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Analytical Construction of Weil Curves Over Function Fields

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Let $f: H \to \mathbb{C}$ be a cuspidal modular form of weight two for the Hecke congruence subgroup $\Gamma_0(N)$ of the modular group $SL(2,\mathbb{Z})$, and suppose it is a new eigenform with rational eigenvalues for the Hecke algebra. With f one can associate an elliptic curve $E = E_f$ defined over $\mathbb{Q}$ with conductor $\mathrm{cond}(E) = N$ and a $\mathbb{Q}$-morphism p from the modular curve $\overline{X_0(N)} = \Gamma_0(N)\backslash H \cup \{\text{cusps}\}$ to E such that $f(z)dz = p^*(\omega)$, where ω is an invariant differential on E. Such a curve E is called a *Weil curve*, and a *strong Weil curve* if p is maximal.

In different levels of clarity and concreteness, Y.Taniyama, G. Shimura and A. Weil conjectured that each elliptic curve $E/\mathbb{Q}$ appears as a Weil curve in an essentially unique way. Modulo known facts, the truth of the conjecture yields canonical bijections between the sets of

a) normalized cusp forms f as above;

b) one-dimensional isogeny factors of $J_0^{\mathrm{new}}(N)$ = new part of the Jacobian of $\overline{X_0(N)}$;
and

c) isogeny classes of elliptic curves $E/\mathbb{Q}$ with $\mathrm{cond}(E) = N$.

Andrew Wiles perhaps (**) has proven the above conjecture in the case where N is squarefree, which would have remarkable arithmetic consequences.

In the present article, I want to describe a similar relationship between elliptic curves and modular/automorphic forms, but where the base field $\mathbb{Q}$ is replaced by a rational function field $K = \mathbb{F}_q(T)$ over a finite field $\mathbb{F}_q$. In contrast with the number theoretical situation, the assertion (see (1.13)) corresponding to Shimura-Taniyama-Weil's conjecture **STW** is proven, including bijections between the sets that over K substitute the sets a), b),

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(**) After some turns, Wiles' proof seems now to work (time of proofreading Nov. 94).

c) above. Here c) deals with elliptic curves E/K with split multiplicative reduction at infinity. The assertion comes out by combining deep results of Grothendieck, Jacquet-Langlands, and Drinfeld, and is treated in detail in [10]. Actually, the function field analogue of **STW** is valid for arbitrary global function fields. But in order to avoid complications coming from class numbers, and since some more concrete results on the relationship between cusp forms and elliptic curves are established only over rational function fields, I will restrict to this case.

We will have need for some ingredients that replace e.g. the complex upper half-plane H, the modular curves $\overline{X_0(N)}$, and the modular forms for subgroups of $SL(2,\mathbb{Z})$. In fact, there are different substitutes, leading to

- a rigid analytic theory (Drinfeld modular curves and forms, values in characteristic $p = \mathrm{char}(\mathbb{F}_q)$, see [8]), and
- an automorphic theory (as in Jacquet-Langlands [15], values in characteristic zero),

the two being related through congruences.

I will give a brief sketch of these theories in section one. In section two, I will discuss theta functions for subgroups Γ of $GL(2,A)$, where $A = \mathbb{F}_q[T]$ is the ring of integers in K. They provide the link between the characteristic p and characteristic zero theories, which technically is given by diagram (2.7). Now we can analytically construct the Jacobian J_Γ of a Drinfeld modular curve as a torus divided by the period lattice of theta functions. There results a description of the group $\phi(\Gamma) = \phi(J_\Gamma)$ of connected components of the Néron model of J_Γ through automorphic data.

In section 3, the strong Weil curve E_φ associated with an automorphic Hecke eigenform φ is described. It is a Tate curve at the place ∞ of K, whose period can be calculated by means of theta functions. In particular, its Néron type at infinity is determined. Except for Theorems 3.2 and 3.17 and their corollaries, this is carried out in detail in the joint paper [10] with M. Reversat, to which I refer for proofs. So far, everything can be generalized to arbitrary function fields (of transcendence degree one over $\mathbb{F}_q$). In the case of a rational function field, however, one can do much better. Among others, we calculate the degree n_φ of the Weil uniformization p_φ onto E_φ, and show that its prime divisors are precisely the congruence primes of φ.

There are two reasons why the proofs of these last-mentioned results presently fail to work for general function fields: First, it is not known whether the map $j\colon \bar{\Gamma} \to \underline{H}_!(\mathcal{T},\mathbb{Z})^\Gamma$ of (1.9) is always bijective (see also the discussion in [10]). Second, in the case of a class number > 1, the argument (3.17) showing the perfect duality (after tensoring with $\mathbb{Z}[p^{-1}]$) of the Hecke algebra $\mathcal{H}$ with $\underline{H}_!(\mathcal{T}, \mathbb{Z})$ must apparently be replaced by some adelic argument, which at the least will require additional efforts.

In the fourth and last section, some examples are given that illustrate how "automorphic" (and easily calculable) properties of φ correspond to properties of the Weil curve E_φ.

I am convinced that evaluating that relationship will produce important progress in questions like the Birch-Swinnerton-Dyer conjecture (or rather the Artin-Tate conjecture) for elliptic curves over function fields. While completing this paper, I enjoyed the hospitality of the Institute for Advanced Study in Princeton, to whose staff and members I would like to express my sincere gratitude.

Notation

$\mathbb{F}_q$	= finite field of characteristic p with q elements
A	$= \mathbb{F}_q[T]$, T indeterminate
K	$= \mathbb{F}_q(T)$, with completion at infinity
K_∞	$= \mathbb{F}_q((\pi))$, $\pi = T^{-1}$
O_∞	$= \mathbb{F}_q[\![\pi]\!]$ ∞-adic integers
C	= completed algebraic closure of K_∞ w.r.t. its normalized ($\lvert T\rvert = q$) absolute value $\lvert . \rvert$
G	= group scheme $GL(2)$ with center Z
$\Gamma(1)$	$= G(A) = GL(2, A)$
Γ	= congruence subgroup of $\Gamma(1)$, in most cases
Γ	$= \Gamma_0(\mathfrak{n}) := \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1) \mid c \equiv 0 \pmod{\mathfrak{n}}\}$, where $\mathfrak{n} \subset A$ non-zero ideal = positive divisor of K coprime with ∞
$\bar{\Gamma}$	$= \Gamma^{\mathrm{ab}}$ modulo torsion
$\mathcal{K}$	$= G(O_\infty) = GL(2, O_\infty)$
$\mathcal{I}$	$= \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{K} \mid c \equiv 0 \pmod{\pi}\}$ Iwahori subgroup
$\mathcal{T}$	= Bruhat-Tits tree of $PGL(2, K_\infty)$ with sets $X(\mathcal{T})$ of vertices and $Y(\mathcal{T})$ of oriented edges
$T_\mathfrak{m}$	= Hecke operator ($\mathfrak{m}$ positive divisor)
$\mathcal{H}$	= Hecke algebra over $\mathbb{Z}$, acting on
$\underline{H}$	$= \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$

1. Modular and automorphic forms over K

(1.1) The K-analogue of the complex upper half-plane H is the *Drinfeld upper half-plane*

$$\Omega = \mathbb{P}^1(C) - \mathbb{P}^1(K_\infty) = C - K_\infty.$$

It has a natural structure of one-dimensional analytic space over C and even over K_∞ ([12], [5], [10]). The group $G(K_\infty)$ acts on Ω through fractional linear transformations: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$. The subgroup $\Gamma(1)$ and thus each congruence subgroup $\Gamma \subset \Gamma(1)$ acts with finite stabilizers, whence the quotient $\Gamma \setminus \Omega$ exists as an analytic space over K_∞. In fact, we have the following result due to Drinfeld [4].

THEOREM 1.2. *There exists a smooth affine algebraic curve M_Γ defined over a finite abelian extension K' of K such that $\Gamma \setminus \Omega$ is isomorphic as an analytic space with the analytification M_Γ^{an} of M_Γ.*

Let $\bar{M}_\Gamma$ be the smooth projective model of M_Γ. The curves $\bar{M}_\Gamma$ are called *Drinfeld modular curves*; their study is the content of [8], to which we refer for the following. On $\bar{M}_\Gamma$, there are two kinds of distinguished points: the *cusps* $\bar{M}_\Gamma(C) - M_\Gamma(C)$, which are in canonical bijection with the finite set $\Gamma \setminus \mathbb{P}^1(K)$ of orbits of Γ on $\mathbb{P}^1(K)$, and the *elliptic points*, which are the classes mod Γ of those $z \in \Omega$ whose stabilizers Γ_z are strictly larger than $\Gamma \cap Z(K)$. Since $\bar{M}_\Gamma$ is uniquely determined by the closed analytic space $\Gamma \setminus \Omega \cup \{\text{cusps}\}$, we henceforth will make no difference between e.g. the point set $\Gamma \setminus \Omega$, the associated analytic space $\cong M_\Gamma^{\mathrm{an}}$, and M_Γ.

