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A generalization of a theorem of Erdös on asymptotic basis of order 2

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Abstract – Let $\mathcal{T}$ be a system of disjoint subsets of $\mathbb{N}^*$. In this paper we examine the existence of an increasing sequence of natural numbers, A, that is an asymptotic basis of all infinite elements T_j of $\mathcal{T}$ simultaneously, satisfying certain conditions on the rate of growth of the number of representations $r_n(A)$; $r_n(A) := |\{(a_i, a_j) : a_i < a_j; a_i, a_j \in A; n = a_i + a_j\}|$, for all sufficiently large $n \in T_j$ and $j \in \mathbb{N}^*$. A theorem of P. Erdös is generalized.

1. Notation

In this paper, $\mathbb{N}^*$ will always denote the set of integers $\{1, 2, \dots, n, \dots\}$. An increasing sequence of natural numbers, A, is called an *asymptotic basis* of order 2 of a given set T of natural numbers if every sufficiently large $n \in T$ has at least one representation in the form $n = a_i + a_j; a_i < a_j; a_i, a_j \in A$. Let $r_n(A)$ be the number of such representations of $n \in T$ by elements of A.

Definition. *A system $\mathcal{T} = (T)_{j\in\mathbb{N}^*}$ of disjoints subsets of $\mathbb{N}^*$ satisfying $\mathbb{N}^* = \bigcup_{j=1}^{\infty} T_j$ is called a disjoint covering system.*

Definition. *If for an increasing sequence A of natural numbers there exists a disjoint covering system $\mathcal{T}$ such that*

$$(1) \qquad \exists j_0 \;:\; T_j = \emptyset \;\; \forall j \geq j_0 \text{ or } |T_j| = \infty \text{ for infinitively many } j \in \mathbb{N}^*$$

and

$$(2) \quad A \text{ is an asymptotic basis of order 2 of all infinite elements } T_j \text{ of } \mathcal{T},$$

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then A is called an asymptotic pseudo-basis of $\mathbb{N}^*$.

Remark. Let A be an asymptotic pseudo-basis in regard to a disjoint covering system $\mathcal{T}$. For any infinite element T_j of $\mathcal{T}$ let

$$n_j := \min\{m \in T_j \ : \ r_n(A) > 0 \ \ \forall n \in T_j, \ n \geq m\}.$$

Obviously any asymptotic basis A of order 2 of $\mathbb{N}^*$ is an asymptotic pseudo-basis (e.g. for $\mathcal{T} := \mathbb{N}^*, \emptyset, \emptyset, ...$). But unfortunately the converse in general is not true since for any asymptotic pseudo-bases A of $\mathbb{N}^*$ together with a corresponding disjoint covering system $\mathcal{T}$ the set of all n_j that are defined in the above sense is not necessarily bounded.

2. Introduction

More than fifty years ago S. Sidon [5] asked if there exists an asymptotic basis of order 2 of $\mathbb{N}^*$ that is economic in the sense that for every $\varepsilon > 0$ the assumption $\lim\limits_{n\to\infty} \frac{r_n(A)}{n^\varepsilon} = 0$ holds.

In 1953 P. Erdös [1] solved this problem ingeniously. In fact he proved the much sharper:

THEOREM. *There exists an asymptotic basis A of order 2 of* $\mathbb{N}^*$, *satisfying:*

$$A(n) \sim \alpha \ n^{\frac{1}{2}} (\log n)^{\frac{1}{2}} \ , \alpha \in \mathbb{R}, \tag{3}$$

$$\textit{with } A(n) := \sum_{a \in A, 1 \leq a \leq n} 1$$

and

$$\log n \ll r_n(A) \ll \log n. \tag{4}$$

An attractive and still open problem is to decide whether there exists a basis A of $\mathbb{N}^*$ for which there exists $c := \lim\limits_{n\to\infty} \frac{r_n(A)}{\log n}$.
Moreover in [4] I. Rusza asks for a basis for which $r_n(A) \ll \frac{\log n}{\log_2 n}$ holds.

