Séminaire de Théorie des Nombres,
Bordeaux **6** *(1994), 81–94*

On a Theorem of Legendre in the theory of continued fractions

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1. Introduction

Let r be a rational number, $r = A/B$, with $A, B \in \mathbb{Z}$, $B \neq 0$. We shall always assume that $(A, B) = 1$ and that $B > 0$. A rational number A/B has two representations as a continued fraction:

- if $B \neq 1$, then

$$\frac{A}{B} = [a_0; a_1, a_2, \cdots, a_n], \text{ with } a_n \geq 2, \tag{1.1}$$

and

$$\frac{A}{B} = [a_0; a_1, a_2, \cdots, a_n - 1, 1],$$

- if $B = 1$, then

$$\frac{A}{B} = a_0 \quad \text{and} \quad \frac{A}{B} = [a_0 - 1; 1].$$

If the continued fraction expansion of A/B is determined by the Euclidean algorithm, the outcome is the expansion (1.1), i.e. the shortest one. In this note we consider the shortest expansion as the most natural one, we call it the *regular* continued fraction expansion of A/B. The regular continued fraction expansion of an irrational number ξ is infinite and unique. We shall denote it by

$$\xi = [a_0(\xi); a_1(\xi), a_2(\xi), \cdots]. \tag{1.2}$$

Let

$$\left(\frac{p_n(\xi)}{q_n(\xi)}\right)_{n \geq -1}$$

Manuscrit reçu le 29 juin 1992, version définitive le 26 avril 1993.

be the sequence of corresponding convergents, also denoted by $RCF(\xi)$. Hence $A/B \in RCF(\xi)$ means that there exists an integer n, $n \geq -1$, such that

$$\frac{A}{B} = \frac{p_n(\xi)}{q_n(\xi)}.$$

(1.3) Definition. *The approximation coefficient of a rational number A/B with respect to a real irrational number ξ, notation $\theta(\xi, A/B)$, is defined by*

$$\theta(\xi, A/B) := B|B\xi - A|.$$

In his "Essai sur la théorie des nombres", [13], pp. 27–29, Legendre gives a necessary and sufficient condition for a rational number A/B to be a convergent of the irrational number ξ. This necessary and sufficient condition is expressed, in modern notation, by (2.4) and (2.6) of the next section of this note. Legendre concludes the paragraph on the criterion by remarking that in particular it follows that:

$$\theta(\xi, \frac{A}{B}) < \frac{1}{2} \implies \frac{A}{B} \in RCF(\xi). \tag{1.4}$$

In the theory of continued fractions, (1.4) is often called Legendre's Theorem.

The following implication is almost trivial

$$\theta(\xi, \frac{A}{B}) > 1 \implies \frac{A}{B} \notin RCF(\xi). \tag{1.5}$$

The constants $\frac{1}{2}$ and 1 are both best possible. For the constant $\frac{1}{2}$ in (1.4) this means that for every $\varepsilon > 0$ there exist a ξ and an $\frac{A}{B}$ such that $\theta(\xi, \frac{A}{B}) < \frac{1}{2} + \varepsilon$ and $\frac{A}{B} \notin RCF(\xi)$. Similarly for the 1 in (1.5).

In this note we shall add some refinements to Legendre's reasoning and thus find a more detailed version of (1.4). We shall also show that one can prove in the same way a result announced by Fatou in 1904, as well as the analogues of Legendre's Theorem for two other types of continued fractions.

2. Legendre's criterion

We shall from now on assume that ξ and A/B are contained in the unit interval, this being no restriction. Hence the a_0 in (1.1) and the $a_0(\xi)$ in (1.2) are both zero. Let $n \in \mathbb{N}$ and $(a_1, a_2, \cdots, a_n) \in \mathbb{N}^n$. Then we denote the set of irrational numbers ξ with the property that

$$a_1(\xi) = a_1, \ a_2(\xi) = a_2, \ \cdots, \ a_n(\xi) = a_n$$

by

$$\Delta_n(a_1, a_2, \cdots, a_n).$$

Such a set is called a *fundamental interval of order* n, see [2] p. 42.