(1.3) A *Drinfeld modular form* of weight k and type m for Γ (where k is a non-negative integer and m a class modulo $d(\Gamma)$ = cardinality of $\det(\Gamma) \subset \mathbb{F}_q^*$) is a function $f\colon \Omega \to C$ that satisfies:

(i) $f(\gamma z) = \frac{(cz+d)^k}{(\det \gamma)^m} f(z)$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$;

(ii) f is holomorphic (in the rigid sense);

(iii) f is holomorphic at the cusps.

Condition (iii) means that for each cusp s of Γ, f has a power series expansion with respect to a uniformizer t_s of s. Clearly, t_s, or rather t_∞, plays the role of the classical uniformizer $q(z) = e^{2\pi i z}$; it is specified in *loc. cit.* V 2, 3. We let $M_{k,m}^i(\Gamma)$ be the C-vector space of i times cuspidal modular forms of weight k and type m, where f is i times cuspidal if its order w.r.t.

t_s is $\geq i$ for each cusp s of Γ. Then $\dim_C M^i_{k,m}$ is always finite, and

$$(1.4) \qquad \begin{aligned} M^2_{2,1}(\Gamma) &\xrightarrow{\cong} \{\text{holomorphic differential forms on } \bar{M}_\Gamma\}. \\ f &\longmapsto f(z)dz \end{aligned}$$

Note that $f(z)dz$ is holomorphic at a cusp s if and only if $\mathrm{ord}_s f(z) \geq 2$ since $\frac{dt_s(z)}{dz} = \text{const. } t_s^2$, whereas classically, $\frac{dq(z)}{dz} = 2\pi i q$. This explains the "double cuspidal" condition.

EXAMPLE 1.5 (cf. [14], [6], [9]). Let $\Gamma = \Gamma(1) = GL(2, A)$. Then there are three distinguished modular forms g, h, Δ of (weight, type) equal to $(q-1, 0)$, $(q+1, 1)$, $(q^2-1, 0)$, respectively, such that

(i) $\bigoplus_{k \geq 0} M_{k,0}(\Gamma) = C[g, \Delta]$ (D. Goss)

and

(ii) $M(\Gamma) := \bigoplus_{k \geq 0, m \in \mathbb{Z}/(q-1)} M_{k,m}(\Gamma) = C[g, h]$.

Moreover, $h^{q-1} = -\Delta$, and $j := g^{q+1}/\Delta$ identifies $\bar{M}_{\Gamma(1)}$ with $\mathbb{P}^1(C)$. There is precisely one cusp ($j = \infty$) and one elliptic point ($j = 0$).

Formulas for the numbers of cusps and elliptic points and for the genera $g(\Gamma) = \mathrm{genus}(\bar{M}_\Gamma)$, when $\Gamma = \Gamma_0(\mathfrak{n})$ or $\Gamma = \Gamma(\mathfrak{n})$ (= full congruence subgroup of divisor $\mathfrak{n}$), may be looked up in [6] or [8].

(1.6) Let $\mathcal{T}$ be the Bruhat-Tits tree of $PGL(2, K_\infty)$ (cf. [21]). It is a $(q+1)$-regular tree with set of vertices $X(\mathcal{T}) = G(K_\infty)/\mathcal{K}\, Z(K_\infty)$ and set of oriented edges $Y(\mathcal{T}) = G(K_\infty)/\mathcal{I} \cdot Z(K_\infty)$, where the canonical map from $Y(\mathcal{T})$ to $X(\mathcal{T})$ associates with each edge e its terminus $t(e)$. The group $G(K_\infty)$ acts from the left on $\mathcal{T}$. If $\Gamma \subset \Gamma(1)$ is a congruence subgroup, the quotient graph $\Gamma \backslash \mathcal{T}$ is the edge-disjoint union of a finite graph $(\Gamma \backslash \mathcal{T})^0$ and a finite number of half-lines h_s (i.e., h_s is isomorphic with $\bullet\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\ \ldots$) labelled by the cusps $s \in \Gamma \setminus \mathbb{P}^1(K)$ of Γ. There is a canonical surjective $G(K_\infty)$-equivariant map λ from Ω to $\mathcal{T}(\mathbb{Q})$ (= points of the realization $\mathcal{T}(R)$ of $\mathcal{T}$ with rational coordinates), which may be described as follows: Recall [13] that $\mathcal{T}(\mathbb{R})$ corresponds bijectively to the set of similarity classes of non-archimedean norms on the K_∞-vector space K^2_∞. Then $\lambda(z)$ is the class of the norm $\| \, . \, \|_z$, where $\|(u, v)\|_z := |uz + v|$. The map λ and the derived maps $\lambda_\Gamma : \Gamma \setminus \Omega \to (\Gamma \setminus \mathcal{T})(\mathbb{Q})$ may be used to describe admissible coverings of Ω and of $M_\Gamma = \Gamma \setminus \Omega$. E.g. for $v \in X(\mathcal{T})$ and $e \in Y(\mathcal{T})$, $\lambda^{-1}(v)$ is isomorphic with $\mathbb{P}^1(C)$ minus $(q+1)$ discs and $\lambda^{-1}(e)$ with an annulus $\{z \in C \mid |\pi| \leq |z| \leq 1\}$. The orientation of e corresponds to an orientation of $\lambda^{-1}(e)$.

(1.7) For any abelian group B, let $\underline{H}(\mathcal{T}, B)$ be the group of maps $\varphi: Y(\mathcal{T}) \to B$ subject to

(i) $\varphi(\bar{e}) = -\varphi(e)$ ($e \in Y(\mathcal{T})$ with inverse $\bar{e}$)

(ii) $\sum_{t(e)=v} \varphi(e) = 0$ ($v \in X(\mathcal{T})$).

We also put $\underline{H}_{!!}(\mathcal{T}, B)^\Gamma \subset \underline{H}_!(\mathcal{T}, B)^\Gamma \subset \underline{H}(\mathcal{T}, B)^\Gamma \subset \underline{H}(\mathcal{T}, B)$ for the subgroups of those φ that also satisfy (iii) + (iv′), (iii) + (iv), (iii), respectively, where

(iii) $\varphi(\gamma e) = \varphi(e)$ ($\gamma \in \Gamma$);

(iv) φ has compact (= finite) support modulo Γ;

(iv′) φ comes from $\underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma \otimes B \to \underline{H}_!(\mathcal{T}, B)^\Gamma$.

Intuitively, (iv) means that φ vanishes eventually on each of the half-lines h_s, and (iv′), that it vanishes on the whole of h_s. Here and in the sequel, we consider $\varphi \in \underline{H}(\mathcal{T}, B)^\Gamma$ as a B-valued function on the edges $Y(\Gamma \setminus \mathcal{T})$ of the graph $\Gamma \setminus \mathcal{T}$.

(1.8) These groups, for B equal to one of

$$\begin{array}{ccc} \mathbb{Z} & \hookrightarrow & \mathbb{C} \\ \downarrow & & \\ \mathbb{F}_p & \hookrightarrow & C\,, \end{array}$$

are closely related to the geometry of $\bar{M}_\Gamma$. Namely, let $\bar{\Gamma}$ be the group Γ^{ab} (Γ made abelian) modulo its torsion subgroup. It is a finitely generated free abelian group of rank $g = g(\Gamma)$, where

a) $g =$ rank $H_1(\Gamma \setminus \mathcal{T}, \mathbb{Z}) =$ rank $\underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$, and

b) $g =$ genus $\bar{M}_\Gamma$ (see (1.12)).

