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Semigroups in lattices of solvable Lie groups

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Abstract. Let G be a solvable Lie group, $\Gamma\subset G$ a lattice and $S\subset\Gamma$ a semigroup. It is proved that S is a group provided it is not contained in a semigroup with non-empty interior of G and Γ satisfies a condition which is described by means of the complex weights of the adjoint representation of the Lie algebra of G. The methods follow the same pattern as those developed by J. D. Lawson [3] in the analysis of the semigroups with interior points in G, and as such they require a machinery about semigroups in finitely generated groups.

1. Introduction

Let G be a solvable Lie group and $\Gamma \subset G$ a lattice in G. The purpose of this article is to study semigroups in G which are contained in Γ. We recall that a lattice in a locally compact group G is a discrete subgroup $\Gamma \subset G$ such that the homogeneous space G/Γ possesses a finite G-invariant measure, and in case G is a connected solvable Lie group, it is well-known that a discrete subgroup $\Gamma \subset G$ is a lattice if and only if G/Γ is compact (c.f. [4, Thm. 3.1]).

The question posed here is that of finding conditions ensuring that such semigroups are not groups. This question is related to the one of finding the maximal semigroups of Γ. Our approach is to look at the semigroups through the embedding of Γ in G and relate them to the semigroups of G which have non-empty interior. This approach was taken by the authors in [6] for the special case where G is a nilpotent Lie group. There it was shown that a subsemigroup of a lattice of a nilpotent Lie group is a group if it is not contained in any semigroup with non-void interior of the group. This way the maximal semigroups of the lattices were obtained as their intersections with the semigroups with interior points. Our purpose here is to extend these results to more general solvable groups. The situation for solvable groups, however, is somewhat distinct from the nilpotent one because contrary to the latter case, there are solvable groups which contain lattices which in turn contain semigroups which are not groups and despite that are not contained in any semigroup with non-empty interior. The following is a typical example of what might happen.

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Example 1.1. Let G be $I\!R \times I\!R^2$ with group structure

$$(t,v)(s,u) = (t+s, e^{tA}u + v)$$

where A is the 2×2 matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The Lie algebra $\mathbf{g}$ of G is the semi-direct product of $I\!R$ with $I\!R^2$ with the representation given by the multiples of the matrix A. The abelian subgroup $\Gamma = 2\pi\, \boldsymbol{Z} \times \boldsymbol{Z}^2$ is a lattice of G and

$$S = \{(t,(x_1,x_2)) \in \Gamma : x_1 \geq 0\}$$

is a proper semigroup of Γ which is not contained in any semigroup with non-empty interior of G. In fact, the only codimension one subalgebra of $\mathbf{g}$ is the ideal $\{0\} \times I\!R^2$ so it follows from the theory in [3] that the only maximal semigroups with non-empty interior of G are the semigroups

$$S^{\pm} = \{(\pm t, v) \in G : t \geq 0\}$$

which do not contain S so that this semigroup is not contained in a semigroup with non-empty interior. Note that $\Gamma = S \cup S^{-1}$ so that S generates Γ as a group.

This example shows the typical situation where the desired result for the semigroups in Γ does not hold. The point here is the appearance of complex roots in the adjoint representation of $\mathbf{g}$ causing the existence of subspaces in $\mathbf{g}$ which are invariant under the adjoint action of Γ but not of G, that is, which are not ideals. Indeed, Theorem 8.3 below ensures that a generating semigroup of Γ is contained in a semigroup with non-empty interior of G provided Γ is in a certain general position with respect to the complex roots of the adjoint representation of $\mathbf{g}$. In particular, this result holds in case all roots are real so that it encompasses the result of [6] about nilpotent groups.

Our methods are adapted from those developed by J. Lawson [3] for the semigroups with interior points. As the lattices are finitely generated groups, their generating semigroups admit algebraic interior (c.f. Section 2. below) which substitute, in the arguments, the topological interior used for semigroups in topological groups. This way, most of the results of [3] carry on to semigroups in Γ. With these methods an alternative proof of the result of [6] for lattices in nilpotent groups is given (c.f. Theorem 4.1 below), and for the solvable groups treated here they lead ultimately to an analysis of semigroups in the two-dimensional non-abelian Lie group which is made in Section 6.

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2. Semigroups in finitely generated groups

The purpose of this section is to develop some background material about semigroups in finitely generated groups. The interest in these semigroups becomes clear if one recalls the well-known fact that the lattices in solvable Lie groups belong to this class of groups.

Throughout this section we let Γ be a finitely generated group. Let $S \subset \Gamma$ be a semigroup. We say that S is generating in case S is not contained in any proper subgroup of Γ. Equivalently, S is generating provided $\Gamma = \langle S \cup S^{-1} \rangle$ where $\langle A \rangle$ means the semigroup generated by the subset $A \subset \Gamma$. There are some aspects of the semigroups in Γ which can be viewed by analogy with the semigroups with non-empty interior in topological groups. This is implemented by the following concept.

Definition 2.1. Let Γ be a group and $A \subset \Gamma$ a subset. The *algebraic interior* of A is the subset of those $x \in A$ such that $\Gamma = \langle Ax^{-1} \rangle$. A symmetric version of this interior is the *symmetric algebraic interior* which is defined as the subset of those $x \in A$ for which there exists $U \subset \Gamma$ which is symmetric, that is, $U = U^{-1}$ and satisfies $\Gamma = \langle U \rangle$ and $Ux \subset A$.

In what follows we denote the algebraic interior of A by $\mathrm{intalg}(A)$ while the symmetric algebraic interior will be denoted by $\mathrm{intalg}_{\mathbf{S}}(A)$. In this definition it was considered right translations by the inverses of the elements of A. We note that the same algebraic interior is obtained in case one takes instead left translations. This is because $x^{-1}A = (x^{-1}Ax)x^{-1}$ and a subset generates a group if and only if its conjugates are also generating.

The analogy of the algebraic interior of a subset of a finitely generated group with the topological interior in a topological group is clear. In particular, we mention the fact that a semigroup $S \subset \Gamma$ coincides with Γ in case the identity $1 \in \mathrm{intalg}(S)$ as follows immediately from the definition. This provides a method analogous to existing ones for semigroups with interior points in topological groups for deciding whether a semigroup is a group. The existence of algebraic interior is ensured by the following proposition in most of the interesting cases.

Proposition 2.2. *Let Γ be a finitely generated group and $S \subset \Gamma$ a subsemigroup. The following statements are equivalent.*

a) *S is generating.*

b) $\mathrm{intalg}_{\mathbf{S}}(S) \neq \emptyset$.

c) $\mathrm{intalg}(S) \neq \emptyset$.

Proof. The implications $(b) \Rightarrow (c)$ and $(c) \Rightarrow (a)$ are trivial. Assume that S is generating. We have that S is denumerable because Γ is finitely generated. Let

$$S = \{x_1, \ldots, x_n, \ldots\}$$

be an enumeration of S. For integer n let S_n be the semigroup of Γ generated by the symmetric subset

$$U_n = \{x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}\} .$$

Of course, $\Gamma = \bigcup_{n\geq 1} S_n$ and since S_n is an ascending family of semigroups and Γ is finitely generated, it follows that $\Gamma = S_n$ for some integer n. This means that U_n is a symmetric generating subset of Γ. From this, it follows easily that

$$N = \{x_1, x_1^{-1}, x_1x_2, (x_1x_2)^{-1}, \dots, x_1\cdots x_n, (x_1\cdots x_n)^{-1}\}$$

is also a symmetric subset of generators. Now, let

$$x = (x_1 \cdots x_n)x_1 \ .$$

Clearly, $x \in S$ and since the elements of Nx are of the form $(x_i \cdots x_n)x_1$, $i = 1, \dots, n$, we have that $Nx \subset S$ showing that $x \in \mathrm{intalg}_{\mathbf{S}}(S)$. □

Like the interior of a semigroup in a topological group, the algebraic interiors are also ideals of the semigroup. In fact, we have

Proposition 2.3. *For a generating semigroup S of a finitely generated group Γ we have that* $\mathrm{intalg}(S)$ *and* $\mathrm{intalg}_{\mathbf{S}}(S)$ *are (right and left) ideals of S.*

Proof. Take $x \in \mathrm{intalg}(S)$ and $y \in S$. Then

$$yx(x^{-1}S) = yS \subset S$$

so that $x^{-1}S \subset (yx)^{-1}S$. Therefore,

$$\Gamma = \langle x^{-1}S\rangle \subset \langle (yx)^{-1}S\rangle \subset \Gamma$$

which shows that $yx \in \mathrm{intalg}(S)$. Similarly, $(Sx^{-1})xy = Sy \subset S$, so $Sx^{-1} \subset S(xy)^{-1}$, showing that $xy \in \mathrm{intalg}_{\mathbf{S}}(S)$.