(2.1) DEFINITIONS. *The signature* $\varepsilon(A/B)$ *of a rational number* A/B, *is defined as*

$$\varepsilon(\frac{A}{B}) := (-1)^n,$$

with the n *taken from the regular expansion (1.1). This* n *is sometimes called the depth of the rational number* A/B.

Further we define

$$\varepsilon(\xi, \frac{A}{B}) := +1 \quad \text{if} \quad \xi > \frac{A}{B},$$

$$\varepsilon(\xi, \frac{A}{B}) := -1 \quad \text{if} \quad \xi < \frac{A}{B}.$$

Finally, the signature of a rational number A/B *with respect to an irrational number* ξ, *notation* $\delta(\xi, A/B)$, *is defined by*

$$\delta(\xi, \frac{A}{B}) := \varepsilon(\frac{A}{B})\varepsilon(\xi, \frac{A}{B}).$$

Hence $\delta(\xi, A/B)$ is determined by the depth of A/B and the order of ξ and A/B. We shall now formulate a more detailed version of Legendre's Theorem (1.4), in which a distinction is made between $\delta(\xi, A/B) = +1$ and $\delta(\xi, A/B) = -1$.

(2.2) THEOREM. *Let* A/B *be a rational number,* $(A, B) = 1$, $B > 0$ *and let* ξ *be an irrational number.*

If $\delta(\xi, \frac{A}{B}) = +1$, *then*

$$\theta(\xi, \frac{A}{B}) < \frac{2}{3} \quad \Longrightarrow \quad \frac{A}{B} \in RCF(\xi),$$

and

$$\theta(\xi, \frac{A}{B}) > 1 \quad \Longrightarrow \quad \frac{A}{B} \notin RCF(\xi).$$

If on the other hand $\delta(\xi, \frac{A}{B}) = -1$, *then*

$$\theta(\xi, \frac{A}{B}) < \frac{1}{2} \quad \Longrightarrow \quad \frac{A}{B} \in RCF(\xi),$$

and

$$\theta(\xi, \frac{A}{B}) > \frac{2}{3} \quad \Longrightarrow \quad \frac{A}{B} \notin RCF(\xi).$$

All constants are best possible.

(2.3) *Proof*

Let the regular expansion of A/B be given by (1.1) and suppose that n is even. Denote by A'/B' the last but one convergent of (1.1). The set of all ξ with $\xi > A/B$, i.e. with $\delta(\xi, A/B) = 1$, and with $A/B \in RCF(\xi)$ is just the fundamental interval.

$$\Delta_n(a_1, a_2, \cdots, a_n)$$

of order n, i.e. the set

$$(\frac{A}{B}, \frac{A+A'}{B+B'}) \setminus \mathbb{Q}, \text{ of length } B^{-1}(B+B')^{-1}.$$

Hence

$$\frac{A}{B} \in RCF(\xi) \quad \Leftrightarrow \quad \theta(\xi, \frac{A}{B}) < (1 + \frac{B'}{B})^{-1}. \tag{2.4}$$

Now

$$\frac{1}{a_n + 1} < \frac{B'}{B} \leq \frac{1}{a_n} \tag{2.5}$$

and thus

$$\theta(\xi, \frac{A}{B}) < \frac{a_n}{a_n + 1} \quad \Longrightarrow \quad \frac{A}{B} \in RCF(\xi)$$

and

$$\theta(\xi, \frac{A}{B}) > \frac{a_n + 1}{a_n + 2} \quad \Longrightarrow \quad \frac{A}{B} \notin RCF(\xi).$$

Since

$$\min_{a_n \geq 2} \frac{a_n}{a_n+1} = \frac{2}{3} \quad \text{and} \quad \sup_{a_n \geq 2} \frac{a_n+1}{a_n+2} = 1,$$

the assertions are now evident.

