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Elasticity of factorizations in atomic monoids and integral domains

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Résumé — L'élasticité $\rho(R)$ d'un anneau d'intégrité atomique R est définie par : $\rho(R) = \sup\ \{m/n | u_1 \cdots u_m = u_1 \cdots v_n$ pour u_i et v_j irréductibles dans $R]$. Nous étudions ici l'élasticité des anneaux d'intégrité noethériens au moyen des invariants plus fins $\mu_m(R)$ définis par :
$\mu_m(R) = \sup\ \{n | u_1 \cdots u_m = u_1 \cdots v_n$ pour u_i et v_j irréductibles dans $R\}$
Le résultat principal que nous donnons permet de caractériser les anneaux d'entiers des corps de nombres qui ont une élasticité finie. Chemin faisant nous obtenons une série de résultats sur les invariants μ_m et ρ des monoïdes et des anneaux d'intégrité qui ont un intérêt propre.

Abstract — For an atomic domain R, its elasticity $\rho(R)$ is defined by :
$\rho(R) = \sup\{m/n \mid u_1 \cdot \ldots \cdot u_m = v_1 \cdot \ldots \cdot v_n$ for irreducible $u_j, v_i \in R\}$.
We study the elasticity of one-dimensional noetherian domains by means of the more subtle invariants $\mu_m(R)$ defined by :
$\mu_m(R) = \sup\{n \mid u_1 \cdot \ldots \cdot u_m = v_1 \cdot \ldots \cdot v_n$ for irreducible $u_j, v_i \in R\}$.
As a main result we characterize all orders in algebraic number fields having finite elasticity. On the way, we obtain a series of results concerning the invariants μ_m and ρ for monoids and integral domains which are of independent interest.

Introduction

An integral domain R is called *atomic* if every non-zero non-unit of R possesses a factorization into a product of (finitely many) irreducible elements of R. We are interested in the deviation of R from being factorial. One possible measure of this deviation is the elasticity $\rho(R)$, defined by :

$$\rho(R) = \sup \left\{ \frac{m}{n} \mid u_1 \cdot \ldots \cdot u_m = v_1 \cdot \ldots \cdot v_n \text{ for irreducible } u_j, v_i \in R \right\} .$$

Clearly, $\rho(R) \in [1, \infty]$, and if R is factorial, then $\rho(R) = 1$.

The concept of elasticity was introduced by R. J. Valenza [20] for rings of integers in algebraic number fields. Using a different terminology, the elasticity of Dedekind domains was investigated by J. L. Steffan [19]. In

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a systematic way, the elasticity of various classes of integral domains was studied in [1], [2] and [3].

If R is the ring of integers of an algebraic number field, then $\rho(R)$ depends only on the class group of R (cf. Corollary 1). More generally, if R is a Krull domain, then $\rho(R)$ depends only on the pair (G, G_0), where G is the divisor class group of R and G_0 is the set of all divisor classes which contain prime divisors (cf. the Remark after Theorem 2).

If R is an order in an algebraic number field, then $\rho(R) = \infty$ may occur. A necessary and sufficient condition for $\rho(R) < \infty$ is given in Corollary 5. In Theorem 5 we produce estimates for $\rho(R)$, depending on the class group $G = \mathrm{Pic}(R)$ and on the "local" elasticities $\rho(R_\mathfrak{p})$ for primes $\mathfrak{p}$ dividing the conductor of R.

Since factorization properties of a domain only depend on its multiplicative structure, it suggests itself to investigate them in a purely multiplicative context. Thus we derive and formulate our main results in the context of commutative and cancellative monoids. This has the advantage of being more general and – what is more important – of revealing the combinatorial structure of factorization properties. Even though most notions and results of the paper concern monoids, the emphasis is on their ring theoretical applications.

§ 1 Preliminaries; μ_m and ρ

Throughout this paper, a *monoid* H is a multiplicatively written commutative monoid satisfying the cancellation law, with unit element $1 \in H$. We denote by $H^\times$ the group of invertible elements of H; H is called *reduced* if $H^\times = \{1\}$. If H_1 and H_2 are monoids, we denote by $H_1 \times H_2$ their direct product, and we view H_1 and H_2 as submonoids of $H_1 \times H_2$, so that every $a \in H_1 \times H_2$ has a unique decomposition $a = a_1 a_2$, where $a_1 \in H_1$ and $a_2 \in H_2$. For a monoid H, we use the notions of divisibility theory in H as introduced in [17; 2.14].

By a *factorization* of an element $a \in H \setminus H^\times$ we mean a representation of the form $a = u_1 \cdot \ldots \cdot u_r$, where $r \geq 1$ and $u_1, \ldots, u_r \in H$ are irreducible; we call r the *length* of that factorization, we denote by $\mathcal{L}^H(a) \subset \mathbb{N}$ the set of lengths of factorizations of a, and we set

$$l^H(a) = \min \mathcal{L}^H(a), \quad L^H(a) = \sup \mathcal{L}^H(a) \in \mathbb{N} \cup \{\infty\} ;$$

finally, we call

$$\rho^H(a) = \frac{L^H(a)}{l^H(a)} \in \mathbb{Q}_{\geq 1} \cup \{\infty\}$$

the *elasticity of* a *(in* H *).* For $a \in H^\times$, we set $\rho^H(a) = 1$.

A monoid H is called *atomic* if every $a \in H \setminus H^\times$ possesses a factorization. For an atomic monoid H, we call

$$\rho(H) = \sup\{\rho^H(a) \mid a \in H\} \in \mathbb{R}_{\geq 1} \cup \{\infty\}$$

the *elasticity* of H; we say that H has *accepted elasticity* if $\rho(H) = \rho^H(a) < \infty$ for some $a \in H$.