The first relation comes out as follows: Let Γ^* be the factor group of Γ modulo the subgroup generated by the elements of finite order. Then Γ^* is canonically isomorphic with the fundamental group of $\Gamma \setminus \mathcal{T}$ ([21] I Thm. 13, Cor. 1), which yields $(\Gamma^*)^{\mathrm{ab}} \xrightarrow{\cong} H_1(\Gamma \setminus \mathcal{T}, \mathbb{Z})$. Let $\varphi \in H_1(\Gamma \setminus \mathcal{T}, \mathbb{Z})$, regarded as a Γ-invariant function $\varphi: Y(\mathcal{T}) \to \mathbb{Z}$. For $e \in Y(\mathcal{T})$ put $n(e) :=$ index of $\Gamma \cap Z(K)$ in the stabilizer group Γ_e of e in Γ. Then $\varphi^*: e \mapsto n(e)\varphi(e)$ is a well-defined element of $\underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$, and $\varphi \mapsto \varphi^*: H_1(\Gamma \setminus \mathcal{T}, \mathbb{Z}) \to \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$ is injective with finite cokernel. Together, we have a map

$$\Gamma \to \Gamma^* \to (\Gamma^*)^{\mathrm{ab}} \xrightarrow{\cong} H_1(\Gamma \setminus \mathcal{T}, \mathbb{Z}) \hookrightarrow \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$$

that factors through $\bar{\Gamma}$. Let $j: \bar{\Gamma} \hookrightarrow \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$ be the induced map.

THEOREM 1.9. *If Γ is a Hecke congruence subgroup $\Gamma_0(\mathfrak{n})$ then j is an isomorphism.*

The proof is given in [17], using an explicit description of the graph $\Gamma \backslash \mathcal{T}$. This method neither works for more general groups Γ nor for more general function rings A than $\mathbb{F}_q[T]$. However, the statement remains true for congruence subgroups Γ which are p'-torsion free. In general, the cokernel of j is finite with order prime to p (see [10] for details).

(1.10) From now on, we assume that $\Gamma = \Gamma_0(\mathfrak{n})$ with some divisor $\mathfrak{n}$ of A. In [7] and [10] it is shown how $\underline{\mathrm{H}}_!(\mathcal{T}, \mathbb{C})^\Gamma$ may be interpreted as a space of automorphic forms in the sense of Jacquet and Langlands [15]. In particular, that space is equipped with

a) a "Petersson" scalar product (. , .);

b) Hecke operators $T_\mathfrak{m}$ for each divisor $\mathfrak{m}$ of A;

c) a canonical integral structure $\underline{\mathrm{H}}_!(\mathcal{T}, \mathbb{Z})^\Gamma$.

The scalar product comes from the L^2-norm on the discrete set $Y(\Gamma \setminus \mathcal{T})$, where the volume of an edge $\tilde{e} \in Y(\Gamma \setminus \mathcal{T})$ is given by $\frac{q-1}{2}\#(\Gamma_e)^{-1} = \frac{1}{2}\, n(e)^{-1}$ ($e \in Y(\mathcal{T})$ above $\tilde{e}$, see (1.8)).

The Hecke operator $T_\mathfrak{m}$ is derived from a correspondence on $Y(\Gamma \setminus \mathcal{T}) = \Gamma \setminus G(K_\infty)/\mathfrak{I} \cdot Z(K_\infty)$. Regarding the latter as an adelic double coset ([7] 3.3), $T_\mathfrak{p}$ for $\mathfrak{m} = \mathfrak{p}$ prime and coprime with $\mathfrak{n}$ may be defined as in [24] Ch. VI. For $\mathfrak{p}$ a divisor of $\mathfrak{n}$, the definition has to be slightly modified. One possible definition (compatibilities are checked in [10] Sec. 9) is as follows: Consider φ as a function on $G(K_\infty)$. Then

$$T_\mathfrak{m}\varphi(g) = \sum \varphi\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right),$$

where the sum is over $a, b, d \in A$ such that a, d are monic, $(ad) = \mathfrak{m}$, $(a, \mathfrak{n}) = 1$, and deg $b <$ deg d.

These Hecke operators $T_\mathfrak{m}$ have the usual properties: They commute mutually, satisfy $T_{\mathfrak{m}\cdot\mathfrak{m}'} = T_\mathfrak{m} \cdot T_\mathfrak{m}$ for $\mathfrak{m}$ and $\mathfrak{m}'$ coprime, for a prime $\mathfrak{p}$, $T_{\mathfrak{p}^i}$ is a polynomial with integral coefficients in $T_\mathfrak{p}$, and $T_\mathfrak{m}$ is Hermitian w.r.t. the Petersson product if $\mathfrak{m}$ is coprime with $\mathfrak{n}$. Furthermore, the integral structure $\underline{\mathrm{H}}_!(\mathcal{T}, \mathbb{Z})^\Gamma$ is stable under the $\mathbb{Z}$-algebra $\mathcal{H}$ generated by the $T_\mathfrak{m}$, and is integral w.r.t. the Petersson product.

(1.11) Now the correspondence $T_\mathfrak{m}$ may also be defined on $\bar{M}_\Gamma$, using the interpretation of M_Γ as a coarse moduli scheme for Drinfeld A-modules.

If e.g. $\mathfrak{m}$ and $\mathfrak{n}$ are coprime, $T_{\mathfrak{m}}$ associates with each point x on M_Γ (= rank-two Drinfeld module with a certain level structure) the collection of all $x' \in \bar{M}_\Gamma$ for which there exists a cyclic $\mathfrak{m}$-isogeny from x to x' (see [8] for details). Let $J_\Gamma = J_0(\mathfrak{n})$ be the Jacobian of $\bar{M}_\Gamma$ (recall that $\Gamma = \Gamma_0(\mathfrak{n})$). Thus $T_{\mathfrak{m}}$ induces an endomorphism of J_Γ, and of the ℓ-adic cohomology $H^1(\bar{M}_\Gamma, \mathbb{Q}_\ell)$. These operators have essentially the same properties as those listed for $T_{\mathfrak{m}}$, and are therefore labelled by the same symbol "$T_{\mathfrak{m}}$". The following deep result is due to Drinfeld (Thm. 2 of [4], specialized to our situation).

Theorem 1.12. *There is a canonical isomorphism between $H^1(\bar{M}_\Gamma, \mathbb{Q}_\ell)$ and $\underline{H}_!(\mathcal{T}, \mathbb{Q}_\ell)^\Gamma \otimes sp$. The isomorphism is compatible with the actions of a) the Hecke operators $T_{\mathfrak{m}}$ and b) the local Galois group $Gal(K_\infty^{\mathrm{sep}}/K_\infty)$ of K_∞.*

Here sp is the two-dimensional special ℓ-adic ($\ell \neq p$) representation of $\mathrm{Gal}(K_\infty^{\mathrm{sep}}/K_\infty)$ ([2] 3.1). It acts on $H^1(\bar{M}_\Gamma, \mathbb{Q}_\ell)$ since $\bar{M}_\Gamma$ is defined over K. We will not go into details of (1.12) and its interpretation as a reciprocity law ([4], [3], [10]), but will just use the principle that the splitting of $H^1(\bar{M}_\Gamma, \mathbb{Q}_\ell)$ and thus of J_Γ under the action of the Hecke algebra $\mathcal{H}$ is encoded in $\underline{H}_!(\mathcal{T}, \mathbb{Q}_\ell)^\Gamma = \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma \otimes \mathbb{Q}_\ell$, which is effectively computable.

(1.13) Now let $\underline{H}_!^{\mathrm{new}}(\mathcal{T}, \mathbb{Q})^\Gamma \subset \underline{H}_!(\mathcal{T}, \mathbb{Q})^\Gamma$ be the new part, i.e., the orthogonal complement of the different embeddings of $\underline{H}_!(\mathcal{T}, \mathbb{Q})^{\Gamma_0(\mathfrak{m})}$ into $\underline{H}_!(\mathcal{T}, \mathbb{Q})^\Gamma$, where $\mathfrak{m}$ runs through the strict divisors of $\mathfrak{n}$. Correspondingly, let $J_0^{\mathrm{new}}(\mathfrak{n})$ be the new part of $J_0(\mathfrak{n}) = J_\Gamma$, which is well-defined up to isogeny. Then also $H^1(J_0^{\mathrm{new}}(\mathfrak{n}), \mathbb{Q}_\ell) \xrightarrow{\cong} \underline{H}_!^{\mathrm{new}}(\mathcal{T}, \mathbb{Q}_\ell)^\Gamma \otimes \mathrm{sp}$.

We will consider elliptic curves E/K with conductor $\mathfrak{n}\cdot\infty$, and with split multiplicative reduction at ∞. By combining results of Grothendieck and Jacquet-Langlands, E gives rise to an automorphic representation, whose new vector $\varphi = \varphi_E$ appears in $\underline{H}_!^{\mathrm{new}}(\mathcal{T}, \mathbb{Q}_\ell)^\Gamma$. By the above, E appears as an isogeny factor of $J_0^{\mathrm{new}}(\mathfrak{n})$; conversely, every factor of dimension one of $J_0^{\mathrm{new}}(\mathfrak{n})$ is an elliptic curve defined over K with the properties stated above. For a detailed discussion, see [10], Section 8. Applying well-known facts from the theory of automorphic forms (notably "multiplicity one"), there result canonical bijections between the sets of

a) normalized Hecke eigenforms φ in $\underline{H}_!^{\mathrm{new}}(\mathcal{T}, \mathbb{Q})^\Gamma$ with rational eigenvalues;

b) one-dimensional isogeny factors of $J_0^{\mathrm{new}}(\mathfrak{n})$;

c) isogeny classes of elliptic curves E/K with $\mathrm{cond}(E) = \mathfrak{n} \cdot \infty$, and with split multiplicative reduction at ∞.