Let $\xi < \frac{A}{B}$ (and n still be even). Then the set of all ξ with $\frac{A}{B} \in RCF(\xi)$ is the fundamental interval of order $n+1$:

$$\Delta_{n+1}(a_1, a_2, \cdots, a_n - 1, 1) = (\frac{2A - A'}{2B - B'} \ , \ \frac{A}{B}) \setminus \mathbb{Q}$$

which has length $B^{-1}(2B - B')^{-1}$. Instead of (2.4) we now have

$$\text{(2.6)} \qquad \frac{A}{B} \in RCF(\xi) \quad \Leftrightarrow \quad \theta(\xi, \frac{A}{B}) < (2 - \frac{B'}{B})^{-1}$$

and the assertions follow in this case, using again (2.5), from

$$\inf_{a_n \geq 2} \frac{a_n+1}{2a_n+1} = \frac{1}{2} \quad \text{and} \quad \max_{a_n \geq 2} \frac{a_n}{2a_n - 1} = \frac{2}{3}$$

respectively.

The proof for the case where n is odd is almost the same; therefore we omit it here. We see that the constant $1/2$ in Legendre's Theorem is due to rational numbers A/B with $\delta(\xi, A/B) = -1$ and with a very large last partial quotient a_n. ♦

3. A metrical observation

(3.1) Definition. *The sequence of regular continued fraction approximation coefficients of a real irrational number ξ is defined by*

$$\theta_n(\xi) := \theta(\xi, \frac{p_n(\xi)}{q_n(\xi)}), \ \ n \geq 1, \quad \textit{see (1.3).}$$

For almost all ξ, in the sense of Lebesgue, the sequence $(\theta_n(\xi))_{n \geq 1}$ is distributed in the unit interval according to the function F, where

$$\text{(3.2)} \qquad F(\lambda) = \begin{cases} \dfrac{1}{\log 2}\lambda, & 0 \leq \lambda \leq \dfrac{1}{2}, \\ \dfrac{1}{\log 2}(-\lambda + \log 2\lambda + 1), & \dfrac{1}{2} \leq \lambda \leq 1, \end{cases}$$

see [3]. The irregular behaviour of F at $\lambda = 1/2$ can be explained by the constant $1/2$ in Legendre's Theorem (1.4), see [6]. In view of Theorem (2.2) one may ask why F does not have an irregularity at $\lambda = 2/3$. The answer is given by the next theorem which shows that there are in fact *two* irregularities at $\lambda = 2/3$, canceling each other.

(3.3) THEOREM. *For almost all ξ one has*

$$\lim_{n\to\infty} \frac{1}{n}\sharp\left\{j; 1 \le j \le n,\ \theta_j(\xi) < \lambda,\ \delta(\xi, \frac{p_j}{q_j}) = +1\right\} = F_{+1}(\lambda),$$

with

$$F_{+1}(\lambda) = \begin{cases} \frac{1}{2\log 2}\lambda, & 0 \le \lambda \le \frac{2}{3}, \\ \frac{1}{\log 2}(-\lambda + \log 2\lambda + 1 - \log\frac{4}{3}), & \frac{2}{3} \le \lambda \le 1, \end{cases}$$

and

$$\lim_{n\to\infty} \frac{1}{n}\sharp\left\{j; 1 \le j \le n,\ \theta_j(\xi) < \lambda,\ \delta(\xi, \frac{p_j}{q_j}) = -1\right\} = F_{-1}(\lambda),$$

with

$$F_{-1}(\lambda) = \begin{cases} \frac{1}{2\log 2}\lambda, & 0 \le \lambda \le \frac{1}{2}, \\ \frac{1}{\log 2}(-\frac{3}{2}\lambda + \log 2\lambda + 1), & \frac{1}{2} \le \lambda \le \frac{2}{3}, \\ \frac{1}{\log 2}\log\frac{4}{3}, & \frac{2}{3} \le \lambda \le 1. \end{cases}$$

(3.4) *Proof*

From the alternating way in which the sequence $(\frac{p_n(\xi)}{q_n(\xi)})_{n\ge 1}$ converges to ξ and from the fact that in the definition of $\varepsilon(\xi, \frac{p_n(\xi)}{q_n(\xi)})$ the n is taken from the shortest expansion of $\frac{p_n(\xi)}{q_n(\xi)}$ as a continued fraction, it follows that

$$\delta(\xi, \frac{p_n(\xi)}{q_n(\xi)}) = 1 \quad \text{if} \quad a_n(\xi) \ne 1,\ n \ge 1$$

and that

$$\delta(\xi, \frac{p_n(\xi)}{q_n(\xi)}) = -1 \quad \text{if} \quad a_n(\xi) = 1,\ n \ge 1.$$