For an integral domain R, we denote by $R^\bullet = R \setminus \{0\}$ its multiplicative monoid. We set $\mathcal{L}^R = \mathcal{L}^{R^\bullet}$, $L^R = L^{R^\bullet}$, $l^R = l^{R^\bullet}$, $\rho^R = \rho^{R^\bullet}$, and we call $\rho(R) = \rho(R^\bullet)$ the *elasticity of* R. We say that R has *accepted elasticity* if $R^\bullet$ has.

For an atomic monid H and $m \in \mathbb{N}$, we set

$$\mu_m(H) = \sup\{L^H(a) \mid a \in H \setminus H^\times,\ m \in \mathcal{L}^H(a)\}\ ,$$

$$\mu_m^*(H) = \sup\{L^H(a) \mid a \in H \setminus H^\times,\ m = l^H(a)\}\ ;$$

the invariants $\mu_m(H)$ were introduced in [12] and also investigated in [7] and [16]. They are connected with the elasticity as follows.

Proposition 1. *Let* H *be an atomic monoid,* $H \neq H^\times$.

i) *For every* $m \in \mathbb{N}$ *we have*

$$\mu_m^*(H) \leq \mu_m(H) = \sup\{L^H(a) \mid a \in H \setminus H^\times,\ m \geq l^H(a)\}\ ;$$

if H *contains a prime element, then* $\mu_m^*(H) = \mu_m(H)$.

ii) *We have*

$$\rho(H) = \sup\Big\{\frac{\mu_m^*(H)}{m} \,\Big|\, m \in \mathbb{N}\Big\} = \sup\Big\{\frac{\mu_m(H)}{m} \,\Big|\, m \in \mathbb{N}\Big\} = \lim_{m \to \infty} \frac{\mu_m(H)}{m}\ .$$

iii) *The following assertions are equivalent:*

a) H *has accepted elasticity.*

b) *There exists some* $N \in \mathbb{N}$ *such that* $\rho(H) = \mu_{Nm}(H)/Nm$ *for all* $m \in \mathbb{N}$.

c) $\rho(H) = \mu_m(H)/m$ *for some* $m \in \mathbb{N}$.

Proof. **i)** By definition,

$$\mu_m^*(H) \leq \mu_m(H) \leq \overline{\mu_m} = \sup\big\{L^H(a) \,\big|\, a \in H \setminus H^\times,\ m \geq l^H(a)\big\}\ .$$

If $a \in H \backslash H^\times$, $m \ge l^H(a)$ and $N = L^H(a)$, let $a = u_1 \cdot \ldots \cdot u_s$ be a factorization of a of length $s \le m$, let $u_{s+1}, \ldots, u_m \in H$ be any irreducible elements, and set $a' = a u_{s+1} \cdot \ldots \cdot u_m$. Then we obtain $m \in \mathcal{L}^H(a')$ and $N \le N + m - s \le L^H(a') \le \mu_m(H)$, which implies $\overline{\mu_m} \le \mu_m(H)$.

Now let $p \in H$ be a prime element, $a \in H$, $m \in \mathcal{L}^H(a)$ and $N = L^H(a)$. If $l^H(a) = s \le m$, then $l^H(ap^{m-s}) = m$, $L^H(ap^{m-s}) = N + m - s$ and consequently $N \le N + m - s \le \mu_m^*(H)$, which implies $\mu_m(H) \le \mu_m^*(H)$.

ii) The first equality holds by definition; **i)** implies $\mu_m(H) = \max\{\mu_j(H) \mid 1 \le j \le m\}$, and therefore the second equality holds.

For the proof of the limit assertion, observe that $\mu_{n+r}(H) \ge \mu_n(H) + r$ and $\mu_{nr}(H) \ge r\mu_n(H)$ for all $n, r \in \mathbb{N}$. Let $A < \rho(H)$ be a real number and $N \in \mathbb{N}$ such that $\mu_N(H) > A$. For $n \in \mathbb{N}$, $n \ge N$, set $n = Nq + r$, where $q, r \in \mathbb{N}_0$, $r < N$, and obtain

$$\frac{\mu_n(H)}{n} \ge \frac{q\mu_N(H) + r}{qN + r} \ge \frac{q\mu_N(H) + N - 1}{qN + N - 1} ,$$

which implies $\mu_n(H)/n > A$ for all sufficiently large n; thus the limit assertion holds true.

iii) It suffices to prove that **a)** implies **b)**. Let $a \in H$ be such that $\rho(H) = \rho^H(a)$ and $N = l^H(a)$. Then we have $mL^H(a) \le L^H(a^m) \le \mu_{mN}(H)$, since $mN \in \mathcal{L}^H(a^m)$, and hence

$$\frac{\mu_{mN}(H)}{mN} \ge \frac{L^H(a)}{l^H(a)} = \rho^H(a) = \rho(H) \ge \frac{\mu_{mN}(H)}{mN}$$

for all $m \in \mathbb{N}$. $\square$

Proposition 1 has a counterpart for minimal lengths. For an atomic monoid $H \ne H^\times$ and $m \in \mathbb{N}$, we define the following quantities:

$$\begin{aligned} \sigma_m(H) &= \min\bigl\{ l(a) \bigm| a \in H\backslash H^\times,\ m \in \mathcal{L}^H(a) \bigr\} , \\ \sigma'_m(H) &= \min\bigl\{ l^H(a) \bigm| a \in H\backslash H^\times,\ m \le L^H(a) \bigr\} , \\ \sigma''_m(H) &= \min\bigl\{ l^H(a) \bigm| a \in H\backslash H^\times,\ m = L^H(a) \bigr\} ; \end{aligned}$$

they are connected with the elasticity as follows.