It is this statement which we label as the analogue of **STW** over the function field K. The relation between an eigenform φ and an associated elliptic curve E is as follows: If $\mathfrak{p}$ is a prime not dividing $\mathfrak{n}$, then the eigenvalue $\lambda_{\mathfrak{p}}$ of $T_{\mathfrak{p}}$ on φ satisfies

$$\lambda_{\mathfrak{p}} = q_{\mathfrak{p}} + 1 - \#(E(\mathbb{F}_{\mathfrak{p}})), \tag{1.14}$$

where $\mathbb{F}_{\mathfrak{p}} := A/\mathfrak{p}$ has $q_{\mathfrak{p}}$ elements and $E(\mathbb{F}_{\mathfrak{p}})$ is the group of $\mathbb{F}_{\mathfrak{p}}$-valued points of the reduction. But note that, so far, **STW**$/K$ is a sheer existence statement. Whereas c) $\rightsquigarrow$ a) is easy to obtain (by Grothendieck, the L-functions associated to E are polynomials in q^{-s}, so one catches φ_E by computing a finite number of reductions of E) and b) $\rightsquigarrow$ c) is more or less tautological, a) $\rightsquigarrow$ b) is the difficult part. In the next section I will show how to construct E as quotient of $J_0(\mathfrak{n})$, φ being given. Note that the condition "split multiplicative reduction" is equivalent to being a Tate curve; so E will be described through its Tate period at infinity.

2. Theta functions and the Jacobian

(2.1) Let f be a nowhere vanishing holomorphic function on Ω, $e = (v, w)$ an edge of $\mathcal{T}$, and

$$r(f)(e) := \log_q \left(\|f\|^{\mathrm{sp}}_{\lambda^{-1}(v)} / \|f\|^{\mathrm{sp}}_{\lambda^{-1}(w)} \right),$$

where $\|\ .\ \|^{\mathrm{sp}}_U$ is the spectral norm on the admissible subset U of Ω. Intuitively, $r(f)(e)$ measures the growth of f along e. Then $r(f)(e) \in \mathbb{Z}$ and $r(f)\colon e \mapsto r(f)(e)$ defines an element of $\underline{\mathrm{H}}(\mathcal{T}, \mathbb{Z})$, as results from the rigid residue formula [5]. More precisely, $r\colon f \mapsto r(f)$ yields a $G(K_\infty)$-equivariant exact sequence (*loc. cit.*)

$$0 \to C^* \to \mathcal{O}_\Omega(\Omega)^* \to \underline{\mathrm{H}}(\mathcal{T}, \mathbb{Z}) \to 0,$$

which is basic for what follows. Next, for each differential form ω on Ω and $e = (v, w)$, we let $\mathrm{res}(\omega, e)$ be the residue of ω w.r.t. the oriented annulus $\lambda^{-1}(e)$. Again from the residue theorem, $\mathrm{res}(\omega)\colon e \mapsto \mathrm{res}(\omega, e)$ is in $\underline{\mathrm{H}}(\mathcal{T}, C)$, and the diagram

$$\begin{array}{ccc} \mathcal{O}_\Omega(\Omega)^* & \xrightarrow{\ r\ } & \underline{\mathrm{H}}(\mathcal{T}, \mathbb{Z}) \\ \Big\downarrow {\scriptstyle d\log} & & \Big\downarrow {\scriptstyle \mathrm{red}} \\ \left\{ \begin{array}{l} \text{holomorphic} \\ \text{differentials on } \Omega \end{array} \right\} & \xrightarrow{\ \mathrm{res}\ } & \underline{\mathrm{H}}(\mathcal{T}, C) \end{array} \tag{2.2}$$

is commutative, where $d\log\colon f \mapsto \frac{df}{f}$ and red is reduction mod$p$.

(2.3) A *holomorphic theta function* for $\Gamma \subset \Gamma(1)$ is an invertible holomorphic function $u\colon \Omega \to C$ such that for each $\gamma \in \Gamma$ there exists $c_u(\gamma) \in C^*$ with $u(\gamma z) = c_u(\gamma)u(z)$, and which is holomorphic and non-zero at the cusps of Γ (see [10] Sec. 5; this makes sense in view of the functional equations of u). Let $\Theta_h(\Gamma) \subset \mathcal{O}_\Omega(\Omega)^*$ be the group of holomorphic theta functions. For $u \in \Theta_h(\Gamma)$, $d\log u = \frac{u'(z)}{u(z)}\,dz$ is Γ-invariant, which means that $u'(z)/u(z)$ lies in $M_{2,1}(\Gamma)$, and in fact, in $M_{2,1}^2(\Gamma)$. Next, we construct holomorphic theta functions for $\Gamma = \Gamma_0(\mathfrak{n})$. Let $\alpha \in \Gamma$, $\omega \in \Omega$ an arbitrary base point, and put

$$u_\alpha(z) = \prod_{\gamma\in\Gamma} \left(\frac{z-\gamma\omega}{z-\gamma\alpha\omega}\right).$$

Products of this type have been introduced and studied in a different context by Manin-Drinfeld and by Gerritzen-van der Put. The following theorem is largely due to Radtke [18].

THEOREM 2.4. *(i) The above product converges locally uniformly to an invertible function u_α on Ω that does not depend on the choice of $\omega \in \Omega$.*

(ii) u_α is a holomorphic theta function, whose multiplier $c_\alpha : \gamma \mapsto c_{u_\alpha}(\gamma)$ is a homomorphism $\Gamma \to C^*$ that only depends on the class of α in $\overline{\Gamma} := \Gamma^{\mathrm{ab}}/tor(\Gamma^{\mathrm{ab}})$.

(iii) The association $(\alpha,\beta) \mapsto c_\alpha(\beta)$ is symmetric and defines a symmetric bilinear map from $\bar\Gamma \times \bar\Gamma$ to $K_\infty^* \hookrightarrow C^*$.

The relationship between theta functions and automorphic forms is described by the following results, whose proofs are given in [10]. Let j and r be the maps introduced in (1.8) and (2.1).

THEOREM 2.5. *For $\alpha \in \Gamma$, $r(u_\alpha) = j(\alpha)$.*

In other words, the diagram

$$\begin{tikzcd} & \bar\Gamma \arrow[dl, "\bar u"'] \arrow[dr, "j"] & \\ \Theta_h(\Gamma)/C^* \arrow[rr, "\bar r"] & & \underline{H}_!(\mathcal{T},\mathbb{Z})^\Gamma \end{tikzcd},$$

whose arrows are induced from $\alpha \mapsto u_\alpha$ and $r\colon \mathcal{O}_\Omega(\Omega)^* \to \underline{\mathrm{H}}(\mathcal{T}, \mathbb{Z})$, is well-defined and commutative. Here we wrote $j(\alpha)$ for j (class of α in $\bar{\Gamma}$). Similar notation will be used in what follows.

Since $\ker(r) = C^*$ and j is bijective, we have

Corollary 2.6. *$\bar{u}$ and $\bar{r}$ are bijective.*

Let $M^2_{2,1}(\Gamma, \mathbb{F}_p)$ be the $\mathbb{F}_p$-subspace of $M^2_{2,1}(\Gamma)$ of forms f such that $f(z)dz$ has its residues in $\mathbb{F}_p$. Dimension considerations show the properties indicated in the commutative diagram

$$\begin{array}{ccc} \Theta_h(\Gamma)/C^* & \xrightarrow{\cong} & \underline{\mathrm{H}}_!(\mathcal{T}, \mathbb{Z})^\Gamma \\ \Big\downarrow{\scriptstyle \text{log derivative}} & & \Big\downarrow{\scriptstyle \text{red}} \\ M^2_{2,1}(\Gamma_1, \mathbb{F}_p) & \underset{\text{res}}{\xrightarrow{\cong}} & \underline{\mathrm{H}}_{!!}(\mathcal{T}, \mathbb{F}_p)^\Gamma \\ \Big\downarrow & & \Big\downarrow \\ M^2_{2,1}(\Gamma) & \underset{\text{res}}{\xrightarrow{\cong}} & \underline{\mathrm{H}}_{!!}(\mathcal{T}, C)^\Gamma \,. \end{array} \tag{2.7}$$

Here the upper vertical arrows are $u(z) \mapsto u'(z)/u(z)$ and reduction $\bmod p$ (both surjective), and the lower vertical arrows come from base extension $\mathbb{F}_p \rightsquigarrow C$.

