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On a characterization of the analytic and meromorphic functions defined on some rigid domains

par Capi Corrales Rodrigáñez

O. Introduction

Let C be a non-singular projective curve (i.e. an integral, regular scheme of finite type and dimension 1) defined over a complete non-archimedean algebraically closed field k of characteristic 0. Let C^{an} be the analytification of C, and $X = C \setminus \{d_1, \cdots, d_s\}$, where d_i is a point of C, or $X = C \setminus \cup_{i=1,\cdots,s} D_i$, where D_i is isomorphic to a disc in $\mathbb{P}^1_k$. Let $u, v \in \mathcal{O}_{C^{an}}(X)$ (holomorphic functions on X) or $u, v \in \mathcal{M}_{C^{an}}(X)$ (meromorphic functions on X).

In this work we are interested in the following problem:
to find the smallest integer r such that whenever $\alpha_1, \cdots, \alpha_r \in k$ and $u^{-1}(\{\alpha_i\}) = v^{-1}(\{\alpha_i\})$ for $1 \leq i \leq r$, then $u = v$.

In other words, how many fibers determine the holomorphic (respectively meromorphic) functions on X ?

This study could be thought of as a continuation of the article by Adams and Straus (see [A-S]), who considered the same problem in the case of the projective line $\mathbb{P}^1_k$.

In section 1 we prove the following results:

Theorem. *Let k be complete with respect to a non-archimedean valuation. Suppose that k is algebraically closed and has characteristic 0. Let C denote a projective, non-singular and irreducible curve over k. Let X be the analytic subspace of C obtained by deleting from C a finite number of points and closed discs. Then X has the following property.*

Suppose that u, v are holomorphic functions on X (resp. meromorphic functions); suppose that the zero set of u is infinite. Let a_1, a_2, a_3

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(resp. $a_1, \cdots, a_5$) denote distinct elements of k; suppose that $u^{-1}(\{a_i\}) = v^{-1}(\{a_i\})$ for $i = 1, 2, 3$ (resp. $i = 1, \cdots, 5$). Then $u = v$.

The strategy we take is the following: using the Riemann-Roch theorem, we construct a function $T \in \mathcal{R}$, the set of rational functions on C, having exactly $\{d_1, \cdots, d_s\}$ at its set of poles; then $X := C \setminus \{d_1, \cdots, d_s\} = \bigcup_{\rho} X_\rho$ with $X_\rho := \{x \in C,\ |T(x)| \leq |\rho|\}$. It turns out that, for $\rho \in k^*$ large enough, we get $X \setminus X_\rho = \{x \in X,\ |\rho| \leq |T(x)| < \infty\} = \bigcup_{i=1}^{s} V_i \setminus \{d_i\}$, where $d_j \in V_j$, and $V_i \cong \mathbb{B}_k^1 := \{x \in \mathbb{P}_k^1,\ |T(x)| \leq 1\}$. This fact allows us to make use of the results obtained by Adams and Straus for functions defined on puntured discs and annuli of the projective line (see [A-S], theorem 4-6).

In section 2 we consider the algebraic situation, i.e. the case in which u and v are rational functions defined on the curve. We finish with a series of remarks and questions in section 3.

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1. On the number of fibers over elements of k which determine, in certain cases, the elements of $\mathcal{O}_{C^{an}}(X)$ and $\mathcal{M}_{C^{an}}(X)$

For the analytification of algebraic varieties we refer to [F-M], p. 161.

THEOREM. *Let k be complete with respect to a non-archimedean valuation. Suppose that k is algebraically closed and has characteristic 0. Let C denote a projective, non-singular and irreducible curve over k. Let X be the analytic subspace of C obtained by deleting from C a finite number of points and closed discs. Then X has the following property.*

Suppose that u, v are holomorphic functions on X (resp. meromorphic functions); suppose that the zero set of u is infinite. Let a_1, a_2, a_3 (resp. $a_1, \cdots, a_5$) denote distinct elements of k; suppose that $u^{-1}(\{a_i\}) = v^{-1}(\{a_i\})$ for $i = 1, 2, 3$ (resp. $i = 1, \cdots, 5$). Then $u = v$.

Proof.
The theorem is known to be true in the following two cases:

1. X is a punctured disk (Theorem 5, [A-S]).
2. X is a "semi-open annulus", i.e. isomorphic to $\{z \in k |\ r \leq |z| < 1\}$ (theorem 4, [A-S]). It suffices now to prove the following lemma.