PROPOSITION 2. *Let H be an atomic monoid, $H \neq H^\times$. Then we have $\sigma'_m(H) \le \sigma_m(H) \le \sigma''_m(H)$ for all $m \in \mathbb{N}$, and*

$$\frac{1}{\rho(H)} = \inf\Big\{\frac{\sigma_m(H)}{m} \,\Big|\, m \in \mathbb{N}\Big\} = \inf\Big\{\frac{\sigma'_m(H)}{m} \,\Big|\, m \in \mathbb{N}\Big\} = \inf\Big\{\frac{\sigma''_m(H)}{m} \,\Big|\, m \in \mathbb{N}\Big\}$$

$$= \lim_{m\to\infty} \frac{\sigma_m(H)}{m} = \lim_{m\to\infty} \frac{\sigma'_m(H)}{m} \ .$$

Proof. By definition, we have $\rho(H)^{-1} = \inf\{\sigma''_m(H)/m \mid m \in \mathbb{N}\}$, $\rho(H)^{-1} \le \sigma'_m(H)/m$ and $\sigma'_m(H) \le \sigma_m(H) \le \sigma''_m(H)$ for all $m \in \mathbb{N}$. Therefore it remains to prove the limit assertions.

For any $n, r \in \mathbb{N}$, we have $\sigma_{n+r}(H) \le \sigma_n(H) + r$ and $\sigma_{nr}(H) \le r\sigma_n(H)$. Let $A > \rho(H)^{-1}$ be a real number and $N \in \mathbb{N}$ such that $\mu_N(H) < A$. For $n \in \mathbb{N}$, $n \ge N$, set $n = Nq+r$, where $q, r \in \mathbb{N}_0$, $r < N$, and obtain

$$\frac{\sigma_n(H)}{n} \le \frac{q\sigma_N(H) + r}{qN + r} \le \frac{q\sigma_N(H) + N - 1}{qN + N - 1} \ ,$$

which implies $\sigma_n(H)/n < A$ for all sufficiently large n. Thus the limit assertion for $\sigma_m(H)/m$ follows; that for $\sigma'_m(H)/m$ is proved in the same way. □

The following finiteness result is of central importance.

THEOREM 1. *Let H be a monoid such that $H/H^\times$ is finitely generated. Then H has accepted elasticity.*

Proof. [3; Theorem 7].

Remark. Let H be a monoid having tame factorizations of degree N as defined in [7]; then we have

$$\rho(H) \le N \ .$$

This follows from [7], Remark 2 on p. 688, where the more precise result

$$\mu_H(m) \le 1 + (m-1)N$$

is asserted. Since every finitely generated monoid has tame factorizations [7; Prop. 2], this implies again $\rho(H) < \infty$ for every finitely generated monoid H.

§ 2 Divisor homomorphisms and coproducts

Recall from [11], Definition 2.3 that a monoid homomorphism $\varphi : H \to S$ is called a *divisor homomorphism* if $x, y \in H$ and $\varphi(x) \mid \varphi(y)$ implies $x|y$. If $\varphi : H \to S$ is a divisor homomorphism, then φ induces an isomorphism $\varphi^* : H/H^\times \xrightarrow{\sim} \varphi H$, and $\varphi H \subset S$ is a saturated submonoid, i. e., $a, b \in \varphi H,\ c \in S$ and $a = bc$ implies $c \in \varphi H$. The factor monoid $S/\varphi H$ consists of all congruence classes for the congruence $\sim_\varphi$, defined by $a \sim_\varphi b$ if $a\varphi(x) = b\varphi(y)$ for some $x, y \in H$; its quotient group is denoted by $\mathcal{C}(\varphi)$ and is called the *class group* of φ. $\mathcal{C}(\varphi)$ is an abelian group; we usually write it additively, and for $a \in S$ we denote by $[a] \in S/\varphi H \subset \mathcal{C}(\varphi)$ the class of a. Since $\varphi H \subset S$ is saturated, we have $[a] = 0$ if and only if $a \in \varphi H$.

The class group of a divisor homomorphism admits the following group-theoretical description. For a monoid H, let Q be a quotient group of H, and call $\mathbf{G}(H) = Q/H^\times$ the *group of divisibility* of H (corresponding with the notions in ring theory). Clearly, $\mathbf{G}$ is a functor from monoids to abelian groups. If $\varphi : H \to S$ is a divisor homomorphism, then $\mathbf{G}(\varphi) : \mathbf{G}(H) \to \mathbf{G}(S)$ is a group monomorphism, and the natural map $S \to \mathbf{G}(S)$ induces a functorial group isomorphism $\mathcal{C}(\varphi) \xrightarrow{\sim} \operatorname{coker}\mathbf{G}(\varphi)$.

For any set P, let $\mathcal{F}(P)$ be the multiplicative free abelian monoid with basis P. Let G be an additive abelian group and $G_0 \subset G$ a subset; for an element $S = g_1 \cdot \ldots \cdot g_s \in \mathcal{F}(G_0)$ we call $\sigma(S) = s \in \mathbb{N}_0$ the *size* and $\iota(S) = g_1 + \cdots + g_s \in G$ the *content* of S; $\sigma : \mathcal{F}(G_0) \to \mathbb{N}_0$ and $\iota : \mathcal{F}(G_0) \to G$ are monoid homomorphisms. The monoid

$$\mathcal{B}(G_0) = \{S \in \mathcal{F}(G_0) \mid \iota(S) = 0\}$$

is called the *block monoid over* G_0; it is a Krull monoid and was investigated in [9] and [10]. *Davenport's constant* $D(G_0)$ is defined as $D(G_0) = 0$ if $\mathcal{B}(G_0) = \{1\}$, and

$$D(G_0) = \sup\{\sigma(B) \mid B \in \mathcal{B}(G_0) \text{ irreducible }\} \in \mathbb{N} \cup \{\infty\}$$

otherwise. If G_0 is finite, then $D(G_0) < \infty$; if $G_0 = \{0\}$, then $D(G_0) = 1$; if $G_0 \neq \{0\}$ and $\mathcal{B}(G_0) \neq \{1\}$, then $D(G_0) \geq 2$. If $\#G = \infty$, then $D(G) = \infty$ by [9; Prop. 2]. For a survey and recent results concerning $D(G)$, see [13].