Now, let $x \in \mathrm{intalg}_{\mathbf{S}}(S)$ and $y \in S$. Then there exists a symmetric system of generators of Γ, say U, such that $Ux \subset S$. It follows that $Uxy \subset Sy \subset S$ showing that $xy \in \mathrm{intalg}_{\mathbf{S}}(S)$. On the other hand, let $U' = yUy^{-1}$. Then U' is also a symmetric generating subset, and since $U'yx = yUx$ we have that $U'yx \subset yS \subset S$ so that yx also belongs to $\mathrm{intalg}(S)$. □

These two propositions contain the facts about the algebraic interiors which will be needed afterwards. However, in order to compare these interiors with the topological interior, we include the following further comments about the algebraic interiors.

Proposition 2.4. *Suppose Γ is finitely generated. Then $S \subset \Gamma$ is a generating semigroup if and only if* $\mathrm{intalg}(S)$ *is generating.*

Proof. Of course, S is generating if $\mathrm{intalg}(S)$ is generating. On the other hand, $\mathrm{intalg}(S) \neq \emptyset$ if S is generating. Let $x \in \mathrm{intalg}(S)$ and take $y \in S$. We have that

$$y = (yx)x^{-1} \in (\mathrm{intalg}(S))(\mathrm{intalg}(S))^{-1}$$

because $\mathrm{intalg}(S)$ is an ideal. This shows that S is contained in the group

$$\langle (\mathrm{intalg}(S)) \cup (\mathrm{intalg}(S))^{-1}\rangle$$

so that $\mathrm{intalg}(S)$ is also generating. □

This proposition permits to define recursively $S^{\circ(n)} = \mathrm{intalg}(S^{\circ(n-1)})$ with $S^{\circ(0)} = S$. In case S is generating, each one of these semigroups is generating and $S^{\circ(n)}$ is an ideal of $S^{\circ(n-1)}$. The following proposition improves this statement.

Proposition 2.5. *$S^{\circ(n)}$ is an ideal of S for each $n \geq 0$ if S is generating.*

Proof. By induction on n. For $n = 0$ there is nothing to prove. Assuming that $S^{\circ(n)}$ is an ideal of S, take $x \in S^{\circ(n+1)}$ and $y \in S$. We have that $yS^{\circ(n)}$ and $S^{\circ(n)}y$ are contained in $S^{\circ(n)}$ because this is an ideal. Hence

$$x^{-1}S^{\circ(n)} = (yx)^{-1}yS^{\circ(n)} \subset (yx)^{-1}S^{\circ(n)}$$

and

$$S^{\circ(n)}x^{-1} = S^{\circ(n)}y(xy)^{-1} \subset S^{\circ(n)}(xy)^{-1} .$$

Since either $\langle x^{-1}S^{\circ(n)}\rangle$ and $\langle S^{\circ(n)}x^{-1}\rangle$ coincide with Γ, we conclude from these inclusions that

$$\Gamma = \langle S^{\circ(n)}(xy)^{-1}\rangle = \langle (xy)^{-1}S^{\circ(n)}\rangle$$

which shows that xy and yx belong to $S^{\circ(n+1)}$ as desired. □

Although these results are stated for the algebraic interiors only, it is not hard to get similar statements for the symmetric algebraic interior. The last proposition shows that the successive algebraic interiors form a non increasing sequence of ideals of S. In contrast to the topological interior, this sequence may be strictly decreasing as happens, for example, with the semigroup

$$S = \{x \in \boldsymbol{Z} : x \geq 0\}$$

of the integer group, for which

$$S^{\circ(n)} = \{x \in \boldsymbol{Z} : x \geq n\} .$$

On the other hand, there are semigroups of finitely generated groups which are "open" in the sense that they coincide with their algebraic interiors. The following is an example of such a semigroup.

Example 2.6. Take $\Gamma = \boldsymbol{Z}^2$ and put

$$S = \{(x, y) : -x + \sqrt{2}y > 0\} .$$

Of course, S is a generating semigroup of $\boldsymbol{Z}^2$. Let us check that $S = \mathrm{intalg}(S)$. Put $S' = S \cup \{0\}$. This is a maximal semigroup of $\boldsymbol{Z}^2$ because it is the intersection of a half-plane in $\mathbb{R}^2$ with $\boldsymbol{Z}^2$ (c.f.[6]). Take $\alpha \in S$. Then $f(\alpha) > 0$ where f is the linear functional $f(x, y) = -x + \sqrt{2}\, y$. Hence by the irrationality of $\sqrt{2}$, there exists $\beta \in S$ such that $0 < f(\beta) < f(\alpha)$. This implies that $f(\beta - \alpha) < 0$ so that $\beta - \alpha \notin S'$. Now, it is clear that $S' \subset \langle -\alpha + S\rangle$ and $\beta - \alpha \in \langle -\alpha + S\rangle$ so that the maximality of S' implies that $\langle -\alpha + S\rangle = \boldsymbol{Z}^2$. Therefore, $\alpha \in \mathrm{intalg}(S)$ showing that $S = \mathrm{intalg}(S)$.

The following is an example of a generating semigroup for which the symmetric algebraic interior is properly contained in the algebraic interior.

Example 2.7. In the group $\boldsymbol{Z}^2$ take the semigroup

$$S = \{(x,y) \in \boldsymbol{Z}^2 : x \geq 0 \text{ and } \mid y \mid \leq \frac{1}{4}x\} .$$

We have that $(1,0) \in \operatorname{intalg}(S)$ because the subset $\{(-1,0),(3,1),(3,-1)\}$ generates $\boldsymbol{Z}^2$ and is contained in $-(1,0)+S$. On the other hand, the symmetric subsets of $-(1,0)+S$ are contained in the x-axis so that $(1,0) \notin \operatorname{intalg}_{\mathbf{S}}(S)$.

Next, we have the following useful statement about maximal semigroups in Γ. By a maximal semigroup we understand, as is usual, a semigroup which is not a group and which is maximal with this property.

Proposition 2.8. *Let Γ be a finitely generated group and $S \subset \Gamma$ a generating proper semigroup. Then S is contained in a maximal semigroup of Γ.*

Proof. Let $\mathcal{M}$ be the family of all proper semigroups of Γ containing S. We order $\mathcal{M}$ by inclusion. Let $(T_i)_{i \in I}$ be a totally ordered subset of $\mathcal{M}$. Clearly, $T = \bigcup_{i \in I} T_i$ is a semigroup containing S. Moreover, T is a proper semigroup. In fact, if T were Γ we would have that $\Gamma = T_i$ for some i because Γ is finitely generated. But this contradicts the fact that T_i is a proper semigroup. Therefore the maximality theorem ensures that S is contained in a maximal semigroup. □

Related to the above results is the following fact about the group generated by a semigroup. This fact can be stated in a more general context.

Proposition 2.9. *Let G be a group and $H \subset G$ a subgroup. Let also $S \subset G$ be a semigroup which is generating and suppose that S is transitive in G/H in the sense that $Sx = G/H$ for all $x \in G/H$. Then $S \cap H$ is generating in H.*

Proof. Denote by L the semigroup of H generated by $S \cap H$ and take $x \in H$. Since S is generating, there are $x_1, \ldots, x_n$ in S such that

$$x = x_1^{-1} x_2 x_3^{-1} \cdots x_{n-1} x_n^{-1}$$

where x_1 or x_n are admitted to be the identity. We wish to show that $x \in L$. We use induction on n. In case, in the above expression $n = 1$, it is obvious that $x \in L$. On the other hand, by the transitivity of S on G/H, there exists $\bar{x} \in S$ such that $\bar{x}x_1 \in H$ or equivalently, $x_1^{-1}\bar{x}^{-1} \in H$. By rewriting the above expression as

$$x = x_1^{-1} \bar{x}^{-1} \bar{x} x_2 \cdots x_n^{-1}$$

we get that

$$(\bar{x}x_1)x = y x_3^{-1} \cdots x_{n-1} x_n^{-1} \in H$$

with $y = \bar{x}x_2 \in S$. By induction, the right hand side of this expression belongs to L hence $(\bar{x}x_1)x \in L$ showing that $x \in L$ because $\bar{x}x_1 \in L$. □

3. Archimedean semigroups

In what follows, we shall need some facts related to archimedean semigroups.

Definition 3.1. We say that a semigroup S in a group G is *right* [respectively *left*] *archimedean* if for any $x \in S$ with $x^{-1} \notin S$ and $y \in G$ there exists a positive integer n such that $yx^n \in S$ [respectively $x^n y \in S$]. The semigroup is archimedean if it is both left and right archimedean.