Theorem 2.8. *For $\alpha, \beta \in \Gamma$, we have*

$$\log_q |c_\alpha(\beta)| = (j(\alpha), j(\beta))$$

(Petersson scalar product in $\underline{\mathrm{H}}_!(\mathcal{T}, \mathbb{Z})^\Gamma$).

Corollary 2.9. *The symmetric bilinear form on $\bar{\Gamma} \times \bar{\Gamma} : (\alpha, \beta) \mapsto \log_q |c_\alpha(\beta)|$ is positive definite, and*

$$\begin{aligned} \bar{c}\colon \bar{\Gamma} &\longrightarrow \mathrm{Hom}(\bar{\Gamma}, K^*_\infty) \\ \alpha &\longmapsto c_\alpha \end{aligned}$$

is injective.

We now proceed to construct the Jacobian $J_\Gamma = J_0(\mathfrak{n})$ of $\bar{M}_\Gamma$ ($\Gamma = \Gamma_0(\mathfrak{n})$). Let T_Γ/O_∞ be the split torus with character group $\bar{\Gamma}$, i.e., $T_\Gamma = \mathrm{Hom}(\bar{\Gamma}, \mathbb{G}_m)$). We may regard $\bar{\Gamma}$ via $\bar{c}$ as a subgroup of $T_\Gamma(K_\infty) = \mathrm{Hom}(\bar{\Gamma}, K^*_\infty)$. Now the assertion of the last corollary implies that the quotient $T_\Gamma/\bar{c}(\bar{\Gamma})$ (which always exists as an analytic group variety) is in fact an abelian variety (see [16], [11]).

Theorem 2.10. *The Jacobian J_Γ of $\bar{M}_\Gamma$ is canonically isomorphic with $T_\Gamma/\bar{c}(\bar{\Gamma})$. I.e., for each complete extension L of K_∞ in C, there is an exact sequence of L-analytic groups* $0 \to \bar{\Gamma} \xrightarrow{\bar{c}} \mathrm{Hom}(\bar{\Gamma}, L^*) \to J_\Gamma(L) \to 0$.

A proof of the theorem is given in [10] in a more general context, along with a precise description of the Abel-Jacobi map. In the next section, we shall use our description to construct strong Weil curves as quotients of J_Γ. Presently, we derive a consequence about the group of connected components of the Néron model of J_Γ. Let $\mathcal{J}$ be the Néron model of J_Γ at ∞, $\mathcal{J}^0 \subset \mathcal{J}$ the connected component of the identity, and $\phi(\Gamma) := \phi(J_\Gamma) := \mathcal{J}/\mathcal{J}^0$ the group of connected components (see [1]). The Petersson product $(\ .\ ,\ .\ )$ on $\underline{H} := \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$ is integral, which yields a canonical injection $i \colon \underline{H} \hookrightarrow \mathrm{Hom}(\underline{H}, \mathbb{Z})$.

Corollary 2.11. *The group $\phi(\Gamma)$ of connected components of the Néron model of J_Γ is canonically isomorphic with* $\mathrm{Hom}(\underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma, \mathbb{Z})/i(\underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma)$.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \mathrm{Hom}(\bar{\Gamma}, O_\infty^*) \arrow[d] & & \\
0 \arrow[r] & \bar{\Gamma} \arrow[r, "\bar{c}"] \arrow[d, "v \circ \bar{c}"] & \mathrm{Hom}(\bar{\Gamma}, K_\infty^*) \arrow[r] \arrow[d, "v"] & J_\Gamma(K_\infty) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Hom}(\bar{\Gamma}, \mathbb{Z}) \arrow[r, "="] & \mathrm{Hom}(\bar{\Gamma}, \mathbb{Z}) \arrow[r] \arrow[d] & 0\ , & \\
 & & 0 & &
\end{tikzcd}$$

where v is derived from the valuation v_∞. Since $\bar{\Gamma}$ is free, the middle column is exact. Identifying $\bar{\Gamma}$ with $\underline{H} = \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$ by means of j, $v \circ \bar{c}$ becomes the map i, and the snake lemma yields the exact sequence

$$0 \to \mathrm{Hom}(\bar{\Gamma}, O_\infty^*) \to J_\Gamma(K_\infty) \to \mathrm{Hom}(\underline{H}, \mathbb{Z})/i(\underline{H}) \to 0.$$

Recall that $\mathrm{Hom}(\bar{\Gamma}, O_\infty^*) = T_\Gamma(O_\infty)$. It follows from the construction given in [16] that $T_\Gamma(K_\infty)/\bar{c}(\bar{\Gamma}) \cdot T_\Gamma(O_\infty) = J_\Gamma(K_\infty)/\mathrm{Hom}(\bar{\Gamma}, O_\infty^*) \xrightarrow{\cong} \phi(J_\Gamma) = \phi(\Gamma)$, thus the right hand term of the above sequence equals $\phi(\Gamma)$. $\square$

The meaning of Corollary 2.11 is that $\phi(\Gamma)$ (or rather its size) is the "regulator" of the lattice $\underline{H}_!(\mathcal{T},\mathbb{Z})^\Gamma$ in the space $\underline{H}_!(\mathcal{T},\mathbb{R})^\Gamma$ of automorphic forms, equipped with the Petersson product. A similar assertion on modular symbols occurs in [23], Thm. 14.

3. The Weil Uniformization

In the present section, we construct the strong Weil curve associated to a primitive Hecke eigenform $\varphi \in \underline{H}_!^{\mathrm{new}}(\mathcal{T},\mathbb{Z})^\Gamma$ with rational eigenvalues. (φ primitive: $\varphi \notin n\underline{H}_!(\mathcal{T},\mathbb{Z})^\Gamma$ for $n > 1$.) We identify $\bar{\Gamma}$ with $\underline{H} = \underline{H}_!(\mathcal{T},\mathbb{Z})^\Gamma$ by means of j, writing both groups additively. Let $\Delta \subset K_\infty^*$ be the subgroup $\{c_\varphi(\alpha) \mid \alpha \in \Gamma\}$. As is shown in [10], Prop. 9.5.1, Δ has a subgroup $t^{\mathbb{Z}}$ of finite index, where $v_\infty(t) > 0$. Choosing $v_\infty(t)$ minimal, we have

$$\Delta = \mu_d \times t^{\mathbb{Z}}, \tag{3.1}$$

where d is a divisor of $q-1$ and $\mu_d \subset K_\infty^*$ the group of d-th roots of unity. In our case $A = \mathbb{F}_q[T]$ however, we can show:

Theorem 3.2. *In the above situation $d = 1$. In other words, the group Δ itself has the form $t^{\mathbb{Z}}$ with some uniquely determined $t \in K_\infty^*$, $v_\infty(t) > 0$.*

Forget the theorem for the moment, and consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \bar{\Gamma} \arrow[r] \arrow[d] & \mathrm{Hom}(\bar{\Gamma}, C^*) \arrow[r] \arrow[d, "ev"] & J_\Gamma(C) \arrow[r] \arrow[d, "pr_\varphi"] & 0 \\
0 \arrow[r] & \Delta \arrow[r] & C^* \arrow[r] & C^*/\Delta \arrow[r] & 0 ,
\end{tikzcd} \tag{3.3}$$

where $ev\colon f \mapsto f(\varphi)$ is evaluation on $\varphi \in \underline{H}_!(\mathcal{T},\mathbb{Z})^\Gamma = \bar{\Gamma}$. Note that raising to d-th powers gives $C^*/\Delta \xrightarrow{\cong} C^*/t^{d\mathbb{Z}} = \mathrm{Tate}(t^d)(C)$. By GAGA, the right hand arrow pr_φ is a morphism of abelian varieties, so the elliptic curve $\mathrm{Tate}(t^d)$ is an isogeny factor of J_Γ. For each prime $\mathfrak{p} \nmid \mathfrak{n}$, the $\mathfrak{p}$-th Hecke operator $T_\mathfrak{p}$ acts on $\mathrm{Tate}(t^d)$ as multiplication by $\lambda_\mathfrak{p}$ = eigenvalue of φ under $T_\mathfrak{p}$ (details being worked out in [10] Section 9). It follows (*loc. cit.*) that pr_φ and its target $E_\varphi := \mathrm{Tate}(t^d)$ are defined over K. From (1.14) we see that E_φ belongs to the isogeny class determined by φ. In fact, E_φ is the strong Weil curve, as is immediate from the construction. Now let P_0 be a fixed K-rational point on $\bar{M}_\Gamma = \Gamma_0(\mathfrak{n}) \setminus \Omega$, say, the cusp ∞, and $P \mapsto$ divisor class of $P - P_0$ the corresponding embedding