Lemma. *Let X be defined as in the theorem. Then X is connected. X has the form $X = X_0 \cup X_1 \cup \cdots \cup X_r$ with $r \geq 1$, X_0 affinoid, each X_i for $i \geq 1$ being either a punctured disc or a semi-open annulus.*

Since u has finitely many zeros on the affinoid X_0, there is an X_i $(i \geq 1)$ where u has infinitely many zeros. Using the known cases, one finds that $u - v$ is identically zero on X_i. It follows that $u - v$ is zero on a connected component of X. Since X is connected, this means that $u = v$ holds on X.

Proof (of the lemma).

1) We suppose that $X = C \setminus \{d_1, d_2, \cdots, d_s\}$. As d_i is a regular point of C, it is well known that d_i is contained in an affinoid subset D_i of C^{an} with $D_i \simeq \mathbb{B}^1_k$ (the unit disc). Let $f \in \mathcal{R}(C)$ (rational functions on C) be such that its poles are exactly $\{d_1, d_2, \cdots, d_s\}$. For π small enough we have

$$\{x \in C,\ |\frac{1}{f}(x)| \leq |\pi|\} = D'_1 \cup D'_2 \cup \cdots \cup D'_s$$

where D'_i is a subdisc of D_i. We put

$$X_0 := \{x \in C,\ |f(x)| \leq |\frac{1}{\pi}|\},\ X_i := D'_i.$$

We also have

$$X = \underset{\rho}{\cup}\{x \in C,\ |f(x)| \leq |\rho|\}.$$

For the connexity we use the following step.

2) *Let $f \in \mathcal{R}(C)$, $Y = \{x \in C,\ |f(x)| \leq 1\}$, and suppose that $\{x \in C,\ |\frac{1}{f}(x)| \leq 1\}$ is an union of closed discs. Then Y is connected.*

Let $\{x \in C,\ |f(x)| \leq 1\} = D_1 \cup D_2 \cup \cdots \cup D_s$, where D_i is a disc with center a_i, $a_i \notin Y$. Suppose $Y = V \cup W$, union of admissible sets. Let C_i be the circumference of D_i with center a_i, then $V \cap D_i = C_i$ or $\emptyset$ and $W \cap D_i = C_i$ or $\emptyset$. Up to reordering indices, we have $C_i = V \cap D_i$ for $1 \leq i \leq t$, $C_j = W \cap D_j$ for $t+1 \leq j \leq s$. It follows that $V^* := V \cup \{\underset{i}{\cup} D_i\}, W^* := W \cup \{\underset{j}{\cup} D_j\}$ are admissible sets of C^{an} with $C = V^* \cup W^*$. As C^{an} is connected by GAGA, one has $V = \emptyset$ or $W = \emptyset$, i.e. Y is connected.

It follows that X defined in 1) is connected.

3) *Let D be a closed subdisc of C^{an}. Then D is strictly contained in a closed subdisc D' of C^{an}.*

One has $\mathcal{O}_{C^{an}}(D) = k < g > \simeq k < z >$, for some $g \in \mathcal{O}_{C^{an}}(D)$. Since $\mathcal{O}_{C^{an}}(D) \cap \mathcal{R}(C)$ is dense in $\mathcal{O}_{C^{an}}(D)$ (see [F-M] theorem 4), there exists $f \in \mathcal{R}(C)$ such that $\mathcal{O}_{C^{an}}(D) = k < f >$ (choose $f \in \mathcal{O}_{C^{an}}(D) \cap \mathcal{R}(C)$ such that $\|f - g\|_D < 1$). Also, there exists $Z \supset D$, affine open set of C with $\mathcal{O}_C(Z) = k[f_2, f_3, \cdots, f_m]$ and $D = \{z \in Z,\ |f_i(z)| \leq 1,\ 2 \leq i \leq m\}$. We consider $Z' = \{x \in C,\ f \in \mathcal{O}_{C,x}\}$; we then have

$$\mathcal{O}_C(Z \cap Z') = k[f, f_2, \cdots, f_m, f_{m+1}, \cdots, f_t].$$

Up to multiplication of $f_{m+1}, \cdots, f_t$ by a constant, we may assume $\|f_i\|_D < 1$ for $m+1 \leq i \leq t$. It follows that

$$D = \{z \in Z,\ |f_i(z)| \leq 1,\ 1 \leq i \leq t\}$$

and that $\mathcal{O}_C(Z \cap Z') = k[f_1, f_2, \cdots, f_m, f_{m+1}, \cdots, f_t]$, with $f = f_1$. Since

$$\mathcal{O}_{C^{an}}(D) = k < f_1 >, \tag{1}$$

we have $f_i = \sum_{n=0}^{\infty} a_{n,i} f_1^n$, with $\lim_{n \to \infty} |a_{n,i}| = 0,\ 2 \leq i \leq t$.