Our purpose here is to look at invariant maximal semigroups in general groups. We shall obtain results similar to those of Lawson [3, section 7] with the difference that instead of topological concepts we make use of the archimedean property of a semigroup. We use the following standard terminology and notation related to a semigroup $S \subset G$ (c.f. [2, 3]): $H(S) = S \cap S^{-1}$ is the largest subgroup contained in S, while $\operatorname{core}(S) = \bigcap\{gH(S)g^{-1} : g \in G\}$ is the largest normal subgroup contained in S (c.f. [3, Prop.1.3]). The semigroup S is said to be invariant provided $gSg^{-1} \subset S$ for all $g \in G$ and S is total in G in case $G = S \cup S^{-1}$.

Before looking at the maximal invariant semigroups we note the following two facts about the archimedean semigroups. First, S is right archimedean if and only if S^{-1} is left archimedean, so that S is archimedean if both S

4. Nilpotent groups

Before embarking into the analysis of the semigroups in the lattices of solvable Lie groups, it is needed to consider the nilpotent Lie groups. Semigroups in lattices of nilpotent Lie groups were studied in [6]. The main result of that paper which will be needed below can be stated as follows.

Theorem 4.1. *Let N be a simply connected nilpotent Lie group and $\Gamma \subset N$ a lattice in N. Let $S \subset \Gamma$ be a generating semigroup and assume that S is not contained in any proper semigroup with non-empty interior of N. Then $S = \Gamma$.*

This is essentially the statement of Theorem 4.1 of [6]. We shall present here an alternative simpler proof of this theorem based on the developments of the previous section and on a result by J. Lawson [3] about maximal semigroups in nilpotent groups.

Proof. Since Γ is finitely generated, we can assume that S is maximal in Γ (c.f. Proposition 2.8). Let $[N,N]$ be the derived group of N and consider the canonical projection

$$\pi : N \longrightarrow N/[N,N] \ .$$

It follows from section 2 of [5] that $\pi(\Gamma)$ is a lattice in the simply connected abelian group $N/[N,N]$. We have that $\pi(S)$ is not contained in a proper semigroup with non-void interior of $N/[N,N]$ otherwise S would be contained in such a semigroup in N. Therefore, [6, Prop.2.1] implies that $\pi(S)$ is a group.

On the other hand, since S is assumed to be maximal, Theorem 8.3 in [3] ensures that $[\Gamma,\Gamma]$ is contained in S. Clearly, $[\Gamma,\Gamma] \subset [N,N]$ so that $[\Gamma,\Gamma]$ is a normal subgroup of $\Gamma \cap [N,N]$. Moreover, we have the following fact

Lemma 4.2. $[\Gamma,\Gamma]$ *is of finite index in* $\Gamma \cap [N,N]$.

Proof. ¿From [5, Thms. 2.1,2.3] we have that $[\Gamma,\Gamma]$ and $\Gamma \cap [N,N]$ are lattices in $[N,N]$. Now, we have the canonical fibration

$$[N,N]/[\Gamma,\Gamma] \longrightarrow [N,N]/(\Gamma \cap [N,N])$$

given by the fact that $[\Gamma,\Gamma] \subset \Gamma \cap [N,N]$. This fibration is a covering and the fiber is isomorphic to $(\Gamma \cap [N,N])/[\Gamma,\Gamma]$ and since $[N,N]/[\Gamma,\Gamma]$ is compact, we have a finite covering which shows that $[\Gamma,\Gamma]$ is of finite index in $\Gamma \cap [N,N]$ as desired. □

Since a semigroup which contains a group of finite index is a group, it follows that $S \cap [N,N]$ is a group. Combining this with the fact that $\pi(S)$ is a group, we get that S itself is a group (c.f. [6, Lemma 3.4]). This contradicts the fact that S is maximal showing that S is contained in a proper semigroup with non-empty interior of N. □

As a complement to this proof, we mention that in general $[\Gamma,\Gamma]$, although of finite index, may be properly contained in $\Gamma \cap [N,N]$ as shows the following example.

Example 4.3. Let N be the Heisenberg group of upper triangular 3×3 matrices with ones on the main diagonal. Alternatively, N is $\mathbb{R}^3$ with the product given by

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + xy') \; .$$

In N let Γ be the lattice

$$\Gamma = \{(2n, 2m, p) : n, m, p \text{ are integers}\} \; .$$

It is easily checked that any commutator of elements of Γ is of the form $(0, 0, 2p)$ for some integer p. Since $\Gamma \cap [N, N] = \{(0, 0, p) : p \text{ is integer}\}$ this shows that $[\Gamma, \Gamma]$ is properly contained in $\Gamma \cap [N, N]$.

The above theorem can be restated in the following convenient form.

Corollary 4.4. *Keep the notations and assumptions of Theorem 4.1. Then S is a group if and only if $S/[N, N]$ is a group.*

Proof. Clearly, $S/[N, N]$ is a group if S is a group. Reciprocally, if $S/[N, N]$ is a group, the above proof shows that S is not contained in any maximal semigroup of Γ so that S is a group by Proposition 2.8. $\square$

Finally, we have the following result proved in [6, Prop. 3.6] which will be needed later in the proof of Lemma 7.2. In its statement we leave implicit the fact that a connected and simply connected abelian Lie group is just the additive group of a vector space.

Proposition 4.5. *Let N be a simply connected nilpotent Lie group with Lie algebra $\mathbf{n}$. Assume that $\mathbf{n}^2 = 0$ and let $S \subset N$ be a semigroup which is not contained in any semigroup with non-empty interior of N. Suppose that $S/[N, N]$ is rational in the sense that it is contained in the rational vector space spanned by a basis of $N/[N, N]$. Then $S \cap [N, N]$ is not contained in any halfspace of $[N, N]$.*

5. Invariant cones

In what follows, we let G stand for a simply connected solvable Lie group and let Γ be a lattice in G. Denote by $\mathbf{g}$ the Lie algebra of G and let $\mathbf{n}$ be the nil radical of $\mathbf{g}$, that is, $\mathbf{n}$ is the largest nilpotent ideal of $\mathbf{g}$. We denote by N the connected subgroup of G whose Lie algebra is $\mathbf{n}$ and refer to it as the nil radical of G.

Our final purpose is to show that a semigroup of Γ which is not contained in any semigroup with non-empty interior of G is in fact a group. The main idea of our approach is to break a semigroup $S \subset \Gamma$ into its quotient S/N modulo N and into its intersection $S \cap N$ with the nil radical. The point here is that N is a nilpotent simply connected closed and normal Lie subgroup of G which contains the derived group $[G, G]$ and which is nicely located with respect to Γ in the sense that both $\Gamma \cap N$ and Γ/N are lattices in N and G/N respectively. This fact permits to recover S from its decomposition into S/N and $S \cap N$. In what follows we say that S satisfies hypothesis **H** provided it is not contained in any semigroup with non-empty interior of G. As to the projections, we have.

Lemma 5.1. *With G, Γ and N as above, let $\pi : G \to G/N$ be the canonical projection. Let $S \subset \Gamma$ be a generating semigroup and suppose that S satisfies* **H**. *Then $\pi(S) = \pi(\Gamma)$ is a lattice in G/N.*

Proof. The fact that Γ/N is a lattice is a classical fact: since $N \cap \Gamma$ is a lattice it follows that $N\Gamma$ is closed in G (c.f. [5, Thm. 1.13]) so that $\pi(\Gamma)$ is closed in G/N and since $\pi(G)/\pi(\Gamma) \approx G/\Gamma$, which is compact, we have that Γ/N is a lattice in G/N.

Now, let S' be a semigroup with non-empty interior of G/N containing $\pi(S)$. Then $\pi^{-1}(S')$ has non-void interior in G and contains S so by assumption $\pi^{-1}(S') = G$. This shows that there is no proper semigroup with non-void interior of G/N containing $\pi(S)$. By [6, Prop. 2.1] $\pi(S)$ is a subgroup of $\pi(\Gamma)$ and since S is generating we conclude that $\pi(S) = \pi(\Gamma)$. □

This lemma implies that a semigroup S which satisfies the assumptions is a group if and only if $S \cap N$ is a group. This follows from the easily proved fact that a subsemigroup is a group provided its intersection and its quotient modulo a normal subgroup are groups (c.f. [6, Lemma 3.4]). For later reference we state this fact.

Corollary 5.2. *Keep the notations and assumptions of the previous lemma. Then S is a group if and only if $S \cap N$ is a group.*

As another consequence of the previous lemma, we have the following fact which will be needed later.

Corollary 5.3. *With the notations and assumptions as in Lemma 5.1, we have that $I \cap N \neq \emptyset$ for every right or left ideal I of S. In particular, the ideals* $\mathrm{intalg}(S)$ *and* $\mathrm{intalg}_{\mathbf{S}}(S)$ *meet N.*

Proof. Let I be a right ideal of S and take $x \in I$. By the lemma, S/N is a group so that there exists $y \in S$ such that $xy \in N$. Since I is a right ideal, it follows that $xy \in I \cap N$. For the left ideals the proof is the same. □

With these facts in mind, we proceed to look at the semigroup $S \cap N$. For this, it is assumed first that N is abelian. After a detailed analysis of this case is made it is indicated how to reduce the general case to it.