$i\colon \bar{M}_\Gamma \hookrightarrow J_\Gamma$. Then the strong Weil uniformization of E_φ is $p_\varphi := pr_\varphi \circ i$, described by

$$\begin{tikzcd}
\Omega \arrow[rr, "u_\varphi"] \arrow[d] & & C^* \arrow[d] \\
\Gamma \setminus \Omega \arrow[d, equal] & & C^*/\Delta \arrow[d, equal] \\
M_\Gamma(C) \hookrightarrow \bar{M}_\Gamma(C) \arrow[rr, "p_\varphi"] & & E_\varphi(C)\ .
\end{tikzcd} \tag{3.4}$$

If w is a coordinate on C^* and $\omega = \frac{dw}{w}$ the associated invariant differential on E_φ, we have

$$p_\varphi^*(\omega) = f(z)dz, \tag{3.5}$$

where $f(z) \in M_{2,1}^2(\Gamma)$ is the modular form $f(z) = \frac{u'_\varphi(z)}{u_\varphi(z)}$ (see (1.4)). But note that, unlike the classical case, knowledge of $f(z)$ does *not* suffice to determine E_φ, due to the loss of information caused by reduction mod p (compare (2.7), and Ex. (4.4)!).

(3.6) We are left to prove Theorem 3.2. In what follows, $\underline{\mathbb{H}} := \underline{\mathbb{H}}_!(\mathcal{T}, \mathbb{Z})^\Gamma = \bar{\Gamma}$ and $\varphi^\perp = \{\alpha \in \underline{\mathbb{H}} \mid (\varphi, \alpha) = 0\}$. Consider the following numbers:

$$\begin{aligned}
d \quad & \text{as given by (3.1), i.e., } d = \#(\mathrm{tor}(\Delta)); \\
n := \quad & \text{degree of the strong Weil uniformization } p_\varphi; \\
m := \quad & \min\{(\varphi, \alpha) > 0 \mid \alpha \in \underline{\mathbb{H}}\}; \\
r := \quad & [\underline{\mathbb{H}} \colon \mathbb{Z}\varphi \oplus \varphi^\perp] = \tfrac{(\varphi,\varphi)}{m} = \text{congruence number of } \varphi.
\end{aligned}$$

The theorem will come out by comparing these numbers. Some of our arguments are inspired by [19] and [25].

Let $p_\varphi^*\colon E_\varphi = \mathrm{Jac}(E_\varphi) \to \mathrm{Jac}(\bar{M}_\Gamma) = J_\Gamma$ be induced by Picard functoriality. Then p_φ^* is injective because p_φ is a strong Weil uniformization, and

$$pr_\varphi \circ p_\varphi^*\colon E_\varphi \to E_\varphi \text{ is multiplication by } n. \tag{3.7}$$

PROPOSITION 3.8. $n = d \cdot r$.

Proof. In what follows, working with tori and abelian varieties, we write down the C-valued points only. Let $E^\varphi = \operatorname{im}(p_\varphi^*) \subset J_\Gamma$. It is the subvariety of J_Γ that corresponds to the integrable subtorus $\operatorname{Hom}(\bar\Gamma/\varphi^\perp, C^*)$ of $\operatorname{Hom}(\bar\Gamma, C^*) = T_\Gamma(C)$. Hence $E^\varphi(C) = \operatorname{Hom}(\bar\Gamma/\varphi^\perp, C^*)/\Lambda$, where

$$\Lambda = \bar c(\bar\Gamma) \cap \operatorname{Hom}(\bar\Gamma/\varphi^\perp, C^*) = \{c_\alpha \mid c_\alpha|\varphi^\perp = 1\}.$$

The map $pr_\varphi \mid E^\varphi \colon E^\varphi \to E_\varphi$ is therefore given by

$$(*)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda & \longrightarrow & \operatorname{Hom}(\bar\Gamma/\varphi^\perp, C^*) & \longrightarrow & E^\varphi(C) & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow{\scriptstyle ev} & & \big\downarrow{\scriptstyle \mathrm{pr}_\varphi|E^\varphi} & & \\ 0 & \longrightarrow & \Delta & \longrightarrow & C^* & \longrightarrow & E_\varphi(C) & \longrightarrow & 0\,, \end{array}$$

where as usual, $ev(f) = f(\varphi)$. Now $ev \mid \Lambda$ is injective. For let c_α be in the kernel. Then $c_\alpha = 1$ on $\mathbb{Z}\varphi \oplus \varphi^\perp$, so c_α has values in the group of r-th roots of unity. But this means the theta function u_α is bounded and hence constant (i.e., $c_\alpha = 1$), since Ω is a Stein domain. Furthermore, $ev(\Lambda) = \{c_\alpha(\varphi)|c_\alpha|\varphi^\perp = 1\} = c_\varphi(\varphi)^{d\mathbb{Z}} = t^{\mathrm{rd}\mathbb{Z}}$ has index $d^2 \cdot r$ in $\Delta = \mu_d \times t^{\mathbb{Z}}$. For any natural number i, let i^* be its p-free part. We then obtain r^* for the order of $\ker(ev) = \operatorname{Hom}(\bar\Gamma/\mathbb{Z}\varphi + \varphi^\perp, C^*)$, and $n \cdot n^*$ for the order of $\ker(pr_\varphi \mid E^\varphi) = \ker(\text{mult. by } n)$. Chasing diagrams in $(*)$ yields $n \cdot n^* = d^2 \cdot r \cdot r^*$, which implies $n = d \cdot r$ as stated. □

Let $\mathcal{H}$ be the Hecke algebra on $\underline{\mathrm{H}} = \underline{\mathrm{H}}_!(\mathcal{T}, \mathbb{Z})^{\Gamma_0(\mathfrak{n})}$ as defined in (1.10). Note that it does *not* agree with the Hecke algebra considered in [10], for two reasons:

a) $\mathcal{H}$ acts on the whole of $\underline{\mathrm{H}}$, not only on its new part;

b) it also contains operators $T_\mathfrak{p}$ with $\mathfrak{p} \mid \mathfrak{n}$.

In particular, $\mathcal{H} \otimes \mathbb{Q}$ is in general not semisimple. We also regard $\mathcal{H}$ as an operator algebra on $J_\Gamma = J_0(\mathfrak{n})$, which yields inclusions

$$\mathcal{H} \hookrightarrow \operatorname{End} J_\Gamma \hookrightarrow \operatorname{End} \underline{\mathrm{H}} \hookrightarrow \operatorname{End} \underline{\mathrm{H}} \otimes \mathbb{Q}.$$

Let $e \in \mathcal{H} \otimes \mathbb{Q}$ be the idempotent that corresponds to $\varphi \in \underline{\mathrm{H}}$. Then

$$(3.9)\qquad \begin{aligned} r &= \text{denominator of } e \text{ in } \operatorname{End} \underline{\mathrm{H}} \text{ (i.e., } r = \text{least} \\ &\quad\ \text{natural number such that re} \in \operatorname{End}\underline{\mathrm{H}}) \end{aligned}$$

and

$$n = \text{denominator of } e \text{ in End } J_\Gamma. \tag{3.10}$$

Here (3.9) is trivial, and the (easy) proof of (3.10) may be found in [25], proof of Thm. 3. Let s be the denominator of e in $\mathcal{H}$. From the above inclusions, we have

$$r \mid n \mid s. \tag{3.11}$$

Next, consider the Fourier coefficients $c(\mathfrak{d}) = c_\psi(\mathfrak{d})$ for $\psi \in \underline{H}_!(\mathcal{T}, \mathbb{C})^\Gamma$, as introduced in [24] Ch. III. The arguments $\mathfrak{d}$ are divisors on K, i.e., $\mathfrak{d} = \mathfrak{f} \cdot \infty^i$, where $\mathfrak{f}$ is a finite divisor (= fractional ideal of A), and $c(\mathfrak{d})$ vanishes if $\mathfrak{d}$ fails to be non-negative. The "constant" Fourier coefficient c_0 of ψ vanishes by virtue of the cusp condition (1.7)(iv). The $c(\mathfrak{d})$ describe the restriction of ψ (regarded as a function on $GL(2, K_\infty)$) to the subgroup of matrices $\left(\begin{smallmatrix} x & y \\ 0 & 1 \end{smallmatrix}\right)$ in $GL(2, K_\infty)$. For the convenience of the reader, we give (without proof) the relevant formulas for $c(\mathfrak{d})$. These may be derived from the adelic formulas of [24] p. 20 by tedious but standard calculations.