On the other hand, since $f_2, f_3, \cdots, f_t$ are algebraic over $k(f)$ and thus over $k < f >$, we can use [B-D-R] theorem 1, and guarantee the existence of ρ, $|\rho| > 1$, with $\lim_{n \to \infty} |a_{n,i}|\ |\rho|^n = 0,\ 2 \leq i \leq t$. Let $\rho_1 = \rho$, $\rho_i := \max\{|a_{n,i}|\ |\rho|^n,\ n \in \mathbb{N}\}$, for $2 \leq i \leq t$, and

$$D' = \{x \in Z \cap Z',\ |f_i(x)| \leq |\rho_i|,\ 1 \leq i \leq t\}. \tag{2}$$

It follows easily from (1) and (2) and from $\mathcal{O}_{C^{an}}(D') = k < \frac{f_1}{\rho_1}, \frac{f_2}{\rho_2}, \cdots, \frac{f_t}{\rho_t} >$ that the homorphism $\gamma : k < \frac{f}{\rho} > \longrightarrow \mathcal{O}_{C^{an}}(D')$ is surjective, and since $k < \frac{f}{\rho} >$ is integral of dimension 1, the map γ must also be injective.

4) We suppose that $X = C \setminus \{D_1 \cup \cdots \cup D_s\}$, where D_i are closed discs. By 3) D_i is strictly contained in a subdisc D_i' of C^{an}. By [F-M] theorem 1, there exists $f \in \mathcal{R}(C)$ such that $\bigcup_{i=1}^{s} D_i = \{x \in C, f \in \mathcal{O}_{C,x}$ and $|f(x)| \leq 1\}$. Let

$$X_\varepsilon := \{x \in C, \frac{1}{f} \in \mathcal{O}_{C,x} \ \text{ and } \ |\frac{1}{f}(x)| \leq 1 - \varepsilon\}.$$

Then for $\mathcal{E}$ small enough , $D_i' \cap X_\varepsilon$ is a semi-open annulus. Furthermore by 2) X_ε is connected, and thus X is connected.

2. The algebraic case

THEOREM. *Let k be an algebraically closed field, char $k = 0$, C a projective non-singular curve of genus g defined over k, u (resp. v) a surjective morphism from C onto $\mathbb{P}^1_k$. Let $\alpha_0, \alpha_1, \cdots, \alpha_{2g+3}$ be distinct elements of $\mathbb{P}^1_k$. If $u^{-1}(\{\alpha_i\}) = v^{-1}(\{\alpha_i\})$, then we have $u = v$.*

Proof.

Case 1: case when $g = 0$. We may assume $u, v \in \mathcal{R}(C)\backslash k, a_0 = 0,\ a_1 = \infty$, and deg (divisor of zeros of u) $\geq$ deg(divisor of zeros of v). Let $\omega = \frac{du}{dt}.(u - v)$. We will show $\omega = 0$. Suppose $u - v \neq 0$.

i) Let $z \in X$ with $z \neq \infty$, and $v_z(u) \geq 0$ (v_z is the valuation defined by z).

- If $v_z(u) < 0$, then $v_z(\frac{du}{dt}) = v_z(u) - 1$, and $v_z(u - v) \geq 1$. Thus $v_z(\omega) \geq v_z(u)$.

- If $v_z(u) = 0$, then $v_z(\frac{du}{dt}) \geq 0$, and $v_z(u - v) \geq 0$; thus $v_z(\omega) \geq v_z(u)$. Consequently,

(3) $$v_z(w) \geq v_z(u), \quad \text{if} \quad v_z(u) \geq 0.$$

Analogously, if $v_z(u - a_i) \geq 0,\ v_z(w) \geq v_z(u - a_i)$, for $i = 2, 3$. We conclude that

(4) $$v_z(w) \geq v_z(u(u - a_2)(u - a_3)), \quad \text{if}\ v_z(u(u - a_2)(u - a_3)) \geq 0 \ \text{and} \ z \neq \infty.$$

- If $v_\infty(u) \geq 0$, then $v_\infty(\frac{du}{dt}) \geq v_\infty(u) + 1$. This, together with (4), shows that if $v_\infty(u) \geq 0$ then

(5) $$\deg(\text{divisor of zeros of } \ w) > \ \deg(\text{divisor of zeros of } \ u(u - a_2)(u - a_3)),$$