Since some of the facts to be proved hold for semigroups in finitely generated subgroups, we state then in this more general context for later reference. Also, in the following discussion there is no reason to restrict ourselves to the nil radical. So we consider instead, an abelian ideal $\mathbf{a}$ which contains the derived algebra $\mathbf{g}'$. Similarly, we let A stand for the connected subgroup whose Lie algebra is $\mathbf{a}$. A way of verifying that $S \cap A$ is a group is by showing that the cone it generates is a subspace (c.f. [6, Prop. 2.1]). So we put

$$W = cl\{x \in \mathbf{a} : \exists t > 0,\ \exp(tx) \in S\} .$$

We have

Lemma 5.4. *Let $\Gamma \subset G$ be a finitely generated subgroup. Assume that $\Gamma \cap A$ is not contained in any connected proper subgroup of A. Let $S \subset \Gamma$ be a generating semigroup such that S/A is a group. Then W is a closed convex and generating cone in* $\mathbf{a}$.

Proof. For $x, x' \in W$ there are sequences x_n, x'_n in $\mathbf{a}$ and rationals t_n and s_n such that $\exp(x_n)$ and $\exp(x'_n)$ are in S and $t_n x_n$ and $s_n x'_n$ converge to x and x' respectively. From this and the fact that S is a semigroup, it follows that $W + W \subset W$. Since it is evident that $\mathbb{R}^+ W \subset W$, we have that W is a closed convex cone. On the other hand, by Proposition 2.9 $S \cap A$ generates $\Gamma \cap A$ so by assumption it follows that W is not contained in any proper subspace of $\mathbf{a}$ so that it is generating. □

The key property of W is that it is invariant under the adjoint action of Γ. The proof of this fact given in the next proposition is similar to the corresponding result for semigroups with non-empty interior (c.f. [3, Thm. 9.1]) with the difference that we take the algebraic interior of the semigroup instead of its topological interior.

Proposition 5.5. *With A as above let $\Gamma \subset G$ be a finitely generated subgroup. Let $S \subset \Gamma$ be a generating semigroup such that S/A is a group. Then W is invariant under the adjoint action of Γ in* $\mathbf{a}$.

Proof. As in Corollary 5.2 we have that both $\mathrm{intalg}_{\mathbf{S}}(S)$ and $\mathrm{intalg}(S)$ meet A. Also, W is generated by $\mathrm{intalg}(S) \cap A$. In fact, $\exp(x + ny) \in \mathrm{intalg}(S)$ if $\exp x \in \mathrm{intalg}(S)$ and $\exp y \in S$ and the ray defined by $x + ny$ approaches the ray defined by y as $n \to \infty$. Therefore it is enough to show that $Ad(g)x \in W$ if $g \in \Gamma$ and $\exp x \in \mathrm{intalg}(S) \cap A$. Pick $x, y \in \mathbf{a}$ such that $\exp x \in \mathrm{intalg}(S)$ and $h = \exp y \in \mathrm{intalg}_{\mathbf{S}}(S)$. Since $\exp(-x)S$ generates Γ we can find $s_1, \ldots, s_k \in S$ such that

$$\exp(-y) = \exp(-x)s_1 \exp(-x)s_2 \cdots \exp(-x)s_k.$$

Let $\tilde{x} = x + Ad(s_1)x + \cdots + Ad(s_1 s_2 \cdots s_{k-1})x$ and $\tilde{s} = s_1 s_2 \cdots s_k$. Since $\mathbf{a}$ is an ideal we have that $\tilde{x} \in \mathbf{a}$, and the above expression gives

$$\exp(-y) = \exp(-\tilde{x})\tilde{s}.$$

Therefore $\tilde{s} = \exp(\tilde{x} - y) \in S \cap A$ so for all $n \in \mathbb{N}$, $\exp(\tilde{x} - y + nx) \in S \cap A$. Hence there exists $z_n \in W$ with $\exp(z_n) \in S$ such that

$$\tilde{x} + nx = y + z_n$$

for every integer n.

Let U be a symmetric generating subset of Γ such that $Uh \subset S$, then for $g \in U$, we have

$$\begin{aligned} \exp(Ad(g)(\tilde{x} + nx) + y) &= g \exp(\tilde{x} + nx) g^{-1} h \\ &= g \exp(y + z_n) g^{-1} h \\ &= g h \exp(z_n) g^{-1} h \\ &\in (Uh) S (Uh) \\ &\subset S \ . \end{aligned}$$

Therefore $\frac{1}{n}(Ad(g)(\tilde{x}+nx)+y) = Ad(g)x + \frac{1}{n}(y + Ad(g)\tilde{x}) \in W$ for all $n \in I\!N$. Passing to the limit as $n \to \infty$ we obtain that $Ad(g)x \in W$ for all $g \in U$. Since U generates Γ, it follows that W is Γ-invariant. □

Clearly, these results hold for semigroups in a lattice Γ which satisfy hypothesis **H**. The rest of this section is devoted to a discussion of the cones in $\mathbf{a}$ which are invariant under the adjoint action of a lattice Γ in G. The first comment we make, is that G acts on $\mathbf{a}$ as an abelian group. In fact, denote by ρ the adjoint action of G on $\mathbf{a}$. Since A is abelian it belongs to $\ker\rho$. Hence this representation factors through the quotient G/A defining a representation of G/A on $\mathbf{a}$, also denoted by ρ. Clearly, $\rho(G) = \rho(G/A)$ and this is abelian because A contains the derived group $[G,G]$. Therefore, $\rho(G)$ is abelian. In particular, the image of Γ under this representation is also abelian. Now, it is a classical fact that any abelian group of linear maps is a Frobenius-Perron group. This means that any pointed invariant cone W (i.e., a cone that does not contain positive dimensional subspaces) in the space of any finite-dimensional continuous representation of the group contains a ray which is also invariant under the group (c.f. [3, Def. 10.2]).

Taking now the dual representation ρ^* of G on $\mathbf{a}^*$, $\rho^*(g)(\lambda) = \lambda \circ \rho(g^{-1})$, $g \in G, \lambda \in \mathbf{a}^*$ instead of ρ we have that $\rho^*(\Gamma)$ is abelian and that the dual cone

$$W^* = \{\lambda \in \mathbf{a}^* : \lambda(X) \geq 0 \text{ for all } X \in W\}$$

is invariant under $\rho^*(\Gamma)$. Also, the fact that W is generating in $\mathbf{a}$ implies that W^* is a pointed cone in $\mathbf{a}^*$ if W is proper. Hence, there exists in W^* a ray which is left invariant under $\rho^*(g)$ for all $g \in \Gamma$. This means that there is $\lambda \in \mathbf{a}^*$ which is a common eigenvector of all $\rho^*(g), g \in \Gamma$. However, it is readily checked that λ is an eigenvector of $\rho^*(g)$ if and only if the hyperplane $\ker\lambda \subset \mathbf{a}$ is invariant under $\rho(g)$. Moreover, the fact that $\lambda \in W^*$ implies that W is entirely contained in one of the half-spaces defined by $\ker\lambda$ in case W is different from $\mathbf{a}$. We have got thus the following fact.

Proposition 5.6. *Keep the notations and assumptions as above and suppose moreover that W is a proper cone in $\mathbf{a}$. Then there exists a hyperplane, say $V \subset \mathbf{a}$ which is invariant under the adjoint action of Γ and such that W is contained in one of the half-spaces bounded by V.*

In this statement both the cone and the hyperplane V, although invariant under Γ, are not in general invariant under G. The example given in the introduction is characteristic of this situation. This suggests the introduction of the following concept.

Definition 5.7. Let $\Gamma \subset G$ be a subgroup and $\mathbf{a} \subset \mathbf{g}$ an abelian ideal containing the derived algebra $\mathbf{g}'$. We say that Γ is in *general position* with respect to $\mathbf{a}$ if every Γ-invariant half-space of $\mathbf{a}$ is also G-invariant.

Remark. The requirement, in this definition, for a subgroup to be in general position is weaker than the requirement that the Γ-invariant hyperplanes in $\mathbf{a}$ are ideals. In fact, it may happen that Γ leaves invariant a hyperplane V but interchanges the half-spaces bounded by V. □

In what follows, we shall search for conditions under which a subgroup is in general position. Such conditions will be given in terms of the weights of the adjoint representation in $\mathbf{a}$. Before discussing this, we will need the following elementary lemma from linear algebra.