Let $\pi = T^{-1}$ be the uniformizer at ∞, and write $\mathfrak{d} = \mathfrak{f} \cdot \infty^i$, where $\mathfrak{f}$ is generated by the monic element f of A. Let further $\eta: K_\infty \to \mathbb{C}^*$ be the character

$$\eta: \sum a_i \pi^i \longmapsto \eta_0 \circ \mathrm{Tr}(a_1),$$

where $\mathrm{Tr}: \mathbb{F}_q \to \mathbb{F}_p$ is the trace and η_0 is any nontrivial character of $\mathbb{F}_p$. Then

$$\text{(3.12)} \quad \text{(i)} \qquad \psi\left(\begin{pmatrix} \pi^j & , & y \\ 0 & , & 1 \end{pmatrix}\right) = \sum_{t \in K^*} c((t)\infty^{j-2})\eta(ty),$$

where

$$\text{(ii)} \quad c(\mathfrak{f} \cdot \infty^i) = c(\mathfrak{f}) q^{-i},$$

$$\text{(iii)} \quad c(\mathfrak{f}) = q^{-\deg \mathfrak{f} - 1} \sum_{u \in \pi O_\infty / \pi^{\deg \mathfrak{f} + 2} O_\infty} \psi\left(\begin{pmatrix} \pi^{\deg \mathfrak{f} + 2} & , & u \\ 0 & , & 1 \end{pmatrix}\right) \eta(-fu).$$

Note that by (i), the $c(\mathfrak{d})$ determine ψ since the matrices $\left(\begin{smallmatrix} \pi^j & , & y \\ 0 & , & 1 \end{smallmatrix}\right)$ represent modulo orientation all the edges $Y(\Gamma \backslash \mathcal{T}) = \Gamma \backslash G(K_\infty)/\mathcal{I} \cdot Z(K_\infty)$. Condition (ii) reflects the flow condition (ii) of (1.7). Hecke operators and Fourier coefficients are related as follows: If $\psi' = T_\mathfrak{m} \psi$ then

$$c_{\psi'}((1)) = q^{\deg \mathfrak{m}} c_\psi(\mathfrak{m}). \tag{3.13}$$

The fact that ψ is an eigenform for $T_{\mathfrak{p}}$ with eigenvalue $\lambda_{\mathfrak{p}}$ yields ($\mathfrak{p}$, $\mathfrak{f}$ coprime, $d := \deg \mathfrak{p}$)

$$c(\mathfrak{f} \cdot \mathfrak{p}^j) = c(\mathfrak{f}) \cdot \gamma_j, \tag{3.14}$$

where the γ_j are given by the power series identity

$$\sum_{j \geq 0} \gamma_j X^j = \begin{cases} (1 - q^{-d}\lambda_{\mathfrak{p}} X + q^{-d} X^2)^{-1} & \text{if } \mathfrak{p} \nmid \mathfrak{n} \\ (1 - q^{-d}\lambda_{\mathfrak{p}} X)^{-1} & \text{if } \mathfrak{p} \mid \mathfrak{n}. \end{cases}$$

In the last case, $\lambda_{\mathfrak{p}} = \pm 1$ if $\mathfrak{p} \,\|\, \mathfrak{n}$ and $\lambda_{\mathfrak{p}} = 0$ if $\mathfrak{p}^2 \mid \mathfrak{n}$. The sum in (3.12)(i) may be simplified to

$$\text{(3.12)(i}'\text{)} \quad \psi\left(\begin{pmatrix} \pi^j & , & y \\ 0 & , & 1 \end{pmatrix}\right) = \sum_{0 \leq k \leq j-2} q^{k+2-j} \sum_{f \in A, \deg f = k,\, f \text{ monic}} c(\mathfrak{f}) \nu(fy)$$

with $\nu(y) = -1$ if y has a term of order π in its π-expansion, and $\nu(y) = q-1$ if it has no term of order π. Similarly, the sum in (3.12)(iii) splits (besides the term corresponding to $u = 0$) into partial sums of type

$$\sum_{c \in \mathbb{F}_q^*} \psi\left(\begin{pmatrix} \pi^{\deg \mathfrak{f}+2} & , & cu \\ 0 & , & 1 \end{pmatrix}\right) \eta(-fcu).$$

But the double class of $\begin{pmatrix} \pi^{\deg \mathfrak{f}+2} & , & u \\ 0 & , & 1 \end{pmatrix}$ does not change if u is replaced by cu. Hence the value of that partial sum is simply

$$\psi\left(\begin{pmatrix} \pi^{\deg \mathfrak{f}+2} & , & u \\ 0 & , & 1 \end{pmatrix}\right) \nu(fu),$$

and $c(\mathfrak{f})$ is a linear combination with coefficients in $\mathbb{Z}[p^{-1}]$ of values of ψ.

Corollary 3.15. *The Fourier coefficients $c_\psi(\mathfrak{d})$ of $\psi \in \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$ lie in $\mathbb{Z}[p^{-1}]$. Conversely, if $\psi \in \underline{H}_!(\mathcal{T}, \mathbb{C})^\Gamma$ has its Fourier coefficients in $\mathbb{Z}[p^{-1}]$, then ψ also takes its values in $\mathbb{Z}[p^{-1}]$.*

Another trivial consequence of the above is

$$c_\psi((1)) = -\psi\left(\begin{pmatrix} \pi^2 & , & \pi \\ 0 & , & 1 \end{pmatrix}\right) \tag{3.16}$$

for $\psi \in \underline{H}_!(\mathcal{T}, \mathbb{C})^\Gamma$.

Now consider the bilinear pairing ($\underline{H} = \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$)

$$\begin{aligned} \mathcal{H} \times \underline{H} &\longrightarrow \mathbb{Z} \\ T, \psi &\longmapsto (T, \psi) := c_{T\psi}((1)) \end{aligned}$$

(Fourier coefficient of $T\psi$ w.r.t. the trivial divisor (1)).

THEOREM 3.17. *The pairing is non-degenerate and becomes a perfect pairing after tensoring with* $\mathbb{Z}[p^{-1}]$.

In other words, the determinant of $(\ .\ ,\ .\ )$ with respect to any $\mathbb{Z}$-bases of $\mathcal{H}$ and $\underline{\mathrm{H}}$ is a power of p.

Proof.

1) Let $\psi \in \underline{\mathrm{H}}$ be in the right kernel. Then $0 = c_{T_{\mathfrak{m}}\psi}((1)) = q^{\deg \mathfrak{m}} c_\psi(\mathfrak{m})$ for all positive finite divisors $\mathfrak{m}$, which gives $\psi = 0$ by (3.12)(i$'$).

2) Let $T \in \mathcal{H}$ be in the left kernel. Then for all $\psi \in \underline{\mathrm{H}}$ and all positive finite divisors $\mathfrak{m}$, $0 = c_{TT_{\mathfrak{m}}\psi}((1)) = c_{T_{\mathfrak{m}}T\psi}((1)) = q^{\deg \mathfrak{m}} c_{T\psi}(\mathfrak{m})$, i.e., $T\psi = 0$ and therefore $T = 0$.

3) Let now $\ell \neq p$ be a prime number and $\psi \in \underline{\mathrm{H}}$ such that $(T, \psi) \equiv 0 \pmod{\ell}$ for all $T \in \mathcal{H}$. Then for all $\mathfrak{m}$, $(T_{\mathfrak{m}}, \psi) = c_{T_{\mathfrak{m}}\psi}((1)) = q^{\deg \mathfrak{m}} c_\psi(\mathfrak{m})$, and all the $c_\psi(\mathfrak{m})$ are congruent to $0 \pmod{\ell}$. By (3.12)(i$'$) this implies that ψ is divisible by ℓ.

The theorem follows from 1, 2 and 3. $\square$

LEMMA 3.18. *The number s/r (see (3.11)) divides the determinant of the pairing $(\ .\ ,\ .\ )$ of (3.17).*

Proof. $se \in \mathcal{H}$ is primitive and $(se)(\psi) \in (s/r)\underline{\mathrm{H}}$ for all $\psi \in \underline{\mathrm{H}}$, so $(se, \psi) \in (s/r)\mathbb{Z}$. $\square$

Proof of Theorem 3.2. The number d equals n/r by (3.8), so divides the p-power s/r ((3.17) + (3.18)). On the other hand, d is a divisor of $q-1$. $\square$

We state separately what has been proven along with Theorem 3.2. Let φ be a primitive Hecke newform with rational eigenvalues, and let (φ, φ), m and $r = (\varphi, \varphi)/m$ be the numbers defined in (3.6).