3. On asymptotic pseudo-bases

In this paper we prove the following:

THEOREM. *For any $k \in \mathbb{N}^*$ there exists a disjoint covering system $\mathcal{T}^{(k)} = \{T_1^{(k)}, T_2^{(k)}, ...\}$ satisfying:*

$$\forall \ j \in \mathbb{N}^* \ : \ T_j^{(k)} \textit{ is an infinite element of } \mathcal{T}^{(k)} :$$

$$\log_{k-1} n \gg T_j^{(k)}(n) \gg \log_{k-1} n \ (n \to \infty) \tag{5}$$

$$(\textit{where } \log_0 n := id(n) = n),$$

and an asymptotic pseudo-basis A satisfying:

$$A(n) \sim 2\alpha(\log_k n)^{\frac{1}{2}} n^{\frac{1}{2}} \tag{6}$$

and

$$\begin{aligned} &c_1 \log_k n \leq \ r_n(A) \leq c_2 \log_k n, \\ &\forall \ n \in T_j^{(k)} \textit{ that are sufficiently large,} \\ &\textit{and } \ \forall \ j \in \mathbb{N}^* \textit{ where } T_j^{(k)} \textit{ is an infinite element of } \mathcal{T}^{(k)}, \end{aligned} \tag{7}$$

where α, c_1 and c_2 are global real constants not depending on j.

Remark. The above theorem generalizes (3,4), which is just the special case $k = 1$ (e.g. with $\mathcal{T} := \mathbb{N}^*, \emptyset, \emptyset, ...$).

The proof of the above theorem is based on a slight modification of Erdös' proof of (3,4). Therefore like the proof of (3,4), it is based on a probabilistic method and not constructive.

3.1 Inductive construction of suitable disjoint covering systems

First of all, for any $k \in \mathbb{N}^*$, we are going to construct a special disjoint covering system $\mathcal{T}^{(k)}$ satisfying (1) and (5).

The case $k = 1$.

For $k = 1$ let $\mathcal{T}^{(1)} := \mathbb{N}^*, \emptyset, \emptyset, \cdots$.
Obviously $\mathcal{T}^{(1)}$ is a disjoint covering system and (1) and (5) hold.

The case $k = 2$.

For $k = 2$ we define $\mathcal{T}^{(2)}$ inductively as follows:

$$T_1^{(2)} := \{1\},$$

$$T_2^{(2)} := \{2^j \ : \ j \in \mathbb{N}^*\}.$$

Now, if $T_1^{(2)}, \cdots, T_r^{(2)}$ are already defined, let:

$$s := \min\{n \in \mathbb{N}^* \ : \ n \notin \bigcup_{i=1}^{r} T_i^{(2)}\}$$

and we define
$$T_{r+1}^{(2)} := \{s^j \ : \ j \in \mathbb{N}^*\}.$$
Now we consider the following equivalence relation on $\mathbb{N}^*$:
$$a \sim b \ : \Longleftrightarrow \exists s, u, v \in \mathbb{N}^* \ : \ a = s^u, \ b = s^v.$$
$\mathcal{T}^{(2)}$ just consists of all equivalence classes concerning the above equivalence relation. Thus $\mathcal{T}^{(2)}$ is a disjoint covering system and obviously (1) holds. For $\mathcal{T}_i^{(2)} \in \mathcal{T}^{(2)} \setminus \{1\}$ there exists $s \in \mathbb{N}^*$ such that
$$\mathcal{T}_i^{(2)} = \{s^j \ : \ j \in \mathbb{N}^* \, , \ s \in \mathbb{N}^* \setminus \{1\}\}.$$
For any sufficiently large $m \in \mathbb{N}^*$ there exists $t \in \mathbb{N}^*$ such that
$$s^t \leq m < s^{t+1}.$$
Thus $T_i^{(2)}(m) = t$ implies that:
$$T_i^{(2)}(m) \leq \frac{1}{\log s} \log m \leq T_i^{(2)}(m) + 1,$$
and consequently
$$\log m \ll T_i^{(2)}(m) \ll \log m.$$
Therefore also (5) holds.

The case $k = 3$.