- If $v_\infty(u) < 0$, then

(6) $$\deg(\text{divisor of zeros of } \ w) \geq \ \deg(\text{divisor of zeros of } \ u(u - a_2)(u - a_3)).$$

ii) Let $z \neq \infty$ with $v_z(u) < 0$. We then have,

$$-v_z(\tfrac{du}{dt}) = -v_z(u) + 1, \text{ and } -v_z(u - v) \leq -v_z(u) + (-v_z(v)) - 1;$$

(7) $$\text{hence, } v_z(\omega) \leq -2v_z(u) + (-v_z(v)), \text{ if } z \neq \infty, \text{ and } v_z(u) < 0.$$

It follows from (7) that

(8) $$\deg(\text{divisor of poles of } \ \omega) \leq \deg(\text{divisor of poles of } \ u^2 v) \text{ if } v_\infty(u) \geq 0.$$

• If $v_\infty(u) < 0$, then $-v_\infty(\frac{du}{dt}) = v_\infty(u) - 1$. This, together with (7) shows that

(9) $\quad \deg(\text{divisor of poles of } \ \omega) < \ \deg(\text{divisor of poles of } \ u^2 v) \ \text{ if } v_\infty(u) < 0.$

iii) We now have:

• if $v_\infty(u) \geq 0$, using (5) and (8),

$$\deg(\text{divisor of zeros of } \ \omega) > 3 \deg(\text{divisor of zeros of } u),$$

$$\deg(\text{divisor of zeros of } \ \omega) \leq 3 \deg(\text{divisor of zeros of } u),$$

and thus a contradiction.

• if $v_\infty(u) < 0$, with (6) and (9)

$$\deg(\text{divisor of zeros of } \ \omega) \geq 3 \deg(\text{divisor of zeros of } u),$$

$$\deg(\text{divisor of zeros of } \ \omega) < 3 \deg(\text{divisor of zeros of } u),$$

and thus a contradiction.

Hence, $u = v$, Q.E.D. $\square$

Case 2: case when $g \geq 1$, i.e. dim $H^0(C, \Omega) \geq 1$.

We may assume $a_0 = \infty$, $a_1 = 0$. Let $d\omega \in H^(C, \Omega)$, and $W := \frac{du}{d\omega}(u-v)$. Let z be such that $v_z(u) \geq 0$; then

$$v_z(W) = v_z(\frac{du}{d\omega}) + v_z(u-v) = v_z(du) - v_z(dw) + v_z(u-v).$$

Since $v_z(du) = v_z(u) - 1$, and $v_z(u-v) \geq 1$, we have $v_z(W) \geq v_z(u) - v_z(dw)$; on the other hand, $d\omega \in H^0(C, \Omega)$ implies that dw only has zeros and is of degree $2g-2$. We thus have, after some calculations,

(10) $\quad \deg(\text{divisor of zeros of } u(u-a_2)\cdots(u-a_{2g+3})) \leq$
$$\deg(\text{divisor of zeros of } W) + 2g - 2,$$

(11) $\quad \deg(\text{divisor of poles of } W) \leq 2g - 2 + 3 \deg(\text{divisor of poles of } u).$

Suppose that $\deg(\text{divisor of poles of } u) \geq \deg(\text{divisor of poles of } v) \geq 2$, (the last inequality comes from $g \geq 1$). It follows from (10) and (11) that

$(2g+3) \deg(\text{divisor of zeros of } u) \leq \ \deg(\text{divisor of zeros of } W) + 2g - 2 =$

$\deg(\text{divisor of poles of } W) + 2g - 2 \leq 4g - 4 + 3 \deg(\text{divisor of poles of } u),$

which is clearly a contradiction. Consequently, $u = v$. $\square$

3. Remarks and questions

3.1. Remarks

1. In their theorem 5, Adams and Straus (see [A-S]) deal with the case in which either u or v has a singularity in the missing point of a punctured disc, and they proof r to be 3 in such a case. What would happen if u and v are both holomorphic on a disc?

The following example shows that, provided the field has enough elements (say, char $k = 0$), one can take r arbitrarily large. That is, for any chosen r we can find $\alpha_1, \cdots, \alpha_r \in k$ distinct, and functions f and g holomorphic on the unit disc, such that $f^{-1}(\{\alpha_i\}) = g^{-1}(\{\alpha_i\})$ for $1 \leq i \leq r$, and still $f \neq g$.