Lemma 5.8. *Let A be a linear operator of a finite-dimensional real vector space U and denote by $\lambda_1, \ldots, \lambda_s$ its real eigenvalues and by $a_1 \pm ib_1, \ldots, a_k \pm ib_k$ the complex ones. Take a real t such that $t \neq \frac{l\pi}{b_j}$ for any integer l and $j = 1, \ldots, k$. Then $u \in U$ is an eigenvector of $\exp(tA)$ if and only if it is an eigenvector of A and hence of $\exp(sA)$ for all $s \in \mathbb{R}$. If $u \in U$ is an eigenvector of $\exp A$ associated to a real eigenvalue of A then it is also an eigenvector of $\exp(sA)$ for all $s \in \mathbb{R}$.*

Proof. ¿From the (real) Jordan canonical form of A it is easy to see that if t is as in the statement then the eigenvectors of $\exp(tA)$ are those for A which are associated to its real eigenvalues and this implies the lemma. □

We consider now the adjoint action of $\mathbf{g}$ on $\mathbf{a}$. We obtain a representation ρ of Lie algebra $\mathbf{g}/\mathbf{a}$ on the vector space $\mathbf{a}$. Its complexification decomposes into weight spaces. Realifying it back, we get a decomposition of $\mathbf{a}$ itself into weight spaces. Let

$$a_1 + ib_1, \ldots, a_k + ib_k, \lambda_1, \ldots, \lambda_s$$

be the weights of the complexification of ρ restricted to $\mathbf{a}$. Here $\lambda_j, j = 1, \ldots, s$ are the real weights and $a_j + ib_j$ are the complex ones with a_j, b_j real linear functionals in $\mathbf{g}/\mathbf{a}$. As $\mathbf{g}/\mathbf{a}$ is abelian, Engel's Theorem ensures that inside a weight space associated to a real weight ρ may be represented by upper triangular matrices while in a weight space associated to a complex weight $a_j + ib_j$ we have

$$\rho(X) = \begin{pmatrix} a_j(X) & -b_j(X) & & & * \\ b_j(X) & a_j(X) & & & \\ & & \ddots & & \\ & & & a_j(X) & -b_j(X) \\ & & & b_j(X) & a_j(X) \end{pmatrix}$$

For the dual representation ρ^* of $\mathbf{g}/\mathbf{a}$ on $\mathbf{a}^*$ we get a similar picture and up to a sign, the weights of the dual representation coincide with the weights of ρ. By identifying $\mathbf{g}/\mathbf{a}$ with G/A we have that the imaginary parts of the weights define the closed subgroups

$$L_j = \{X \in \mathbf{g}/\mathbf{a} : b_j(X) = 2n\pi, n \in \mathbb{Z}\} .$$

With these facts in mind we have

Lemma 5.9. *Let Γ be a subgroup of G and assume that Γ/A is a lattice which is not contained in any of the subgroups L_j. Pick $\mu \in \mathbf{a}^*$ and set*

$$H_\mu = \{g \in G/A : \mu \textit{ is a positive eigenvector of } \rho^*(g)\}$$

(by a positive eigenvector we understand an eigenvector associated to a positive eigenvalue). Suppose that $\Gamma/A \subset H_\mu$. Then $H_\mu = G/A$.

Proof. We have that H_μ is closed because it is an isotropy of the spherical action of G/A on the set of rays of $\mathbf{a}^*$. Therefore it is a Lie subgroup. In order to see that it coincides with G/A, we note first that the above lemma applied to $\rho^*(\exp X)$ inside the weight space associated to the complex weight $a_j + ib_j$ shows that μ does not belong to this weight space if $X \notin L_j$. Therefore, μ does not belong to any complex weight space. This implies that μ is an eigenvector of $\rho^*(g)$ associated to a real eigenvalue, for all $g \in \Gamma/A$. Taking $X \in \mathbf{g}/\mathbf{a}$ such that $g = \exp X$, $g \in \Gamma/A$ we have by the previous lemma again that $\exp tX \in H_\mu$ for all $t \in I\!\!R$. But since Γ/A is a lattice, this implies that $\mathbf{g}/\mathbf{a}$ is contained in the Lie algebra of H_μ showing that $H_\mu = G/A$. □

¿From this lemma we get the following necessary and sufficient condition in order that a subgroup is in general position.

Proposition 5.10. *Let Γ be a subgroup of G and assume that Γ/A is a lattice in G/A. Then Γ is in general position if and only if $\Gamma/A \not\subset L_j$ for all j.*

Proof. Assume that $\Gamma/A \not\subset L_j$ for all j, and let $\mu \in \mathbf{a}^*$ be such that

$$T = \{X \in \mathbf{a} : \mu(X) \geq 0\}$$

is a Γ-invariant half-space. We note that T is invariant under $\rho(g)$, $g \in G$ if and only if μ is a positive eigenvector of $\rho^*(g)$. Therefore the invariance of T under Γ and above lemma imply that T is G-invariant showing that Γ is in general position.

Reciprocally, suppose that $\Gamma/A \subset L_j$ for some j. Then there exists μ in the weight space of $\mathbf{a}^*$ associated to the weight $a_j + ib_j$, which is a positive eigenvector of $\rho^*(g)$ for all $g \in \Gamma$. For this μ the half-space

$$T = \{X \in \mathbf{a} : \mu(X) \geq 0\}$$

is Γ-invariant but is not X-invariant for $X \in G/A$ and $X \notin L_j$. □

6. Semigroups in the affine group

In our analysis of the semigroups in the lattices of solvable groups we shall lead to consider semigroups in the two-dimensional non-abelian Lie group. The purpose of this section is to present the relevant results about these semigroups. As is well-known, the only non-abelian connected two-dimensional Lie group is the component of the identity of the affine group of the real line. We denote this group by Aff^+ and its Lie algebra, which is the two-dimensional solvable, non-abelian Lie algebra by **aff**. Explicitly, Aff^+ is $I\!\!R^+ \times I\!\!R$ with the product given by composition of affine maps, that is,

$$(p,x)(q,y) = (pq, py + x) \qquad p,q > 0; x,y \in I\!\!R .$$

On the other hand, **aff** is $I\!\!R^2$ with the bracket given by

$$[(a,b),(a',b')] = (0, ab' - a'b) .$$

We have that the exponential mapping is

$$\exp t(a,b) = (e^{ta}, \frac{b}{a}(e^{ta} - 1))$$

if $a \neq 0$ and $\exp(0,t) = (1,t)$. Since $\exp t(a,b)$, $a \neq 0$ stays in the line with equation

$$v = \frac{b}{a}u - \frac{b}{a}$$

which passes through $(1,0)$ and has slope b/a, it follows that the one parameter groups of Aff^+ are the lines through the identity $(1,0)$. Of course, these lines are determined by their slopes, and if a non vertical line has slope m then its equation is $v = mu - m$. In what follows we denote the vertical line by N. This is the only non trivial normal subgroup of Aff^+ and is its nil radical.

The product structure in Aff^+ is clarified by the geometry of the cosets of the one-parameter subgroups: the right or left cosets of N are the vertical lines in Aff^+. As to the other subgroups, we have that

$$(u, mu - m)(p, x) = (pu, (x + m)u - m)$$

so that the right coset of the subgroup given by the line with slope m which contains (p,x) is the line with slope $(x+m)/p$. This slope is smaller than m in case $x < mp - m$, that is, in case (p,x) is below the one-parameter group. In a symmetric way, the right cosets above the subgroup are straight lines with slope bigger than the slope of the group so that the group and any of its right cosets become apart when the first coordinate goes to $+\infty$.

On the other hand, the left cosets are given by

$$(p, x)(u, mu - m) = (pu, pmu - pm)$$

so that they are parallel to the subgroup.

These simple computations also show that a left translation maps a non vertical line into a line parallel to it while a right translation changes the slope of the lines.

We consider now semigroups in Aff^+. It follows from [3, Prop. 6.5] that the only maximal semigroups with non empty interior are the half-planes in $\mathbb{R}^+ \times \mathbb{R}$ bounded by an one-parameter group. In order to consider other classes of semigroups, it will be needed the following lemma whose proof is similar to that of [3, Prop 6.5].

Lemma 6.1. *Let $S \subset Aff^+$ and assume that*

a) *S meets the two half-planes bounded by N*

b) *clS does not intercepts the lower half-line of N, that is, $x \geq 0$ if $(1,x) \in S$.*

Put

$$m^+(S) = \inf\{\frac{x}{p-1} : (p,x) \in S \text{ and } p > 1\}$$

$$m^-(S) = \sup\{\frac{x}{p-1} : (p,x) \in S \text{ and } p < 1\} .$$

Then $m^-(S) \leq m^+(S)$. In particular, these quantities are well defined.

Proof. Take $(p, mp-m)$ and $(q, nq-n)$ in S with $0<p<1<q$. It is enough to show that $m \leq n$. Let j, k be positive integers. Then

$$(p, mp-m)^j (q, nq-n)^k = (p^j q^k, n(p^j q^k - 1) + (n-m)(1-p^j))$$

belongs to S. Now, by [3, Lemma 6.4] it is possible to choose j, k such that $|p^j q^k - 1| < \epsilon$ and $p^j < \epsilon$, for any prescribed $\epsilon > 0$. This shows that $(1, n-m) \in clS$. So assumption (b) ensures that $m \leq n$, showing the lemma. □

Geometrically, $m^+(S)$ is the highest slope of the line passing through the identity whose upper half-plane contains the part of S which is on the right side of N. Since $m^-(S)$ interprets in a symmetric way, the above lemma implies that a semigroup which satisfies its assumptions is contained in a semigroup with non-empty interior of Aff^+.