After this remark the proof can be given using the same techniques with which (3.2) is proved in [3]. Details are left to the reader. ♦

4. Extreme mediants and the Theorems of Fatou-Grace and Koksma

DEFINITION. *The sequence of mediants of a real irrational number ξ is the sequence of irreducible fractions of the from*

$$(4.2) \qquad \frac{bp_n(\xi) + p_{n-1}(\xi)}{bq_n(\xi) + q_{n-1}(\xi)}, \; n \geq 0, \; b = 1, 2, \cdots, a_{n+1}(\xi) - 1,$$

ordered in such a way that the denominators form an ascending sequence, compare [9] p. 26. The fractions in (4.2) formed with $b = 1$ are called the first mediants of ξ, those with $b = a_{n+1}(\xi) - 1$ the last mediants of ξ. The first and the last mediants are called extreme or nearest mediants. A first mediant is also a last one if and only if the corresponding partial quotient equals 2.

In 1904, P. Fatou stated that if $\theta(\xi, A/B) < 1$, then A/B is either a convergent or an extreme mediant of ξ, [4]. The first one to publish a proof of this was J. H. Grace [5], see also Koksma [10] and [11]. We will therefore refer to this result as the Theorem of Fatou-Grace. Koksma [11] showed that $\theta(\xi, A/B) < 2/3$ implies that A/B is either a convergent or a first mediant of ξ. We will now prove more detailed versions of these results by the method from section 2.

(4.3) THEOREM. *Let A/B be a rational number, $(A, B) = 1$, $B > 0$ and let ξ be an irrational number.*

- *If $\delta(\xi, \frac{A}{B}) = 1$, then $\frac{A}{B}$ is not a first mediant of ξ.*
- *If $\delta(\xi, \frac{A}{B}) = -1$, then*

  $\theta(\xi, \frac{A}{B}) < 1 \implies \frac{A}{B}$ *is a convergent or a first mediant of ξ,*

  and

  $\theta(\xi, \frac{A}{B}) > 2 \implies \frac{A}{B}$ *is neither a convergent nor a first mediant of ξ.*

Both constants 1 *and* 2 *are best possible.*

Theorems (2.2) and (4.3) yield at once the following result of Koksma ([10] p. 102):

(4.4) THEOREM (KOKSMA). *If A/B is a rational, and ξ an irrational number and if $\theta(\xi, A/B) < 2/3$, then A/B is either a convergent or a first mediant of ξ. The constant $2/3$ is best possible.*

(4.5) **Remark.** The constant 2/3 is due to the constant 2/3 in Theorem (2.2) i.e. from rational numbers A/B with $\delta(\xi, A/B) = 1$ and with last partial quotient 2.

(4.6) *Proof of Theorem (4.3)*

Let A/B have the expansion (1.1) and suppose that n is even. The set of irrational numbers ξ such that A/B is of the form $\frac{p_k(\xi)+p_{k-1}(\xi)}{q_k(\xi)+q_{k-1}(\xi)}$ for some k, is the fundamental interval $\Delta_{n+1}(a_1, a_2, \cdots, a_n - 1, 1)$.

Hence, when $\delta(\xi, A/B) = 1$, A/B is not a first mediant of ξ, whereas when $\delta(\xi, A/B) = -1$, the set of ξ's such that A/B is either a convergent or a first mediant of ξ is just the fundamental interval

$$\Delta_n(a_1, a_2, \cdots, a_n - 1) = (\frac{A - A'}{B - B'}\ ,\ \frac{A}{B}) \setminus \mathbb{Q}$$

which has length $B^{-1}(B - B')^{-1}$. Therefore, if $\delta(\xi, A/B) = -1$, A/B is a convergent or a first mediant if and only if

$$\theta(\xi, \frac{A}{B})\ <\ (1 - \frac{B'}{B})^{-1}.$$

Using (2.5) we then find that

$\theta(\xi, \frac{A}{B}) < \frac{a_n+1}{a_n} \implies \frac{A}{B}$ is a convergent or a first mediant of ξ

and that

$\theta(\xi, \frac{A}{B}) > \frac{a_n}{a_n-1} \implies \frac{A}{B}$ is neither a convergent nor a first mediant of ξ.

