)$, where H_i is defined analogously to the case when i is even. Indeed, P_i is a maximal compact torus of H_i and every such torus is special in $\mathrm{SO}^*(V_i)$. ∎

7. Groups of type D III

In this section we investigate the exponentiality of the groups of type D III listed in Table 1. We set $G^* = \mathrm{Spin}^*(2n)$ and $G = \mathrm{SO}^*(2n)$. Since G is exponential, there are two cases to consider depending on the parity of n. We first deal with the case when n is odd.

Proposition 7.1. *For odd $n \geq 3$,* $\mathrm{Spin}^*(2n)$ *is exponential.*

Proof. We use induction on n. If $n = 3$, then $\mathrm{Spin}^*(6) \simeq \mathrm{SU}(3, 1)$ is exponential by Corollary 3.2. Let $n > 3$ and assume that the assertion is true for all odd integers $< n$. Let $u^* \in \mathrm{Spin}^*(2n)$, $u^* \neq 1$, be a unipotent element, and $u = \phi(u^*)$, where $\phi : G^* \to G$ is the natural projection. The centralizer $Z = Z_G(u)$ is described in Proposition 5.2. Namely, $Z = RC$ where R is the unipotent radical and the Levi factor $C = \prod_{i \geq 0} C_i$. Furthermore, $C_i \simeq \mathrm{SO}^*(2k_i)$ for i even, and $C_i \simeq \mathrm{Sp}(p_i, q_i)$ for i odd.

Since n is odd, there is at least one even integer i such that k_i is odd and, necessarily, $k_i < n$ (for the notations see Section 5). By Lemma 5.3, $\phi^{-1}(C_i) \simeq \mathrm{Spin}^*(2k_i)$ and, by induction hypothesis, $\mathrm{Spin}^*(2k_i)$ is exponential. Since

$$\phi^{-1}(\prod_{i \text{ even}} C_i) \simeq \prod_{i \text{ even}}^{*} \mathrm{Spin}^*(2k_i),$$

this group is exponential by Lemma 2.3.

If i is odd then it is well-known that C_i is simply connected, and so $\phi^{-1}(C_i) = \{\pm 1\} \times C_i^*$ with $C_i^* \simeq C_i$. By Lemma 5.3, for i even, $C_i^* := \phi^{-1}(C_i) \simeq \mathrm{Spin}^*(2k_i)$. Consequently we have

$$C^* = \phi^{-1}(C) = \prod_{i \text{ even}}^{*} C_i^* \times \prod_{i \text{ odd}} C_i^*. \tag{7.1}$$

As each of the groups C_i, i odd, is exponential, it follows from (7.1), that the group $C^* = \phi^{-1}(C)$ is exponential. Hence G^* is exponential by Theorem 2.2. ■

Now we consider the case n is even.

Proposition 7.2. *For even $n \geq 4$, $G' := \mathrm{Spin}^*(2n)/\langle z' \rangle$ is exponential.*

Proof. Let $\phi : G^* \to G$ and $\psi : G^* \to G'$ be the canonical projections. As we noted before, z' does not belong to any maximal $\mathbf{R}$-split torus of G^*. It follows from [6, Corollary 4.7] that G' is the group of real points of the corresponding $\mathbf{R}$-group $\mathrm{Spin}_{2n}(\mathbf{C})/\langle z' \rangle$. Given a unipotent element $u' \in G'$ there exists a unique unipotent element $u^* \in G^*$ such that $u' = \psi(u^*)$. Let $Z_{G'}(u') = R'C'$, where R' is the unipotent radical and C' a Levi factor. Then $Z_{G^*}(u^*) = R^*C^*$ where $R^* \simeq R'$ and $C^* = \psi^{-1}(C')$. We shall prove that C' is w.e.

If $u = \phi(u^*)$ then $Z_G(u) = RC$ where $R = \phi(R^*)$ and $C = \phi(C^*)$. The Levi factor C is described in Proposition 5.2. We shall use the notations introduced in that section.

Consider first the case where $C_i = 1$ for all even i. Then C is simply connected and so $C^* = \{\pm 1\} \times C^{*0}$ with $C^{*0} \simeq C$.