We can prove now the main result of this section which will be crucial in the analysis of the semigroups in general solvable groups.

Proposition 6.2. *Let $S \subset Aff^+$ be a semigroup satisfying*

- a) *S/N is a group, and*
- b) *S is not contained in any semigroup with non-empty interior.*

Then S meets both half-lines of N, that is, there are $x, y > 0$ such that $(1, x)$ and $(1, -y)$ belong to S.

Proof. The group Aff^+/N is isomorphic to the additive group of reals. Because of this, we shall describe the group S/N by

$$S/N = \{\alpha \in I\!\!R : \exists x \in I\!\!R \text{ with } (e^\alpha, x) \in S\} .$$

Take $\alpha \in S/N$ and let Aff_α be the subgroup of Aff^+ which projects onto the subgroup of $I\!\!R$ generated by α:

$$Aff_\alpha = \{(e^{n\alpha}, x) : x \in I\!\!R, n \in \boldsymbol{Z}\} .$$

Also, let $S_\alpha = S \cap Aff_\alpha$ be the subsemigroup of S which projects onto the same group.

We will show that S_α intercepts the lower half-line of N for some α. Suppose to the contrary. Then since S_α projects onto a discrete subgroup of $I\!\!R$ it satisfies the assumptions of the previous lemma. Therefore $m^\pm(S_\alpha)$ are well defined. We put $m(\alpha) = m^+(S_\alpha)$ if $\alpha > 0$ and $m(\alpha) = m^-(S_a)$ if $\alpha < 0$. We have $m(\alpha) \leq m(-\alpha)$ if $\alpha > 0$. Now, putting

$$m^+ = \inf_{\alpha > 0} m(\alpha) \qquad m^- = \sup_{\alpha > 0} m(-\alpha) ,$$

we have that $m^- > m^+$ because otherwise S would be contained in a semigroup with non-empty interior contradicting assumption (b). Hence there are $\alpha, \beta > 0$

with $m(-\beta) > m(\alpha)$. This means that there are $n_1 > 0$ and $x \in \mathbb{R}$ such that $(e^{-n_1\beta}, x)$ belongs to S_β and is below the line with slope $m(\alpha)$, that is,

$$x < m(\alpha)e^{-n_1\beta} - m(\alpha) \ .$$

This inequality is equivalent to

$$e^{n_1\beta}(m(\alpha) + x) < m(\alpha) \ .$$

¿From this inequality and the definition of $m(\alpha)$, we get $(e^{n_2\alpha}, y) \in S_\alpha$ with $n_2 > 0$ such that the slope $m = y/(e^{n_2\alpha} - 1)$ of the one-parameter group containing it satisfies

$$e^{n_1\beta}(m + x) < m(\alpha) \ .$$

The characterization given above of the right cosets shows that for any integer n, the product

$$(e^{n_2\alpha}, y)^n (e^{-n_1\beta}, x)$$

stays in a straight line with slope $e^{n_1\beta}(m + x)$ which is therefore smaller than $m(\alpha)$. Now, since S/N is assumed to be a group, there exists $z \in \mathbb{R}$ such that $(e^{n_1\beta}, z) \in S_\beta$. Also, the fact that a left translation does not affect the slopes of the lines, implies that for any positive integer n the product

$$(e^{n_1\beta}, z)(e^{n_2\alpha}, y)^n (e^{-n_1\beta}, x)$$

stays in a straight line with slope strictly smaller than $m(\alpha)$. This product belongs to S and since it is of the form

$$(e^{nn_2\alpha}, *)$$

it belongs to S_α. Choosing n sufficiently large we then get an element in S_α which is on the right side of N and below the one-parameter group with slope $m(\alpha)$ and this is a contradiction.

Therefore, S meets the lower half-line of N. As to the upper one, we have that $\phi(S)$ satisfies the assumptions of the proposition where ϕ is the automorphism $\phi(p, x) = (p, -x)$. Hence $\phi(S)$ meets the lower half-line, that is, S meets the upper one. □

7. Groups with abelian nil radical

In this section we complete the analysis of the semigroups in lattices in solvable groups with abelian nil radical. We take such a group G and a lattice $\Gamma \subset G$ and assume throughout this section that Γ is in general position with respect to the nil radical $\mathbf{n}$ of $\mathbf{g}$. Also, we take a generating semigroup $S \subset \Gamma$ which is not contained in any semigroup with non-void interior of G. The objective is to show that S is a group.

Let $W \subset \mathbf{n}$ denote the cone generated by $S \cap N$. We know that W is Γ-invariant by Proposition 5.5. If S is not a group, then W is proper. Now we obtain from the assumption that Γ is in general position an ideal $\mathbf{h} \subset \mathbf{n}$ of

$\mathbf{g}$ which has codimension 1 in $\mathbf{n}$ such that W is entirely contained in one of half-spaces bounded by $\mathbf{h}$.

Let H be the connected normal subgroup whose Lie algebra is $\mathbf{h}$ and consider the canonical projection

$$\theta : G \longrightarrow G/H \ .$$

We have that $\theta(S)$ is not contained in any semigroup with interior points in G/H.

Now, there are the following exclusive possibilities

A) G/H is abelian.

B) G/H is nilpotent buto not abelian.

C) G/H is solvable but not nilpotent.

And we have,

Lemma 7.1. *Case* (A) *is impossible.*

Proof. Since G/H is abelian and G simply connected, $G/H \approx I\!\!R^k$ for some k. Also, since H has codimension one, N/H becomes a one-dimensional subspace of G/H. Now, $\theta(S)$ is not contained in any semigroup with non-empty interior of G/H so that the closed convex cone it generates is G/H. Hence for any ray r of G/H there are points of $\theta(S)$ arbitrarily close to r. Returning back to G, this means that there are points of S arbitrarily close to each side of H in N. However, Γ/N is discrete. Hence there are points of S on both sides of H in N which contradicts the fact that W is proper. □

Lemma 7.2. *Case* (B) *is also impossible.*

Proof. We have that the derived algebra $(\mathbf{g}/\mathbf{h})'$ is contained in $\mathbf{n}/\mathbf{h}$. However, as $\mathbf{n}/\mathbf{h}$ is one-dimensional and $\mathbf{g}/\mathbf{h}$ is not abelian the equality $(\mathbf{g}/\mathbf{h})' = \mathbf{n}/\mathbf{h}$ holds. On the other hand, $(\mathbf{g}/\mathbf{h})^2 = [\mathbf{g}/\mathbf{h}, (\mathbf{g}/\mathbf{h})'] = 0$ for otherwise $\mathbf{g}/\mathbf{h}$ would not be nilpotent. Moreover, $(S/H)/(N/H) \approx S/N$ is a lattice in the abelian group G/N. Therefore, $\theta(S)$ satisfies the assumptions of Proposition 4.5 and hence $\theta(S) \cap (N/H)$ is not contained in a half-line of N/H. This means that $S \cap N$ meets both sides of H contradicting the fact that W is a proper cone. □

Case (C) is somewhat more involved. Its analysis requires the following facts about Lie algebras with one-dimensional derived algebras.

Lemma 7.3. *Let* $\mathbf{p}$ *be a Lie algebra such that* $\dim \mathbf{p}' = 1$ *and suppose that* $\mathbf{p}$ *is not nilpotent. Then there exists a unique abelian ideal* $\mathbf{k} \subset \mathbf{p}$ *of codimension two such that* $\mathbf{p}/\mathbf{k}$ *is not abelian. Moreover,* $\mathbf{k}$ *is contained in the nil radical* $\mathbf{n}(\mathbf{p})$ *of* $\mathbf{p}$ *and* $\mathbf{n}(\mathbf{p})$ *is abelian and of codimension one.*

Proof. Denote by ρ the adjoint representation of $\mathbf{p}$ in $\mathbf{p}'$. Since $\dim \mathbf{p}' = 1$, we have that $\ker \rho$ is of codimension zero or one. It happens that $\ker \rho$ is not of codimension zero because otherwise $[X, \mathbf{p}'] = 0$ for all $X \in \mathbf{p}$ which implies that $\mathbf{p}$ is nilpotent. Therefore $\ker \rho$ is an ideal of codimension one in $\mathbf{p}$ which contains $\mathbf{p}'$. Take a basis

$$\{X, Y_1, \ldots, Y_k, Z\}$$

of $\mathbf{p}$ such that $X \notin \ker \rho$, $Z \in \mathbf{p}'$ and $\{Y_1, \ldots, Y_k, Z\}$ is a basis of $\ker \rho$. As $[X, Z] \neq 0$, we can choose X such that $[X, Z] = Z$. The structural constants for this basis is given as follows

$$[X, Z] = Z \quad [Y_i, Z] = 0 \quad [X, Y_i] = a_i Z \quad [Y_i, Y_j] = b_{ij} Z \ .$$