The statements now follow from

$$\inf_{a_n \geq 2} \frac{a_n + 1}{a_n} = 1 \quad \text{and} \quad \max_{a_n \geq 2} \frac{a_n}{a_n - 1} = 2.$$

The proof for the case where n is odd is almost the same. ♦

(4.7) THEOREM. *Let A/B be a rational number, $(A, B) = 1$, $B > 0$ and let ξ be an irrational number.*

- If $\delta(\xi, \frac{A}{B}) = 1$, then

$\theta(\xi, \frac{A}{B}) < 1 \implies \frac{A}{B}$ *is a convergent or a last mediant of* ξ,

and

$\theta(\xi, \frac{A}{B}) > 2 \implies \frac{A}{B}$ *is neither a convergent nor an extreme mediant of* ξ.

- *If* $\delta(\xi, \frac{A}{B}) = -1$, *then*

$\frac{A}{B}$ *is a last mediant of* $\xi \implies \frac{A}{B}$ *is a first mediant of* ξ,

and

$\theta(\xi, \frac{A}{B}) < \frac{2}{3} \implies \frac{A}{B}$ *is a convergent or a last mediant of* ξ,

and

$\theta(\xi, \frac{A}{B}) > 1 \implies \frac{A}{B}$ *is neither a convergent nor a last mediant of* ξ.

All constants are best possible.

(4.8) *Proof*

The set of all irrational numbers ξ such that A/B is a last mediant of ξ consists of the union of the two fundamental intervals

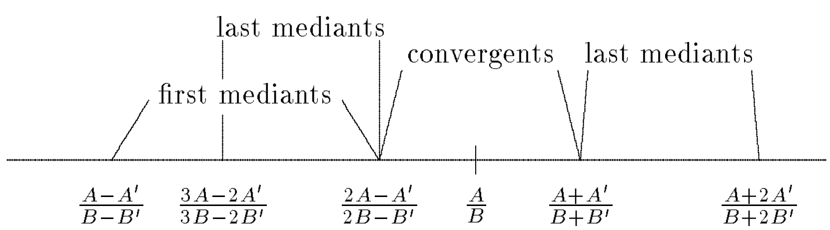
(4.9) $\Delta_{n+1}(a_1, a_2, \cdots, a_{n-1}, a_n - 1, 2)$ and $\Delta_n(a_1, a_2, \cdots, a_{n-1}, a_n + 1)$.

Now

$$\Delta_{n+1}(a_1, a_2, \cdots, a_{n-1}, a_n - 1, 2) \subset \Delta_n(a_1, a_2, \cdots, a_{n-1}, a_n - 1),$$

i.e. the ξ's in the first interval of (4.9) are those for which A/B is a first and a last mediant. After these remarks the proof runs almost the same as the proofs of Theorems (2.2) and (4.3) and may therefore be omitted. ♦

The location of the various intervals occurring in this section and in section 2, is depicted, for even n, in the figure below. For odd n, the order is reversed.



5. Legendre's Theorem for two other types of continued fractions

The above method can be used to obtain similar results for other types of continued fraction expansions. We will illustrate this with two examples:

(1) the continued fractions with odd partial quotients,

(2) the nearest integer continued fractions.

(5.1) The continued fraction expansion with odd partial quotients: every irrational number ξ in the unit interval has a unique expansion of the form

$$(5.2) \qquad \xi = \underline{1}/\overline{b_1} + \underline{\varepsilon_1}/\overline{b_2} + \underline{\varepsilon_2}/\overline{b_3} + \cdots + \underline{\varepsilon_{n-1}}/\overline{b_n} + \cdots$$

with

(5.3) b_n an *odd* positive integer, $\varepsilon_n \in \{-1, 1\}$, $b_n + \varepsilon_n \geq 2$, $n = 1, 2, \cdots$.