Definition. *For $s \in \mathbb{N}^*$ and any non-empty subset M of $\mathbb{N}^*$ we define*
$$s^M := \{s^m \ : \ m \in M\}.$$

We construct $\mathcal{T}^{(3)}$ by dividing every element $\mathcal{T}_i^{(2)}$ of $\mathcal{T}^{(2)}$ except $\{1\}$ into disjoint infinite subsets of $\mathbb{N}^*$.
For any $\mathcal{T}_i^{(2)}$ of $\mathcal{T}^{(2)}$ there exists $s \in \mathbb{N}^*$:
$$\mathcal{T}_i^{(2)} = \{s^j \ : \ j \in \mathbb{N}^*\}.$$
Consequently
$$\mathcal{T}_i^{(2)} = \bigcup_{\mathcal{T}_j^{(2)} \in \mathcal{T}^{(2)}} s^{\mathcal{T}_j^{(2)}}$$

and we define $\mathcal{T}^{(3)}$ as the system of all those sets $s^{\mathcal{T}_j^{(2)}} = \{s^{p^j} \ : \ j \in \mathbb{N}^*\}$ where p is a natural constant. Since $\mathcal{T}^{(2)}$ is a disjoint covering system, $\mathcal{T}^{(3)}$ is a disjoint covering system, too; and as (1) holds for $\mathcal{T}^{(2)}$, $\mathcal{T}^{(3)}$ satisfies (1), too.

For any infinite element $\mathcal{T}_i^{(3)}$ for $\mathcal{T}^{(3)}$ and any sufficiently large number $m \in \mathbb{N}^*$ there exist $s, p, t \in \mathbb{N}^*$ such that

$$\mathcal{T}_i^{(3)} = \{s^{p^j} \ : \ j \in \mathbb{N}^*\},$$

and

$$s^{p^t} \leq m < s^{p^{t+1}}.$$

Then $\mathcal{T}_i^{(3)}(m) = t$ implies $\log_2 m \ll \mathcal{T}_i^{(3)}(m) \ll \log_2 m$.
Consequently $\mathcal{T}^{(3)}$ satisfies also (5).

The general case $k \geq 4$.

Let $\mathcal{T}^{(1)}, \mathcal{T}^{(2)}, \mathcal{T}^{(3)}, \cdots \mathcal{T}^{(k)}$ be already constructed by the above procedure. Thus for every infinite element $\mathcal{T}_i^{(k)}$ of $\mathcal{T}^{(k)}$ there exist $s_1, \cdots, s_{k-1} \in \mathbb{N}^*$ so that

$$\mathcal{T}_i^{(k)} = \{s_1^{\left(s_2^{\left(\cdot^{\cdot^{\cdot^{(s_{k-1}^j)}}}\right)}\right)} \ : \ j \in \mathbb{N}^*\},$$

and according to the above procedure $\mathcal{T}^{(k+1)}$ will be constructed out of $\mathcal{T}^{(k)}$ by dividing every infinite $\mathcal{T}_i^{(k)}$ of $\mathcal{T}^{(k)}$ into disjoint subsets

$$s_1^{\left(s_2^{\left(\cdot^{\cdot^{\cdot^{\left(s_{k-1}^{T_i^{(2)}}\right)}}}\right)}\right)}, \ T_i^{(2)} \in \ \mathcal{T}^{(2)}.$$

It is easy to see that also $\mathcal{T}^{(k+1)}$ is a disjoint covering system satisfying (1) and (5).

3.2 Proof of the existence of an asymptotic pseudo-basis A satisfying (6) and (7) in regard to $\mathcal{T}^{(k)}$ for any fixed $k \in \mathbb{N}^*$.

This part of the proof of the above theorem uses the probabilistic method of Erdös and Rényi [2]. Since [3] contains an excellent exposition of it, we only give a short survey of those of Erdös' and Rényi's ideas our next steps are based on without proof.

Remark. Since, as we mentioned above, the case $k = 1$ is already solved we restrict ourselves to the case $k \geq 2$.

By the method of Erdös and Rényi ([2] and [3]) for any sequence of real numbers $(\alpha_j)_{j\in\mathbb{N}^*}$, $0 \leq \alpha_j \leq 1$, there exists a probability space with probability measure μ on the space Ω of all strictly increasing sequences of natural numbers, satisfying:

(8) the event $B^{(n)} := \{\omega \in \Omega \ : \ n \in \Omega\}$ is measurable, $\mu(B^{(n)}) = \alpha_n$,

(9) and the events $B^{(1)}$, $B^{(2)}$, $\cdots$ are independent.

We denote by ρ_n the characteristic function of the event $B^{(n)}$.