We have that

$$[[X, Y_i], Y_j] + [Y_i, [X, Y_j]] = a_i [Z, Y_j] + a_j [Y_i, Z] = 0$$

so the Jacobi identity implies that $b_{ij} = 0$ and $\ker \rho$ is an abelian ideal of codimension one. Consider now the restriction of $ad(X)$ to $\ker \rho$. Its kernel $\mathbf{k}$ is abelian and of codimension two in $\mathbf{p}$. Moreover it is an ideal because $[X, \mathbf{k}] = 0$ and $\ker \rho$ is abelian. Also, $\mathbf{p}/\mathbf{k}$ is not abelian because $[X, Z] \neq 0$. Hence $\mathbf{k}$ is an ideal as in the statement. And it is the only one because $[X, Y] \neq 0$ or $[Z, Y] \neq 0$ if $Y \notin \mathbf{k}$. Finally, the last statement follows from the fact that the nil radical of $\mathbf{p}$ is exactly $\ker \rho$. □

We note that in the proof above it is shown that $\mathbf{p}$ is isomorphic to $\mathbf{aff} \oplus \mathbb{R}^k$ because the ideal $\mathbf{k}$ turns out to be an abelian direct summand. Therefore we obtain another fact about a Lie algebra with one-dimensional derived algebra.

Corollary 7.4. *Let* $\mathbf{p}$ *be as in the above lemma and let* $\overline{\mathbf{h}} \subset \mathbf{n}(\mathbf{p})$ *be an ideal of* $\mathbf{p}$ *of codimension one in* $\mathbf{n}(\mathbf{p})$. *Then either* $\mathbf{p}' \subset \overline{\mathbf{h}}$ *or* $\overline{\mathbf{h}}$ *is the center of* $\mathbf{p}$, *which is the ideal* $\mathbf{k}$ *of the above lema.*

Now we consider case (C). By the same arguments as in the proof Lemma 7.2 we deduce that $\dim(\mathbf{g}/\mathbf{h})' = 1$. Denote by $\mathbf{m}$ the nil radical of $\mathbf{g}/\mathbf{h}$ and let $\pi : \mathbf{g}/\mathbf{h} \to (\mathbf{g}/\mathbf{h})/(\mathbf{g}/\mathbf{h})'$ be the canonical projection. We have that $\mathbf{m}$ is a codimension one abelian ideal so that $\pi(\mathbf{m})$ is a codimension one subspace of the abelian algebra $(\mathbf{g}/\mathbf{h})/(\mathbf{g}/\mathbf{h})'$. Identifying this algebra with the group $(G/H)/[G/H, G/H]$ we get that the projection of Γ defines a lattice, denoted by $\tilde{\Gamma}$ in $(\mathbf{g}/\mathbf{h})/(\mathbf{g}/\mathbf{h})'$. With these notations, let V be the subspace spanned by $\pi(\mathbf{m}) \cap \tilde{\Gamma}$ and put $\mathbf{q} = \pi^{-1}(V)$.

Now, let Q be the connected subgroup of G/H associated to $\mathbf{q}$. By construction, this is the smallest connected subgroup of the nil radical M of G/H which contains $\theta(S) \cap M$. This being so, let $\overline{W}$ be the cone in $\mathbf{q}$ generated by $\theta(S) \cap Q$ as in Proposition 5.5. We have two possibilities

1. $\overline{W} = \mathbf{q}$. Then as in case (A), for any ray r in $\mathbf{q}$ there exist elements of $\theta(S) \cap Q$ arbitrarily close to r. Returning back to G this means that we can approximate points on both sides of H in N by elements of S. But since Γ/N is discrete this implies that $S \cap N$ meets both sides of H contradicting the fact that W is proper.

2. $\overline{W}$ is a proper cone of $\mathbf{q}$. Then Propositions 5.5 and 5.10 ensure that $\overline{W}$ is contained in a half-space of $\mathbf{q}$ bounded by an ideal $\overline{\mathbf{h}}$ of $\mathbf{g}/\mathbf{h}$. Now, take $X \notin \mathbf{m}$ and form the subalgebra $\mathbf{p}$ spanned by X and $\mathbf{q}$. We have that $\mathbf{q}$ is the nil radical of $\mathbf{p}$ and $\overline{\mathbf{h}}$ is an ideal of $\mathbf{p}$ so by the last lemma there are two possibilities for $\overline{\mathbf{h}}$. We have
   (a) $(\mathbf{g}/\mathbf{h})' = \mathbf{p}' \subset \overline{\mathbf{h}}$. This possibility is ruled out by the fact that modulo the derived group of G/H, $\theta(S)$ is a group because S modulo the nil radical of G is a group. Therefore there are points of $\overline{W}$ on both sides of any hyperplane in $\mathbf{q}$ which contains $(\mathbf{g}/\mathbf{h})'$.
   (b) $\overline{\mathbf{h}} = \mathbf{k}$. Let $\mathbf{k}_1 \subset \mathbf{m}$ be the ideal of $\mathbf{g}/\mathbf{h}$ ensured by the lemma. We have that $\mathbf{k}_1 \cap \mathbf{q} = \mathbf{k}$. Let $K_1 \subset G/H$ be the connected subgroup associated to $\mathbf{k}_1$. As $\mathbf{k}_1$ is of codimension two, and $(\mathbf{g}/\mathbf{h})/\mathbf{k}_1$ is not abelian, we have that $(G/H)/K_1$ is isomorphic to Aff^+. Hence, if we put $S' = \theta(S)/K_1$ then S' is a semigroup in Aff^+ which is not contained in any semigroup with non-void interior in Aff^+. Also, modulo the derived group of Aff^+, S' is a group because modulo $[G/H, G/H]$ the semigroup $\theta(S)$ is a group. Therefore S' satisfies the conditions of Proposition 6.2 showing that it meets both half-lines of the derived group of Aff^+. But this means that $\theta(S)$ is not contained in a half-space of G/H containing K_1, which in turn implies, by the construction of $\mathbf{q}$, that $\theta(S) \cap Q$ is not contained in a half-space of Q bounded by K. Hence this possibility for $\overline{\mathbf{h}}$ is also ruled out.

Theses cases show that (C) also leads to a contradiction covering then all the possibilities and showing the main result of this section which we state now.

Theorem 7.5. *Let G be a simply connected solvable Lie group and denote by N the connected subgroup associated to the nil radical $\mathbf{n}$ of its Lie algebra. Let $\Gamma \subset G$ be a lattice and assume that Γ is in general position with respect to $\mathbf{n}$. Let $S \subset \Gamma$ be a generating semigroup. Then $S = \Gamma$ if it is not contained in any semigroup with non-empty interior of G.*

8. The general case

We will show now how to reduce the analysis of the semigroups in general solvable groups to those which have abelian nil radical. This will require the following lemma on Lie algebras.

Lemma 8.1. *Let $\mathbf{g}$ be a solvable Lie algebra with nil radical $\mathbf{n}$. Let $\mathbf{s} \subset \mathbf{n}$ be an abelian ideal which contains the derived algebra $\mathbf{g}'$ and denote by ρ the representation of the abelian Lie algebra $\mathbf{g}/\mathbf{s}$ on $\mathbf{s}$ induced by the adjoint representation of $\mathbf{g}$ on $\mathbf{s}$. Let $\mathbf{n}'$ be the derived algebra of $\mathbf{n}$ and denote by ρ' the representation of $\mathbf{g}/\mathbf{s}$ on $\mathbf{s}/\mathbf{n}'$ induced by ρ. Then the non-zero weights of (the complexifications of) ρ and ρ' coincide.*

Proof. By complexifying the representations we can work in the field of complex numbers. Let $\{\lambda_1, \ldots, \lambda_p\}$ be the weights of ρ and denote by $\mathbf{s}_{\lambda_j}$ the corresponding weight spaces. It is enough to show that $\mathbf{s}_{\lambda j}$ is not contained in $\mathbf{n}'$ if λ_j is a non-zero weight. For this, fix j such that $\lambda_j \neq 0$ and use Engel's Theorem to get a basis $\{Y_1, \ldots Y_s\}$ of $\mathbf{s}_{\lambda_j}$ such that with respect to it the restriction of $\rho(X)$ to $\mathbf{s}_{\lambda_j}$ is written as

$$\begin{pmatrix} \lambda_j(X) & & * \\ & \ddots & \\ & & \lambda_j(X) \end{pmatrix}$$