THEOREM 2. *Let* $\varphi : H \to S$ *be a divisor homomorphism of atomic monoids,* $H \neq H^\times$ *and* $G = \mathcal{C}(\varphi)$. *Let* G_0 *be the set of all classes* $g \in G$ *containing irreducible elements of* S.

i) *For all* $m \in \mathbb{N}$, *we have* $\mu_m(H) \le \mu_{mD(G_0)}(S)$, *and*

$$\rho(H) \le D(G_0)\rho(S) .$$

ii) *Suppose that* $S = \mathcal{F}(P)$ *for some set* P *and* $D(G_0) \ge 2$. *Then we have*

$$\mu_m(H) \le \frac{1}{2}D(G_0)m$$

for all $m \in \mathbb{N}$, *with equality if* $m \equiv 0 \mod 2$ *and* $G_0 = -G_0$. *Moreover, we have*

$$\rho(G) \le \frac{1}{2}D(G_0) ,$$

with equality if $G_0 = -G_0$.

Proof. See [12; Theorem 1] for the results concerning μ_m; the results concerning ρ are easy consequences. Note that the notion of a divisor homomorphism used in [12] differs slightly from that used here. The proofs given there are valid literally in our case. □

Remark. Theorem 2 applies for Krull monoids and hence for Krull domains; see [10] and [14] for the corresponding background material. In particular, we obtain the estimates given in [1; Theorem 2.2]. For convenience we formulate our result in the most interesting case of rings of integers in algebraic number fields, where the class group is finite and every class contains a prime.

COROLLARY 1. *Let* R *be the ring of integers of an algebraic number field and* G *its class group; then*

$$\rho(R) = \frac{1}{2}D(G) .$$

COROLLARY 2. *Let* G *be an abelian group,* $G_0 \subset G$ *a subset,* $B_0 = \mathcal{B}(G_0)$ *and* $D(G_0) \ge 2$. *Then we have*

$$\rho(B_0) \le \frac{1}{2}D(G_0)$$

with equality if $G_0 = -G_0$.

Proof. By Theorem 1, applied for $\varphi = (B_0 \hookrightarrow \mathcal{F}(G_0))$. □

Recall that a monoid H is called *half-factorial* if it is atomic and any two factorizations of an element $a \in H \setminus H^\times$ have the same length; note that H is half-factorial if and only if $\rho(H) = 1$. If $H \neq H^\times$, then H is half-factorial if and only if $\mu_m(H) = m$ for all $m \in \mathbb{N}$.

Proposition 3. *Let* H *be an atomic monoid,* $H \neq H^\times$ *, and let* F *be a half-factorial monoid. Then we have* $\mu_m(H \times F) = \mu_m(H)$ *for all* $m \in \mathbb{N}$ *, and* $\rho(H \times F) = \rho(H)$.

Proof. It suffices to prove the assertion concerning μ_m . For any $z \in H \setminus H^\times$, we have $\mathcal{L}^H(z) = \mathcal{L}^{H \times F}(z)$, and therefore $\mu_m(H) \leq \mu_m(H \times F)$. Let $a = yq \in H \times F$ be such that $y \in H$, $q \in F$ and $m \in \mathcal{L}^{H \times F}(a)$. If $y \in H^\times$, then $\mathcal{L}^{H \times F}(a) = \{m\}$ and hence $L^{H \times F}(a) = m \leq \mu_m(H)$; if $y \notin H^\times$, then $m = k + l$, where $k \in \mathcal{L}^H(y)$ and $\mathcal{L}^F(q) = \{l\}$, which implies $L^{H \times F}(a) = l + L^H(y) \leq l + \mu_k(H) \leq \mu_{k+l}(H) = \mu_m(H)$. In any case $\mu_m(H \times F) \leq \mu_m(H)$ follows. □

For a family of monoids $(H_\lambda)_{\lambda \in \Lambda}$, we consider their coproduct

$$H = \coprod_{\lambda \in \Lambda} H_\lambda = \left\{ (x_\lambda)_{\lambda \in \Lambda} \in \prod_{\lambda \in \Lambda} H_\lambda \,\middle|\, x_\lambda = 1 \text{ for almost all } \lambda \in \Lambda \right\} ,$$

together with the canonical embeddings $\iota_\lambda : H_\lambda \to H$, defined by

$$\iota_\lambda(x_\lambda) = (\ldots, 1, x_\lambda, 1, \ldots) \in H \ .$$

If all H_λ are atomic, then H is also atomic, and the irreducible elements of H are (up to associates) the elements of the form $\iota_\lambda(u_\lambda)$, where $\lambda \in \Lambda$ and $u_\lambda \in H$ is irreducible.

Proposition 4. *Let* $(H_\lambda)_{\lambda \in \Lambda}$ *be a family of atomic monoids, and* $H = \coprod_{\lambda \in \Lambda} H_\lambda$. *Then we have*

$$\rho(H) = \sup\{\rho(H_\lambda) \mid \lambda \in \Lambda\} \ ;$$

if $\rho(H) = \rho(H_\lambda)$ *for some* $\lambda \in \Lambda$ *such that* H_λ *has accepted elasticity, then so has* H.