We will denote the sequence of convergents associated with the expansion (5.2) by $OCF(\xi)$. A rational number A/B always has a finite expansion

$$(5.4) \qquad A/B = \underline{1}/\overline{b_1} + \underline{\varepsilon_1}/\overline{b_2} + \underline{\varepsilon_2}/\overline{b_3} + \cdots + \underline{\varepsilon_{n-1}}/\overline{b_n}$$

with the same conditions as in (5.3). If in (5.4) one has $b_n = 1$ and $\varepsilon_{n-1} = -1$, then A/B admits two expansions of this type, viz.

$$(5.5) \qquad A/B = \underline{1}/\overline{b_1} + \underline{\varepsilon_1}/\overline{b_2} + \underline{\varepsilon_2}/\overline{b_3} + \cdots + \underline{\varepsilon_{n-2}}/\overline{b_{n-1}} + \underline{-1}/\overline{1}$$

and

$$(5.6) \qquad A/B = \underline{1}/\overline{b_1} + \underline{\varepsilon_1}/\overline{b_2} + \underline{\varepsilon_2}/\overline{b_3} + \cdots + \underline{\varepsilon_{n-2}}/\overline{b_{n-1} - 2} + \underline{1}/\overline{1,}$$

otherwise such an expansion of a rational number is unique. We consider the expansion (5.5) as the most natural one since it is obtained when one repeatedly applies the shift operator for this continued fraction, just as in the case of (1.1). For a description of this operator the reader is referred to [14].

(5.7) THEOREM. *Let A/B be a rational number, $(A, B) = 1$, $B > 0$ and let ξ be an irrational number.*

- *If $\delta(\xi, \frac{A}{B}) + 1$, then*

  $\theta(\xi, \frac{A}{B}) < \frac{1}{10}(5 + \sqrt{5}) = 0,7236 \cdots \implies \frac{A}{B} \in OCF(\xi),$

  and

  $\theta(\xi, \frac{A}{B}) > 1 \implies \frac{A}{B} \notin OCF(\xi).$

- *If on the other hand $\delta(\xi, \frac{A}{B}) = -1$, then*

  $\theta(\xi, \frac{A}{B}) < g^2 \implies \frac{A}{B} \in OCF(\xi),$

  and

  $\theta(\xi, \frac{A}{B}) > G \implies \frac{A}{B} \notin OCF(\xi).$

Here $g := \frac{1}{2}(\sqrt{5}+1)$, *hence* $g^2 = 0,3819\cdots$, *and* $G := g^{-1} = 1,6180\cdots$; *the four constants are best possible.*

It was already known that $\theta(\xi, A/B) > G$ implies $A/B \notin OCF(\xi)$, i.e. the analogue of (1.5). We now also have the analogue of (1.4), that is Legendre's Theorem for the continued fractions with odd partial quotients:

(5.8) Corollary. *Let* A/B *be a rational number,* $(A,B)=1$, $B>0$ *and let* ξ *be an irrational number. If*

$$\theta(\xi, \frac{A}{B}) < \frac{1}{2}(3-\sqrt{5}) = 0,3819\cdots,$$

then A/B *is a convergent of the expansion of* ξ *into its continued fraction with odd partial quotients. The constant* $0,3819\cdots$ *is best possible.*

(5.9) *Proof of Theorem (5.7)*

The proof differs only in technical details from that of Theorem (2.2). Therefore it may suffice to indicate the main differences. First note that for the $\varepsilon(A/B)$ from definition (2.1) one has

$$\varepsilon(\frac{A}{B}) = \varepsilon_1\varepsilon_2\cdots\varepsilon_{n-1}(-1)^n,$$

where n and $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{n-1}$ are given by the expansion (5.4) and, in case we have the two expansions (5.5) and (5.6), by (5.5). This can easily be shown with the techniques from [1], II 7.