We claim that $\mathbf{n}' \cap \mathbf{s}_{\lambda_j}$ is contained in the subspace spanned by $\{Y_1, \ldots, Y_{s-1}\}$. Denote by V this subspace and suppose to the contrary that there exists $Y \in \mathbf{n}' \cap \mathbf{s}_{\lambda_j}$ such that $Y \neq 0 \bmod V$. We can assume that $Y = Y_s \bmod V$. Take $X \in \mathbf{g}$ such that its projection $\overline{X} \in \mathbf{g}/\mathbf{s}$ satisfies $\lambda_j(\overline{X}) \neq 0$. Then modulo V, $[X, Y]$ is equal to $\lambda(X)Y_s$. On the other hand, Y is a sum of the type

$$Y = \sum_k [Z_k, W_k]$$

with $Z_j, W_j \in \mathbf{n}$. By taking brackets with X and applying the Jacobi identity, we get

$$[X, Y] = \sum([[X, Z_k], W_k] + [Z_k, [X, W_k]])$$

which shows that $[X, Y]$ belongs to $[\mathbf{n}, \mathbf{g}']$ which is contained in $[\mathbf{n}, \mathbf{s}]$. However, the fact that the adjoint of an element of $\mathbf{n}$ is nilpotent together with the decomposition in weight spaces provided by Engel's Theorem, show that $[\mathbf{n}, \mathbf{s}]$ has no components in the direction of Y_s. This contradicts the fact that $[X, Y] = Y_s \bmod V$ proving the lemma. □

Now, let $\mathbf{g}$ be a solvable Lie algebra with nil radical $\mathbf{n}$ and put $\mathbf{g}_1 = \mathbf{g}/\mathbf{n}'$ and $\mathbf{s}_1 = \mathbf{n}/\mathbf{n}'$. We have that $\mathbf{s}_1$ is an abelian ideal of $\mathbf{g}_1$ which contains the derived algebra $\mathbf{g}_1'$. Also, as $\mathbf{s}_1$ is abelian, the adjoint representation of $\mathbf{g}_1$ in $\mathbf{s}_1$ factors through the abelian algebra $\mathbf{g}_1/\mathbf{s}_1 \approx \mathbf{g}/\mathbf{n}$. Denote by ρ_1 this representation and let $\lambda_1, \ldots, \lambda_r$ be its weights. Then the nil radical $\mathbf{n}_1$ of $\mathbf{g}_1$ is given by

$$\mathbf{n}_1 = \bigcap_{1 \leq i \leq s} \ker \lambda_i \ ,$$

and $\mathbf{s}_1 \subset \mathbf{n}_1$. Also, from the fact that $\mathbf{s}_1$ is abelian and $\mathbf{g}_1' \subset \mathbf{s}_1$ we have that the representation induced by ρ_1 on $\mathbf{s}_2 = \mathbf{s}_1/\mathbf{n}_1'$ has the same non-zero weights as ρ_1. Of course, the representation on $\mathbf{s}_2$ coincides with the adjoint representation of $\mathbf{g}_2 = \mathbf{g}_1/\mathbf{n}_1'$ factored through $\mathbf{g}_2/\mathbf{s}_2 \approx \mathbf{g}/\mathbf{n}$. By continuing this process we get algebras $\mathbf{g}_i$ with nil radical $\mathbf{n}_i$ and abelian ideals $\mathbf{s}_i$ with $\mathbf{g}_i/\mathbf{s}_i \approx \mathbf{g}/\mathbf{n}$ such that the non-zero weights of the representation ρ_i of $\mathbf{g}_i/\mathbf{s}_i$ on $\mathbf{s}_{i+1} = \mathbf{s}_i/\mathbf{n}_i'$ induced by the adjoint representation of $\mathbf{g}_i$ on $\mathbf{n}_i$ coincide with the non-zero weights of the representation ρ induced by the adjoint representation of $\mathbf{g}$ on $\mathbf{n}/\mathbf{n}'$. By finite dimensionality, we have for some k, that $\mathbf{s}_k = \mathbf{n}_k$ which implies that $\mathbf{n}_k$ is abelian. Since $\mathbf{g}_i$ is obtained by successive quotients starting in $\mathbf{g}$ we obtain ideals $\mathbf{j}_i$ in $\mathbf{g}$ such that $\mathbf{g}_i \approx \mathbf{g}/\mathbf{j}_i$. For these ideals we have that $\mathbf{s}_i = \mathbf{n}/\mathbf{j}_i$ and that

$$\mathbf{n}' \subset \mathbf{j}_1 \subset \cdots \subset \mathbf{j}_k \subset \mathbf{g}' \subset \mathbf{n}.$$

This is because in each step we take the quotient by the derived algebra of the nil radical. We have got thus the

Lemma 8.2. *Let* $\mathbf{g}$ *be a solvable Lie algebra with nil radical* $\mathbf{n}$*. Then there exists an ideal* $\mathbf{j}$ *of* $\mathbf{g}$ *which is contained in* $\mathbf{g}'$ *and hence in* $\mathbf{n}$ *such that the nil radical* $\tilde{\mathbf{n}}$ *of* $\tilde{\mathbf{g}} = \mathbf{g}/\mathbf{j}$ *is abelian. For this ideal we have further that* $\mathbf{g}/\mathbf{j} \approx \mathbf{g}_k/\mathbf{n}_k$ *where* $\mathbf{g}_i$ *is defined inductively by* $\mathbf{g}_0 = \mathbf{g}$ *and* $\mathbf{g}_i = \mathbf{g}_{i-1}/\mathbf{n}'_{i-1}$ *where* $\mathbf{n}_i$ *stands for the nil radical of* $\mathbf{g}_i$ *and* $\mathbf{n}_k$ *is abelian.*

Let ρ_1 *[respectively* ρ_2*] be the representation of* $\mathbf{g}/\mathbf{n} \approx \tilde{\mathbf{g}}/(\mathbf{n}/\mathbf{j})$ *on* $\mathbf{n}/\mathbf{n}'$ *[respectively* $\tilde{\mathbf{n}}$*] induced by the adjoint representation of* $\mathbf{g}$ *on* $\mathbf{n}$ *[respectively* $\tilde{\mathbf{g}}$ *on* $\tilde{\mathbf{n}}$*]. Then the non-zero weights of* ρ_1 *and* ρ_2 *coincide.*

With this lemma it is possible to extend Theorem 7.5 to general solvable groups. First we note that if $\Gamma \subset G$ is a lattice in the simply connected solvable Lie group G then Γ/J is a lattice in G/J if J is the normal connected subgroup of G whose Lie algebra is the ideal $\mathbf{j}$ of the above lemma. This is because $\mathbf{g}/\mathbf{j}$ was constructed by successive quotients by the derived algebras of the nil radical. As is well-known, the group $\Gamma/[N,N]$ is a lattice in $G/[N,N]$ if Γ is a lattice in G.

Theorem 8.3. *Let* G *be a simply connected solvable Lie group with Lie algebra* $\mathbf{g}$*. Denote by* $\mathbf{n}$ *the nil radical of* $\mathbf{g}$ *and by* N *the associated connected group. Let* $\Gamma \in G$ *be a lattice and assume that* $\Gamma/[N,N]$ *is in general position with respect to* $\mathbf{n}/[\mathbf{n},\mathbf{n}]$*. Let* $S \subset \Gamma$ *be a generating semigroup. Then* $S = \Gamma$ *if* S *is not contained in any semigroup with non-empty interior of* G*.*

Proof. Let $\mathbf{j}$ be the ideal of $\mathbf{g}$ given by the above lemma and J the corresponding normal subgroup. We have that S/J is not contained in any semigroup with non-void interior of G/J and that Γ/J is in general position in G/J because the non-zero weights of the adjoint representation of G/N on $\mathbf{n}/\mathbf{n}'$ and of G/J on the nil radical $\tilde{\mathbf{n}}$ of $\mathbf{g}/\mathbf{j}$ coincide. Since $\tilde{\mathbf{n}}$ is abelian we can apply Theorem 7.5 to deduce that $S/J = \Gamma/J$. With this fact in mind, we can climb up the successive quotients and get that $S = \Gamma$. In fact, let J_i be the normal subgroup associated to the ideal $\mathbf{j}_i$. We have the sequence of quotients

$$G \to G/J_1 \to \cdots \to G/J_{i-1} \to G/J$$

where each quotient is obtained from the preceding one by division of the derived group of the nil radical. Now, the fact that S/J is a group implies that $(S \cap N)/J_{i-1}$ is a semigroup in a nilpotent group which turns out to be a group when projected on the quotient by the derived group $[N/J_{i-1}, N/J_{i-1}]$. Therefore Corollary 4.4 ensures that $(S \cap N)/J_{i-1}$ is a group which shows that $S/J_{i-1} = \Gamma/J_{i-1}$. This argument can be reproduced for any of the quotients getting that S is a group so that $S = \Gamma$. □

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