From now on we consider only those sequences of probabilities $(\alpha_j)_{j\in\mathbb{N}^*}$, satisfying :

$$0 < \alpha_j < 1, \tag{10}$$

$$\lim_{j\to\infty} \alpha_j = 0, \tag{11}$$

$$\exists j_0 \ : \ \alpha_{j+1} < \alpha_j \quad \forall j \geq j_0, \tag{12}$$

$$\sum_{j=1}^{\infty} \alpha_j = \infty. \tag{13}$$

Then by a particular variant of the strong law of large numbers, with probability 1,

$$\sum_{j=1}^{n} \alpha_j \sim \omega(n) \ (n \to \infty) \tag{14}$$

holds, where

$$\omega(n) := \sum_{j\in\omega; 1\leq j\leq n} 1. \tag{15}$$

Let

$$\lambda_n := \sum_{1\leq j<\frac{n}{2}} \alpha_j \alpha_{n-j}, \; m_n := \sum_{j=1}^{n} \alpha_j,$$

and

$$\lambda'_n := \sum_{1\leq j<\frac{n}{2}} \alpha_j \alpha_{n-j} (1-\alpha_j\alpha_{n-j})^{-1}.$$

Then we have:

$$\lambda'_n \sim \lambda_n \; (n\to\infty), \tag{16}$$

and

$$\mu(\{\omega : r_n(\omega) = d\}) \leq \frac{\lambda'^d_n}{d!} e^{-\lambda_n}, \; d\in\mathbb{N}. \tag{17}$$

Lemma 1. *A sequence* $(\alpha_j)_{j\in\mathbb{N}^*}$ *of positive real numbers is defined by*

$$\alpha_j := \alpha \frac{(\log_k j)^{c'}}{j^c} \quad \forall j > j_0, \tag{18}$$

where j_0, α, k, c *and* c' *are suitably chosen real constants, satisfying*

$$0\leq c', \qquad 0<c<1, \qquad 0<\alpha, \qquad 1\leq k$$

so that $\log_k(j) > 0$, $\forall j > j_0$ *and (18) and (10 - 13) are compatible. The precise value of* α_j *for small* j *is unimportant in case that their choice ensures that (18) and (10 - 13) are compatible also for* $\alpha_1, \cdots, \alpha_{j_0}$. *Then as* $(n\to\infty)$

$$\lambda_n \sim \frac{1}{2}\alpha^2 \frac{(\Gamma(1-c))^2}{\Gamma(2-2c)} (\log_k n)^{2c'} n^{1-2c} \tag{19}$$

$$m_n \sim \frac{\alpha}{1-c} (\log_k n)^{c'} n^{1-c}. \tag{20}$$

Remark. The above lemma is a slight generalization of Lemma 11 in [3], p 144. Its proof corresponds essentially to that of the above-mentioned Lemma 11 and is therefore left to the reader.

Now let k be a fixed natural number. To prove our theorem, corresponding to Erdös' proof of (3,4), we first choose a number α with $0 < \alpha < 1$, so that

$$\frac{1}{2}\alpha^2\pi > 1 \tag{21}$$

holds, and we define the sequence $(\alpha_j)_{j\in\mathbb{N}^*}$ by

$$\alpha_j = \begin{cases} \dfrac{1}{2} & 1 \le j \le j_0, \\ \alpha\dfrac{(\log_k n)^{\frac{1}{2}}}{j^{\frac{1}{2}}} & j > j_0, \end{cases} \tag{22}$$

where j_0 is a suitably chosen natural number so that $\log_k j > 0 \quad \forall j > j_0$ and $(\alpha_j)_{j\in\mathbb{N}^*}$ satisfies (10 - 13).

Therefore by (14) and by Lemma 1 we have with probability 1

$$\omega(n) \sim 2\alpha\sqrt{\log_k n}\sqrt{n}, \tag{23}$$

$$\lambda_n \sim \frac{\pi}{2}\alpha^2 \log_k n, \tag{24}$$

which because of (21) ensures the existence of a number $\delta > 0$ such that

$$e^{-\lambda_n} \ll (\log_{k-1} n)^{-(1+\delta)}. \tag{25}$$

In view of (17) for any $n \in \mathbb{N}^*$, $d \in \mathbb{N}$:

$$\mu(\{\omega \ : \ r_n(\omega) > e\lambda'_n\}) \le \sum_{d \ge e\lambda'_n} \mu(\{\omega \ : \ r_n(\omega) = d\}) \le \sum_{d\ge e\lambda'_n} \frac{\lambda'^d_n}{d!}e^{-\lambda_n}$$

$$\le \left(\frac{e\lambda'_n}{e\lambda'_n}\right)^{e\lambda'_n} e^{-\lambda_n} = e^{-\lambda_n} \ll \frac{1}{(\log_{k-1} n)^{1+\delta}}.$$

Let $T_i^{(k)}$ be an infinite non-empty element of $\mathcal{T}^{(k)}$.