Fix $r \geq 2$, and let k be a non-archimedean valued field, algebraically closed and with char $k = 0$. Choose $\pi \in k$ with $0 < |\pi| < 1$, and let $g(z) = z^2(z-\pi)$. Let $\alpha_1 = 0$, $\alpha_2 = 1$, and, for $2 \leq i \leq r$, let $\alpha_i \in k$ be such that $|\alpha_i| = 1$; $g(z) - \alpha_i = (z - A_i)(z - B_i)(z - C_i)$ with A_i, B_i, C_i distinct; $\alpha_i \neq \alpha_j$ if $i \neq j$; this is possible because char $k = 0$. We define

$$f(z) := z^2(z-\pi) + \lambda\pi z(z-\pi)\prod_{i=2}^{r}(z-A_i)(z-B_i)(z-C_i),$$

with $z + \lambda\pi \prod_{i=2}^{r}(z-A_i)(z-B_i)(z-C_i)$ vanishing in π, that is,

$1 + \lambda \prod_{i=2}^{r}(\pi - A_i)(\pi - B_i)(\pi - C_i) = 0$, and so $|\lambda| = 1$. It follows that

(i) $f(z) = z^2(z-\pi)u$ with $u \in k < z >^{\times}$,

(ii) $f(z) - \alpha_i = (z-A_i)(z-B_i)(z-C_i)v_i$, with $v_i \in k < z >^{\times}$.

To see that $v_i \in k < z >^{\times}$, we only need to notice that $v_i(z) = 1 + h_i(z)$, with $\|h_i\| < 1$, for $2 \leq i \leq r$. The fact that $u \in k < z >^{\times}$ follows from looking at the Newton polygon of

$$h(z) = z + \lambda\pi \prod_{i=2}^{r}(z-A_i)(z-B_i)(z-C_i),$$

which tells us that the only zero of $h(z)$ in the unit disc is precisely π.

2. In the algebraic case, in the case where $g = 0$ the conditions of the theorem are necessary, as the following example shows: let

$$u = \frac{x^2(x-1)}{(x-a)^2(x-b)},\quad v = -\frac{x(x-1)^2}{(x-a)(x-b)^2}.$$

We have $u^{-1}(\{0\}) = v^{-1}(\{0\})$, $u^{-1}(\{\infty\}) = v^{-1}(\{\infty\})$; we look for π such that $u^{-1}(\{\pi\}) = v^{-1}(\{\pi\})$. We choose π with

$$x^2(x-1) - \pi(x-a)^2(x-b) = \left\{\frac{1-\pi}{1+\pi}\right\}\{x(x-1)^2 + \pi(x-a)(x-b)^2\};$$

this gives us

$$\frac{1}{3(1+\pi)} = \frac{(3\pi-1)^2}{8\pi^2},\ a = \frac{(3\pi-1)(\pi-1)}{4\pi^2},\ b = \frac{(3\pi-1)(\pi+1)}{4\pi^2}.$$

3.2. Questions Two questions remain unanswered:

1. Is the result of the theorem in the algebraic case best possible in case $g \geq 1$?

2. Open question since 1971: would the conclusions of theorems 4-6 in [A-S], (and thus our theorem in 1.) remain valid if the hypothesis on the number of values attained at the same points are reduced from three to two and from five to four respectively? As Adams and Straus observe (see [A-S], *Problem* and theorem 8, p. 424), the answer is yes if we strengthen the hypothesis to saying that two functions attain certain values at the same points with the same multiplicities.

Bibliography

[A-S] W. W. ADAMS, E. G. STRAUS, *Non-archimedean analytic functions taking the same values at the same points*, Illinois J. of Math **15** (1971), 418–424.

[B] N. BOURBAKI, *Eléments de mathématiques. Espaces vectoriels topologiques*, Chap. I, Hermann, Paris, 1953.

[B-D-R] S. BOSCH, B. DWORK, P. ROBBA, *Un théorème de prolongement pour des fonctions analytiques*, Math. Ann. **252** (1980), 165–173.

[B-G-R] S. BOSCH, U. GÜNTZER, R. REMMERT, *Non archimedean analysis*, Grund. der Math 262, Springer Verlag, 1984.

[F] J. FRESNEL, *Géométrie anlytique rigide*, Université Bordeaux I, 1984.

[F-M] J. FRESNEL, M. MATIGNON, *Sur les espaces analytiques quasi-compacts de dimension 1 sur un corps valué complet ultramétrique*, Annali di matematica pura ed applicata (IV) **CXLV** (1986), 159–210.

[F-vdP] J. FRESNEL, M. van der PUT, *Géométrie analytique rigide et applications*, Progress in Math. 18, Birkhäuser, 1981.

[K] R. KIEHL, *Theorem A und Theorem B in der nichtarchimedischen Funktionentheorie*, Invent. Math. **2** (1967), 256–273.

[S] J.-P. SERRE, *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier **6** (1956), 1–42.

Capi Corrales Rodrigáñez
Departamento de álgebra
Facultad de matemáticas
Universidad Complutense de Madrid
Madrid 28040 - Espagne