We claim that $z' \notin C^{*0}$. To prove this claim, we introduce the subspaces

$$V' = \sum_{i \text{ odd}} \sum_{0 \leq s < i/2} V_i(s),$$

and V'' defined similarly except that $i/2 < s \leq i$. They are maximal totally isotropic subspaces and $V = V' \oplus V''$. We have $C \subset H \subset G$, where $H := (((\mathrm{GL}(V') \times \mathrm{GL}(V'')) \cap G) \simeq \mathrm{GL}(V')$. By taking inverse images of the triple $C \subset H \subset G$ under ϕ and by using the fact that H is simply connected, we obtain $C^* \subset \{\pm 1\} \times H^* \subset G^*$, with $H^* \simeq H$. Since maximal compact tori of H^* are special in G^*, we have $z' \notin H^*$, and so $z' \notin C^{*0}$. This proves our claim.

Hence $C' = \psi(C^*) \simeq C$ and so C' is exponential.

Now assume that $C_i \neq 1$ for at least one even integer i. ¿From (7.1) it follows that if at least one of the k_i's is odd, the corresponding $C_i^* \simeq \mathrm{Spin}^*(2k_i)$ is exponential by Prop. 7.1, and by Lemma 2.3, $\phi^{-1}(C)$ and C' are exponential. It remains to consider the case where all k_i's are even.

By Lemma 2.4 the maximal tori of C' are of the form $\psi(P)$ where P is a maximal torus of C^*. Assume that P is not connected. As G is exponential, $\phi(P)$ is connected. Thus $[P : P^0] = 2$ and $-1 \notin P^0$. For i even each $P_i := P \cap C_i^*$ is a maximal torus of C_i^* and $-1 \in P_i \setminus P_i^0$. ¿From the known classification of maximal tori in C_i (see [15]), it follows that P_i is a maximally split maximal torus of C_i^*. Consequently the maximal compact subtorus Q_i of P_i is special in C_i^* for i even. For odd i let Q_i be any compact maximal torus of C_i^*. By Lemma 6.3 the product Q of all Q_i's is contained in a special torus of G^*.

For i even let z_i, z_i' be the central elements of C_i^* defined analogously to z and z'. For i odd let z_i be the non-identity central element of $C_i^* \simeq \mathrm{Sp}(p_i, q_i)$. Let w be the product of all z_i. Then $\phi(w) = -1$ and so $w \in \{z, z'\}$. Since each k_i is even, we have $z_i \in Q_i$ for i even, see [15, Proof of Theorem 4.5], and so $w \in Q$. Since Q is contained in a special torus of G^*, we must have $w = z$. Hence $z \in Q \subset P^0$. As P has only two components, it follows that $\psi(P) = P/\langle z' \rangle$ is connected. This completes the proof of the proposition. ∎

8. Exceptional groups

In this short section we investigate the exponentiality of the four types of exceptional groups listed in Table 1.

Proposition 8.1. *Among the exceptional w.e. groups in* Σ *only the one of type* E IV *is exponential.*

Proof. Let $\mathfrak{g}$ be the Lie algebra of $G(\mathbf{R})$. We claim that for adjoint groups of type E III, E VII, or F II there is a nilpotent element $X \in \mathfrak{g}$ such that $Z_{G(\mathbf{R})}(X)$ is not connected (as a Lie group). Then, by Theorem 2.2, $G(\mathbf{R})$ is not w.e.

This claim follows by inspection of tables in [13, 14], which classify nilpotent orbits in $\mathfrak{g}$ under the action of the adjoint group, and the description of $Z_G(X)$ for a nilpotent element $X \in \mathfrak{g}$ given in [1], see also [12, Section 5].

If G is of type E III then we choose X to lie in the orbit No. 6 in Table X of [13]. When G is of type E VII then we use orbit No. 2 in Table VII of [14]. If G is of type F II there are only two non-zero nilpotent orbits in $\mathfrak{g}$, see [13, Table VIII], and in both cases $Z_{G(\mathbf{R})}(X)$ is disconnected.

Now let G^* be of type E IV. Since the center of G^* is trivial (see [T]), we may assume that G^* is the group of real points of the adjoint $\mathbf{R}$-group of type E IV. There are only two non-zero nilpotent orbits in $\mathfrak{g}$, see [14, p. 204]. The Levi factor of $Z_{G^*}(X)$ is compact and connected for all $X \in \mathfrak{g}$, $X \neq 0$. By Theorem 2.2, G^* is exponential. ∎

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