There exists $s_1, \cdots, s_{k-1} \in \mathbb{N}^*$ so that

$$T_i^{(k)} = \{ s_1^{\left(s_2^{\left(\cdot^{\cdot^{\cdot (s_{k-1}^j)}} \right)} \right)}, j \in \mathbb{N}^* \}.$$

Consequently :

$$\sum_{n \in T_i^{(k)}} \mu(\{\omega \ : \ r_n(\omega) > e \ \lambda_n'\}) \leq \sum_{n \in T_i^{(k)}} e^{-\lambda_n}$$

$$\leq \sum_{j=1}^{\infty} \left(\log_{k-1} s_1^{\left(s_2^{\left(\cdot^{\cdot^{\cdot (s_{k-1}^j)}} \right)} \right)} \right)^{-(1+\delta)}$$

$$\ll \sum_{j=1}^{\infty} (\frac{1}{j})^{1+\delta} < \infty.$$

Therefore the application of the Borel-Cantelli-Lemma proves the existence of a positive real number c_2, such that for any infinite $T_i^{(k)} \in \mathcal{T}^{(k)}$

$$\mu(\{\omega \ : \ r_n(\omega) \leq c_2 \log_k n, \ n \in T_i^{(k)}, \ (n \text{ sufficiently large})\}) = 1. \tag{26}$$

On the other hand for any suitably chosen constant $b < 1$ again in view of (17) we have

$$\mu(\{\omega \ : \ r_n(\omega) < b\lambda_n'\}) \leq \sum_{1 \leq d \leq b\lambda_n'} \mu(\{\omega \ : \ r_n(\omega) = d\})$$

$$\leq \sum_{1 \leq d \leq b\lambda_n'} \frac{\lambda_n'^{d}}{d!} e^{-\lambda_n}$$

$$\leq (\frac{e\lambda_n'}{b\lambda_n'})^{b\lambda_n'} e^{-\lambda_n}$$

$$= [(\frac{e}{b})^b]^{\lambda_n'} e^{-\lambda_n}.$$

Therefore because of (16) there exists c_1, $0 < c_1 < 1$ such that

$$[(\frac{e}{c_1})^{c_1}]^{\lambda_n'} e^{-\lambda_n} \ll (\log_{k-1} n)^{-(1+\frac{\delta}{2})}. \tag{27}$$

Thus for any fixed infinite $T_i^{(k)} \in \mathcal{T}^{(k)}$, with

$$T_i^{(k)} = \{s_1^{\left(s_2^{\left(\cdot^{\cdot^{\cdot(s_{k-1}^j)}}\right)}\right)}, j \in \mathbb{N}^*\},$$

we have

$$\sum_{n \in T_i^{(k)}} \mu(\{\omega \ : \ r_n(\omega) < c_1 \lambda'_n\}) \ll \sum_{j=1}^{\infty} \left(\log_{k-1} s_1^{\left(s_2^{\left(\cdot^{\cdot^{\cdot(s_{k-1}^j)}}\right)}\right)} \right)^{-(1+\frac{\delta}{2})}$$

$$\ll \sum_{j=1}^{\infty} (\frac{1}{j})^{1+\frac{\delta}{2}} < \infty.$$

Again we apply the Borel-Cantelli-Lemma to prove the existence of $c_1 > 0$ such that for any infinite $T_i^{(k)} \in \mathcal{T}^{(k)}$

$$\mu(\{\omega \ : \ r_n(\omega) \geq c_1 \log_k n, \ n \in T_i^{(k)}, \ (n \text{ sufficiently large})\}) = 1. \tag{28}$$

We have shown that ω has each of the desired properties with probability 1 and thus the whole proof is complete.

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