Proof. Since $\iota_\lambda(x_\lambda) \in H$ is irreducible whenever $x_\lambda \in H_\lambda$ is irreducible, we obtain $\rho(H_\lambda) \le \rho(H)$ for all $\lambda \in \Lambda$ (confer [3; Lemma 6]), and hence $\sup\{\rho(H_\lambda) \mid \lambda \in \Lambda\} \le \rho(H)$.

For the proof of the reverse inequality, we suppose that $r = \sup\{\rho(H_\lambda) \mid \lambda \in \Lambda\} < \infty$. For $x = (x_\lambda)_{\lambda \in \Lambda} \in H \setminus H^\times$, we obtain

$$L^H(x) = \sum_{\substack{\lambda \in \Lambda \\ x_\lambda \notin H_\lambda^\times}} L^{H_\lambda}(x_\lambda) \le r \sum_{\substack{\lambda \in \Lambda \\ x_\lambda \notin H_\lambda^\times}} l^{H_\lambda}(x_\lambda) = r l^H(x) ,$$

and therefore $\rho^H(x) \le r$, which implies

$$\rho(H) = \sup\Big\{ \frac{L^H(x)}{l^H(x)} \mid x \in H \Big\} \le r .$$

If $\rho(H) = \rho(H_\lambda) = \rho^{H_\lambda}(x_\lambda)$, where $x_\lambda \in H_\lambda$, then $\rho(H) = \rho^H\big(\iota_\lambda(x_\lambda)\big)$. $\square$

Let H be a monoid and Q a quotient group of H. A family $\mathbf{H} = (H_\lambda)_{\lambda \in \Lambda}$ of submonoids $H_\lambda \subset Q$ is called a *defining family* of H, if

$$H = \bigcap_{\lambda \in \Lambda} H_\lambda ,$$

and, for each $x \in H$, the set $\{\lambda \in \Lambda \mid x \notin H_\lambda^\times\}$ is finite. In this case, the mapping

$$\varphi : H \to \coprod_{\lambda \in \Lambda} H_\lambda / H_\lambda^\times ,$$

defined by $\varphi(x) = (xH_\lambda^\times)_{\lambda \in \Lambda}$, is a divisor homomorphism; see [11; § 3]. We call $\mathcal{C}(\mathbf{H}) = \mathcal{C}(\varphi)$ the class group of $\mathbf{H}$. Using this terminology, Theorem 2 and Proposition 4 immediately entail the following corollary.

Corollary 3. *Let* H *be an atomic monoid,* $\mathbf{H} = (H_\lambda)_{\lambda \in \Lambda}$ *a defining family of* H, *where all* H_λ *are atomic, and* $G = \mathcal{C}(\mathbf{H})$. *Then we have*

$$\rho(H) \le D(G) \sup\{\rho(H_\lambda) \mid \lambda \in \Lambda\} ;$$

if $G = \{0\}$, *then*

$$\rho(H) = \sup\{\rho(H_\lambda) \mid \lambda \in \Lambda\} .$$

Remark. Corollary 3 applies in particular for atomic domains R having a finite character representation

$$R = \bigcap_{\mathfrak{p} \in \mathfrak{S}} R_{\mathfrak{p}}$$

for some set $\mathfrak{S} \subset \operatorname{spec}(R)$ such that all $R_{\mathfrak{p}}$ are atomic. Indeed, in this case the family $(R_{\mathfrak{p}}^{\bullet})_{\mathfrak{p} \in \mathfrak{S}}$ is a defining family of $R^{\bullet}$. The ideal theory of domains having such a finite character representation was investigated in [5]. Perhaps the most interesting case arises if $\mathfrak{S} = \mathfrak{X}^{(1)}(R)$, the set of prime ideals of height 1 of R. Following [1], we call R a *weakly Krull domain* if $R = \bigcap\{R_{\mathfrak{p}} \mid \mathfrak{p} \in \mathfrak{X}^{(1)}(R)\}$ is a finite character representation; see [14] for a description of weakly Krull domains by means of generalized divisor theories.

Let R be a weakly Krull domain and $\mathbf{G}(R) = \mathbf{G}(R^{\bullet})$ is its group of divisibility; then there is an exact sequence

$$(\star) \qquad 1 \to \mathbf{G}(R) \to \coprod_{\mathfrak{p} \in \mathfrak{X}^{(1)}(R)} \mathbf{G}(R_{\mathfrak{p}}) \to C_t(R) \to 0$$

identifying the t-class group $C_t(R)$ with the class group of the family $(R_{\mathfrak{p}}^{\bullet})_{\mathfrak{p} \in \mathfrak{X}^{(1)}(R)}$ (see [15; Theorem 4.6]. Therefore Corollary 3 implies [1; Theorem 2.14 and Cor. 2.15]. The conjecture stated in [1] after Cor. 2.15 (on p. 231) is false; we give a counterexample at the end of this paper.

If R is a one-dimensional noetherian domain, then $\mathfrak{X}^{(1)}(R) = \max(R)$, R and all $R_{\mathfrak{p}}$ are atomic, and $C_t(R) = \operatorname{Pic}(R)$ is the usual class group. In this case, $(\star)$ is proved in [18; Satz (12.6)]. For later use, we state Corollary 3 in this particular case.

COROLLARY 4. *Let R be a one-dimensional noetherian domain and $G = \operatorname{Pic}(R)$. Then we have*

$$\rho(R) \le D(G) \sup\{\rho(R_{\mathfrak{p}}) \mid \mathfrak{p} \in \max(R)\} \ ;$$

if G is trivial, then

$$\rho(R) = \sup\{\rho(R_{\mathfrak{p}}) \mid \mathfrak{p} \in \max(R)\} \ .$$

In § 5 we shall prove substantially stronger results than Corollary 4 under the assumption that the integral closure $\bar{R}$ of R is a finitely generated R-module. In § 3 and § 4 we develop the necessary combinatorial tools in the context of monoids.