Next, denote by $I(A/B)$ the set of irrational numbers ξ such that $A/B \in OCF(\xi)$. If A/B has the expansion (5.4) with $b_n \geq 3$, the end points of $I(A/B)$ are

$$\underline{1}/\overline{b_1} + \underline{\varepsilon_1}/\overline{b_2} + \underline{\varepsilon_2}/\overline{b_3} + \cdots + \underline{\varepsilon_{n-1}}/\overline{b_n+1}$$

and

$$\underline{1}/\overline{b_1} + \underline{\varepsilon_1}/\overline{b_2} + \underline{\varepsilon_2}/\overline{b_3} + \cdots + \underline{\varepsilon_{n-1}}/\overline{b_n-1},$$

from which it follows that

$$I(\frac{A}{B}) = (\frac{A-A'}{B-B'}, \frac{A+A'}{B+B'}) \setminus \mathbb{Q} \quad \text{if} \quad \varepsilon(\frac{A}{B}) = +1$$

and with the end points reversed when $\varepsilon(A/B) = -1$. Here A'/B' denotes the last but one convergent of (5.4).

If A/B has the expansion (5.5), then the end points are

$$\underline{1}/\overline{b_1} + \underline{\varepsilon_1}/\overline{b_2} + \underline{\varepsilon_2}/\overline{b_3} + \cdots + \underline{\varepsilon_{n-2}}/\overline{b_{n-1}} + \underline{-1}/\overline{1}$$

and

$$\underline{1}/\overline{b_1} + \underline{\varepsilon_1}/\overline{b_2} + \underline{\varepsilon_2}/\overline{b_3} + \cdots + \underline{\varepsilon_{n-2}}/\overline{b_{n-1} - 2} + \underline{1}/\overline{1+1}\,,$$

from which it follows that

$$I(\frac{A}{B}) = (\frac{3A - A'}{3B - B'}\ ,\ \frac{A + A'}{B + B'}) \setminus \mathbb{Q} \quad \text{if} \quad \varepsilon(\frac{A}{B}) = +1,$$

with A'/B' the last but one convergent of (5.5) and again with the end points reversed when $\varepsilon(A/B) = -1$.

The analogue of (2.5) is

$$\frac{1}{b_n + G} < \frac{B'}{B} < \frac{1}{b_n - g^2},$$

which follows from the structure of the two-dimensional ergodic system underlying this continued fraction, see [1] and [14]. The rest of the proof is exactly the same as the corresponding part of the proof of Theorem (2.2). ⧫

(5.10) **Remark.** The constant g^2 from Corollary (5.8) corresponds to an irregularity of the distribution function, for almost all ξ, of the sequence of approximation coefficients connected with this continued fraction, see [1], IV, Théorème 1. One could give here results similar to those in section 3.

Recently, the analogue of Legendre's Theorem for the nearest integer continued fraction expansion was found in three different ways, see [7], [8] and [12]. The constant turns out to be the same as in the case of the continued fraction with odd partial quotients: g^2. The method from the previous sections applies to the nearest integer continued fraction as well. The sequence of convergents of this expansion is denoted here by $NICF(\xi)$. We will state the result without giving the proof, which is very similar to the previous ones.

(5.11) THEOREM. *Let A/B be a rational number, $(A,B)=1$, $B>0$ and let ξ be an irrational number.*

- *If $\delta(\xi,\frac{A}{B})=+1$, then*

$$\theta(\xi,\tfrac{A}{B})<g^2 \implies \tfrac{A}{B}\in NICF(\xi),$$

and

$$\theta(\xi,\tfrac{A}{B})>g \implies \tfrac{A}{B}\notin NICF(\xi).$$

- *If on the other hand $\delta(\xi,\frac{A}{B})=-1$, then*

$$\theta(\xi,\tfrac{A}{B})<g^2 \implies \tfrac{A}{B}\in NICF(\xi),$$

and

$$\theta(\xi,\tfrac{A}{B})>\tfrac{1+G}{3+G}=0,5669\cdots \implies \tfrac{A}{B}\notin NICF(\xi).$$

All constants are best possible.

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