COROLLARY 3.19. *The number m is the pole order $-v_\infty(j_E)$ of the j-invariant j_E of the strong Weil curve E associated with φ.*

COROLLARY 3.20. *The congruence number r equals the degree n of the strong Weil uniformization of E.*

Note that the last result is stronger than the corresponding result over $\mathbb{Q}$, which seems to be established for prime conductors only ([25] Thm. 3).

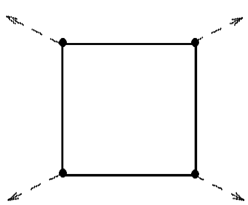
4. Examples

In the following examples we always take $q = 2$, thus $\Gamma = \Gamma_0(\mathfrak{n}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in GL(2, A) \,\middle|\, n|c \right\}$ with some $n \in A = \mathbb{F}_2[T]$, $\mathfrak{n} = (n)$. The Petersson product on $\underline{H}_!(\mathcal{T}, \mathbb{R})^\Gamma$ will be derived from the L^2-norm on $Y(\Gamma \setminus \mathcal{T})$, where each pair of inverse edges contributes the volume 1 (compare (1.10)).

Remark 4.1. Under our hypotheses, two isogeneous elliptic curves E, E' over $K = \mathbb{F}_2(T)$, which are Tate curves over K_∞, and whose j-invariants j, j' satisfy $v_\infty(j) = v_\infty(j')$ are in fact isomorphic.

The number $g = \text{genus}(\bar{M}_\Gamma) = \text{rank } \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$ is calculated in [6], see also [8]. If deg $n \le 2$, $g = 0$, so there are no elliptic curves E/K with conductor $\mathfrak{n} \cdot \infty$ and $E \times K_\infty$ = Tate curve. Up to coordinate changes in T, there are precisely two conductors n such that $g = 1$, given by $n = T^3$ and $n = T^2(T-1)$.

EXAMPLE 4.2. Let $n = T^3$. The graph $\Gamma \setminus \mathcal{T}$ looks



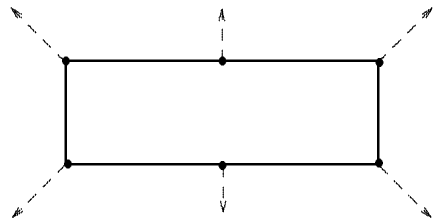
where $\dashrightarrow$ indicates a cusp = infinite half-line (see [7] Sec. 5).

If φ is a basis vector of $\underline{H} = \underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$ then $m = (\varphi, \varphi) = 4$. Now the curve

$$E\colon\ Y^2 + TXY = X^3 + T^2$$

with discriminant $\Delta = T^8$ and invariant $j = T^4$ has split multiplicative reduction at ∞ and conductor T^3, as a routine application of Tate's algorithm [22] shows. Hence E is the strong Weil curve associated with φ, i.e., $E = \bar{M}_\Gamma$ in this case.

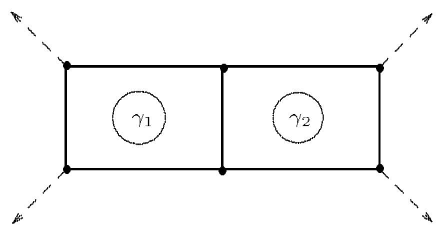
EXAMPLE 4.3. Let $n = T^2(T-1)$. The graph $\Gamma \setminus \mathcal{T}$ is (*loc. cit.*)



If $\underline{H} = \mathbb{Z}\varphi$ then $m = (\varphi, \varphi) = 6$. The strong Weil curve is

$$\begin{aligned} &E\colon \quad Y^2 + TXY + TY = X^3 \\ &\text{with} \quad \Delta = T^4(T-1)^2 \quad \text{and} \quad j = T^8/(T-1)^2. \end{aligned}$$

EXAMPLE 4.4. Let now $n = T(T^2 + T + 1)$. The graph $\Gamma \setminus \mathcal{T}$ looks (*loc. cit.*)



Let γ_1 and γ_2 be the cycles of length 4, oriented counterclockwise, and w_1, w_2 the involutions attached to the prime divisors $\mathfrak{p}_1 = (T)$ and $\mathfrak{p}_2 = (T^2 + T + 1)$ of $\mathfrak{n}$. The Hecke eigenforms $\varphi_1 := \gamma_1 + \gamma_2$, $\varphi_2 := -\gamma_1 + \gamma_2$ in $\underline{H}$ satisfy

$$\begin{aligned} w_1(\varphi_1) &= \varphi_1 = -w_2(\varphi_1) \\ w_2(\varphi_2) &= \varphi_2 = -w_1(\varphi_2). \end{aligned}$$

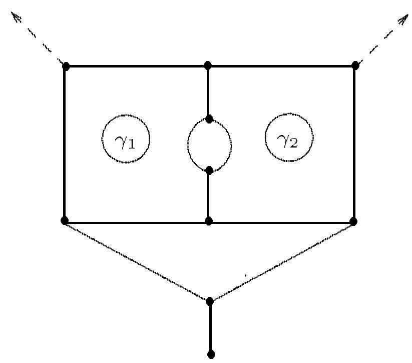
The Jacobian J_Γ is isogeneous to $E_1 \times E_2$, where E_i is the elliptic curve $\bar{M}_\Gamma/w_i$. The numbers m_i are $m_1 = (\varphi_1, \gamma_1) = 3$ and $m_2 = (\varphi_2, \gamma_2) = 5$, the congruence numbers r_i equal 2 for φ_1 and φ_2. Equations are given by

$$\begin{aligned} E_1\colon \quad & Y^2 + (T+1)XY + Y = X^3 + T(T^2 + T + 1) \\ & \Delta = T^3(T^2 + T + 1)^3, \; j = (T+1)^{12}/\Delta, \end{aligned}$$

$$\begin{aligned} E_2\colon \quad & Y^2 + (T+1)XY + Y = X^3 + X^2 + T + 1 \\ & \Delta = T^5(T^2 + T + 1), \; j = (T+1)^{12}/\Delta. \end{aligned}$$

E_1 and E_2 intersect in J_Γ in their common subgroup scheme of two-division points. Since $p = 2$ itself is a congruence prime, the two weight two modular forms u_i'/u_i ($u_i := u_{\varphi_i}$, $i = 1, 2$) agree.

EXAMPLE 4.5. Put $n = T^4 + T^3 + 1$, which is prime in $\mathbb{F}_2[T]$. Here $\Gamma \setminus \mathcal{T}$ is



The canonical involution w acts as the reflexion at the middle axis, and it is easy to see that the $+$-space of w in $\underline{H}_!(\mathcal{T}, \mathbb{R})^\Gamma$ has dimension one with basis vector $\varphi := \gamma_1 - \gamma_2$, which is primitive for $\underline{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$. Now $(\varphi, \varphi) = 16$ and $m = (\varphi, \gamma_1) = 8$, so $r = \deg p_\varphi = 2$ ($p_\varphi \colon \bar{M}_\Gamma \to \bar{M}_\Gamma / w = E$ is the projection). The curve E is given by

$$E\colon Y^2 + TXY + Y = X^3 + X^2$$
$$\Delta = T^4 + T^3 + 1,\ j = T^{12}/\Delta.$$

The Birch-Swinnerton-Dyer conjecture predicts the rank 1 for $E(K)$; in fact, $P = (0, 0) \in E(K)$ has infinite order.

A complete list of strong Weil curves over $\mathbb{F}_2(T)$ for conductors n of degree ≤ 4, along with the calculation of their relevant arithmetic invariants, has been given by A. Schweizer [20]. That list can be extended without difficulty to higher degrees of n and larger constant fields $\mathbb{F}_q$.

The philosophy behind these examples is as follows: Calculations on $\underline{H}$ not only yield the numbers $m = -v_\infty(j_E)$ and $r = \deg p_E$ for the strong Weil curve E in a class, but also provide insight into the position of E "on the modular curve $\bar{M}_\Gamma$". This means that e.g. the image of cusps, elliptic points, Heegner points... of $\bar{M}_\Gamma$ in E can be determined, and also the action of Atkin-Lehner involutions and any other geometric data that exist on $\bar{M}_\Gamma$. Obviously, this enhances the toolbox of methods available for a deeper study of elliptic curves over function fields.

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