§ 3 T-block monoids

We recall the concept of T-block monoids as introduced in [8]. Let G be an (additively written) abelian group, $G_0 \subset G$ a subset, T a reduced monoid and $\iota : T \to G$ a homomorphism. We extend ι to a homomorphism

$$\iota^* : \mathcal{F}(G_0) \times T \to G$$

by setting

$$\iota^*(g_1 \cdot \ldots \cdot g_n t) = \sum_{i=1}^{n} g_i + \iota(t) \in G ,$$

i. e., $\iota^* \mid \mathcal{F}(G_0)$ is the content already considered. Then

$$\mathcal{B}(G_0, T, \iota) = \{\alpha \in \mathcal{F}(G_0) \times T \mid \iota(\alpha) = 0\}$$

is called the *T-block monoid* over G_0 with respect to ι. If in particular $\iota(t) = 0$ for all $t \in T$, then $\mathcal{B}(G_0, T, \iota) = \mathcal{B}(G_0) \times T$, where $\mathcal{B}(G_0)$ is the ordinary block monoid. The usefulness of T-block monoids is shown by the following proposition.

Proposition 5. *Let H be an atomic monoid and $\varphi : H \to S = \mathcal{F}(P) \times T$ a divisor homomorphism, where T is a reduced monoid and P is any set. Let $G = \mathcal{C}(\varphi)$ be the class group of φ, $G_0 = \{g \in G \mid g \cap P \neq \emptyset\}$, and for $t \in T$ let $\iota(t) \in G$ be the class of t. Then $\mathcal{B}(G_0, T, \iota)$ is atomic, $\mu_m(H) = \mu_m(\mathcal{B}(G_0, T, \iota))$ for all $m \in \mathbb{N}$, and $\rho(H) = \rho(\mathcal{B}(G_0, T, \iota))$. If G_0 is finite and T is finitely generated, then H has accepted elasticity.*

Proof. Since φ induces an isomorphism $\varphi^* : H/H^\times \xrightarrow{\sim} \varphi H$ onto a saturated submonoid of S, we may assume that $H \subset S$ is a saturated submonoid and $\varphi = (H \hookrightarrow S)$. For $p \in P$, let $\boldsymbol{\beta}(p) \in G_0$ be the class of p, and define $\boldsymbol{\beta} : S \to \mathcal{F}(G_0) \times T$ by $\boldsymbol{\beta}(p_1 \cdot \ldots \cdot p_n t) = \boldsymbol{\beta}(p_1) \cdot \ldots \cdot \boldsymbol{\beta}(p_n) t$. By [8; Prop. 4], $\boldsymbol{\beta}$ induces a surjective homomorphism

$$\boldsymbol{\beta} : H \to \mathcal{B}(G_0, T, \iota)$$

mapping irreducible elements of H onto irreducible elements of $\mathcal{B}(G_0, T, \iota)$ (whence $\mathcal{B}(G_0, T, \iota)$ is atomic) and satisfying $\mathcal{L}^H(a) = \mathcal{L}^{\mathcal{B}(G_0,T,\iota)}(\boldsymbol{\beta} a)$ for all $a \in H$. This implies $\mu_m(H) = \mu_m(\mathcal{B}(G_0, T, \iota))$ for all $m \in \mathbb{N}$, and consequently $\rho(H) = \rho(\mathcal{B}(G_0, T, \iota))$. Moreover, H has accepted elasticity if and only if $\mathcal{B}(G_0, T, \iota)$ has. If G_0 is finite and T is finitely generated, then $\mathcal{B}(G_0, T, \iota)$ is finitely generated by [8; Prop. 2] and hence has accepted elasticity by Theorem 2. $\square$

Remark. Proposition 2 implies in particular that Krull monoids (and hence Krull domains) have accepted elasticity, provided that the set G_0 of divisor classes containing prime divisors is finite; for this particular case see [3; Theorem 10].

Next we investigate the invariants μ_m and ρ for T-block monoids. If H is a monoid and $E \subset H$, then $[E]$ denotes the submonoid generated by E.

THEOREM 3. *Let G be an abelian group, $G_0 \subset G$ a subset, T a reduced atomic monoid, $T \neq T^\times$, $\iota : T \to G$ a monoid homomorphism, U the set of irreducible elements of T and $G_1 = G_0 \cup \iota(U)$. Let $B_0 = \mathcal{B}(G_0) \neq \{1\}$ be the ordinary block monoid and assume that the T-block monoid $B = \mathcal{B}(G_0, T, \iota)$ is atomic.*

i) *For all $m \in \mathbb{N}$, we have $\mu_m(B_0) \le \mu_m(B) \le \mu_m(T)D(G_1)$, and*

$$\rho(B_0) \le \rho(B) \le \rho(T)D(G_1) \ .$$

ii) *Suppose that $\iota(U) \subset [-G_0]$; then we have $\mu_m(T) \le \mu_m(B)D(G_1)$ for all $m \in \mathbb{N}$, and*

$$\rho(T) \le \rho(B)D(G_1) \ .$$

Proof. It is sufficient to prove the assertions concerning μ_m.

i) If $b_0 \in B_0$, $b \in B$ and $b \,|\, b_0$, then $b \in B_0$, and hence $\mathcal{L}^{B_0}(b_0) = \mathcal{L}^B(b_0)$; this implies $\mu_m(B_0) \le \mu_m(B)$.

By [8, Proposition 1], the injection $B \hookrightarrow S = \mathcal{F}(G_0) \times T$ is a divisor homomorphism; its class group identifies in a natural way with a subgroup of G. Clearly, $G_0 \cup U$ is the set of irreducible elements of S, and therefore G_1 is the set of all classes containing irreducible elements of S. Theorem 2 implies $\mu_m(B) \le \mu_m(S)D(G_1)$, and Proposition 3 implies $\mu_m(S) = \mu_m(T)$.

ii) We may assume that $D(G_1) < \infty$, and we must prove that $n \in \mathbb{N}$, $u_1, \ldots, u_m, v_1, \ldots \ldots, v_n \in U$ and $u_1 \cdot \ldots \cdot u_m = v_1 \cdot \ldots \cdot v_n$ implies $n \le \mu_m(B)D(G_1)$. Let $n \in \mathbb{N}$ and $u_1, \ldots, u_m, v_1, \ldots, v_n \in U$ be given, and $u_1 \cdot \ldots \cdot u_m = v_1 \cdot \ldots \cdot v_n$. We may suppose that there exists some $l \in \{0, \ldots, m\}$ such that $u_1, \ldots, u_l \notin B$ and $u_{l+1}, \ldots, u_m \in B$. For $1 \le j \le l$, $\iota(u_j) \in \iota(U) \subset [-G_0]$ implies the existence of $g_{j,1}, \ldots, g_{j,d_j} \in G_0$ such that

$$\iota(u_j) + \sum_{\nu=1}^{d_j} g_{j,\nu} = 0 \ ,$$

and we may assume that $d_j \in \mathbb{N}$ is minimal with this property. If $b_j = g_{j,1} \cdot \ldots \cdot g_{j,d_j} \in \mathcal{F}(G_0)$, then the element $b_j u_j \in B$ is irreducible (since d_j is minimal), and

$$(b_1 u_1) \cdot \ldots \cdot (b_l u_l) u_{l+1} \cdot \ldots \cdot u_m = v_1 \cdot \ldots \cdot v_n b_1 \cdot \ldots \cdot b_l \ .$$

The zero sum

$$0 = \sum_{\nu=1}^{n} \iota(v_\nu) + \sum_{j=1}^{l} \sum_{\nu=1}^{d_j} g_{j,v}$$

of elements of G_1 splits into $k \geq 0$ zero subsums each of which has at most $D(G_1)$ summands; This implies

$$v_1 \cdot \ldots \cdot v_n b_1 \cdot \ldots \cdot b_l = w_1 \cdot \ldots \cdot w_k \ ,$$

where $w_i \in B$ and $kD(G_1) \geq n + d_1 + \cdots + d_l \geq n$. Factoring each w_i into irreducible elements of B, we obtain

$$(b_1 u_1) \cdot \ldots \cdot (b_l u_l) u_{l+1} \cdot \ldots \cdot u_m = y_1 \cdot \ldots \cdot y_r \ ,$$

where $y_i \in B$ are irreducible and $r \geq k$. This implies

$$\mu_m(B) \geq r \geq k \geq \frac{n}{D(G_1)} \ ,$$

whence the assertion. $\square$

§ 4 Finitely Primary monoids

Definition. A monoid T is called *finitely primary of rank* $s \in \mathbb{N}$ *and of exponent* $\alpha \in \mathbb{N}$, if it is a submonoid of a factorial monoid F containing exactly s mutually non-associated prime elements $p_1, \ldots, p_s$,

$$T \subset F = [p_1, \ldots, p_s] \times F^\times \ ,$$

satisfying the following two conditions:

1. $T^\times = T \cap F^\times$.

2. For any $a = p_1^{\alpha_1} \cdot \ldots \cdot p_s^{\alpha_s} u \in F$ (where $\alpha_1, \ldots, \alpha_s \in \mathbb{N}_0$ and $u \in F^\times$), the following two assertions hold true:

2a) If $a \in T \setminus T^\times$, then $\alpha_1 \geq 1, \ldots, \alpha_s \geq 1$.

2b) If $\alpha_1 \geq \alpha, \ldots, \alpha_s \geq \alpha$, then $a \in T$.

The simplest examples of finitely primary monoids are finitely generated submonoids of $(\mathbb{N}_0,+)$. Indeed, if $T=[d_1,\ldots,d_m]\subset\mathbb{N}_0$ and $d=\gcd(d_1,\ldots,d_m)$, then there exists some $\alpha\in\mathbb{N}$ such that $d\alpha+d\mathbb{N}_0\subset T$; we set $F=d\mathbb{N}_0$ and see that T is finitely primary of rank 1 and exponent α.

Every finitely primary monoid is atomic, and it is primary in the sense of [15]. Our interest in finitely primary monoids comes from their appearance in the theory of one-dimensional domains, which is shown by the following proposition.

Proposition 6. *Let R be a one-dimensional local noetherian domain such that its integral closure $\bar{R}$ is a finitely generated R-module. If $\max(\bar{R})=\{\mathfrak{p}_1,\ldots,\mathfrak{p}_s\}$ and $[R:\bar{R}]=\mathfrak{p}_1^{\beta_1}\cdot\ldots\cdot\mathfrak{p}_s^{\beta_s}$, then $R^\bullet$ is finitely primary of rank s and of exponent $\beta=\max\{\beta_1,\ldots,\beta_s\}$.*

Proof. Being a semilocal Dedekind domain, $\bar{R}$ is principal; if $\max(\bar{R})=\{\mathfrak{p}_1,\ldots,\mathfrak{p}_s\}$, then

$$\bar{R}^\bullet=[p_1,\ldots,p_s]\times\bar{R}^\times\ ,$$

where $\mathfrak{p}_i=p_i\bar{R}$, and $\mathfrak{p}_i\cap R$ is the maximal ideal of R. Since $\bar{R}\supset R$ is integral, we also have $\bar{R}^\times\cap R=R^\times$.

Now let $a=p_1^{\alpha_1}\cdot\ldots\cdot p_s^{\alpha_s}u\in\bar{R}^\bullet$ be given, where $\alpha_1,\ldots,\alpha_s\in\mathbb{N}_0$ and $u\in\bar{R}^\times$. If $a\in R$, then either $\alpha_1=\cdots=\alpha_s=0$ or $\alpha_1\geq1,\ldots,\alpha_s\geq1$; if $\alpha_1\geq\beta,\ldots,\alpha_s\geq\beta$ then $a\bar{R}\subset[R:\bar{R}]\subset R$ and hence $a\in R$. $\square$

Theorem 4. *Let T be a finitely primary monoid of rank s and exponent α, and suppose that $T\subset F=[p_1,\ldots,p_s]\times F^\times$ as in the definition.*

i) *$T/T^\times$ is finitely generated if and only if $s=1$ and $(F^\times:T^\times)<\infty$.*

ii) *If $s=1$, then $\mu_m(T)\leq(2\alpha-1)m$ for all $m\in\mathbb{N}$, and $\rho(T)\leq2\alpha-1$.*

iii) *If $s\geq2$, then $\mu_m(T)=\infty$ for every $m\geq2\alpha$, and $\rho(T)=\infty$.*

Proof. Assume first that $s\geq2$. For $n\in\mathbb{N}$, we consider the sets

$$T_n=\{z\in T\mid z=p_1^{\alpha_1}\cdot\ldots\cdot p_s^{\alpha_s}u,\ \alpha_1\geq n,\ \alpha_2\leq\alpha,\ldots,\alpha_s\leq\alpha,\ u\in F^\times\}\ ,$$

$$T'_n=\{z\in T\mid z=p_1^{\alpha_1}\cdot\ldots\cdot p_s^{\alpha_s}u,\ \alpha_1\leq\alpha,\ \alpha_2\geq n,\ldots,\alpha_s\geq n,\ u\in F^\times\}\ .$$

For every $z\in T_n\cup T'_n$, we have $l^T(z)\leq\alpha$. If $z\in T$ and $z'\in T$, then

$$zz'=p_1^{\alpha_1}\cdot\ldots\cdot p_s^{\alpha_s}u\ ,$$

where $\alpha_1, \dots, \alpha_s \geq n+1$ and $u \in F^\times$. Since $p_1^{\beta_1} \cdot \ldots \cdot p_s^{\beta_s} v \in T$ whenever $\beta_1 \geq \alpha, \dots, \beta_s \geq \alpha$ and $v \in F^\times$, the element zz' factors in T in the form

$$zz' = [(p_1 \cdot \ldots \cdot p_s)^\alpha]^t \cdot [p_1^{\alpha_1 - t\alpha} \cdot \ldots \cdot p_s^{\alpha_s - t\alpha}] ,$$

where $t = [\frac{n+1}{\alpha}] - 1$; this implies

$$L^T(zz') \geq \left[\frac{n+1}{\alpha}\right] .$$

On the other hand,

$$l^T(zz') \leq l^T(z) + l^T(z') \leq 2\alpha ,$$

and therefore $\mu_m(T) \geq [\frac{n+1}{\alpha}]$, whenever $m \geq 2\alpha$. Since n was arbitrary, we obtain $\mu_m(T) = \infty$ if $m \geq 2\alpha$, and consequently also $\rho(T) = \infty$. By Theorem 1, $T/T^\times$ is not finitely generated.

Now we assume $s = 1$. The irreducible elements of T are of the form $p^\gamma w$, where $1 \leq \gamma \leq 2\alpha - 1$ and $w \in T^\times$. If $z \in T \backslash T^\times$, $z = p^\beta u$, where $\beta \in \mathbb{N}$ and $u \in T^\times$, then

$$L^T(z) \leq \beta , \quad l^T(z) \geq \frac{\beta}{2\alpha - 1}$$

and hence $L^T(z) \leq (2\alpha - 1) l^T(z)$, which implies $\mu_m(T) \leq (2\alpha - 1)m$ for all $m \in \mathbb{N}$ and $\rho(T) \leq 2\alpha - 1$.

Let $(u_i)_{i \in I}$ be a set of representatives of $F^\times / T^\times$; then the set

$$T_0 = \{p_1^\beta u_i T^\times \in T/T^\times \mid 1 \leq \beta \leq \alpha, \ i \in I\}$$

generates $T/T^\times$. If $(F^\times : T^\times) = \# I < \infty$, then T_0 is finite and hence $T/T^\times$ is finitely generated. If $(F^\times : T^\times) = \# I = \infty$, then T_0 contains infinitely many irreducible elements, and therefore $T/T^\times$ is not finitely generated. $\square$

Remark. In the context of one-dimensional (noetherian) domains, parts of Theorem 4 are proved in [4; Theorem 2.12].

§ 5 One-dimensional domains

We start with some finiteness conditions for one-dimensional local domains.

Proposition 7. *Let R be a one-dimensional local noetherian domain with maximal ideal $\mathfrak{p}$ and $\bar{R}$ its integral closure. Suppose that $\bar{R}$ is a finitely generated R-module, $\bar{R} \neq R$, and that $\bar{R}$ is also local, with maximal ideal $\bar{\mathfrak{p}}$. Then the following conditions are equivalent:*

a) $(\bar{R}^\times : R^\times) < \infty$.

b) $(R : \mathfrak{p}) < \infty$.

c) $(\bar{R} : \bar{\mathfrak{p}}) < \infty$.

Proof. We set $k = R/\mathfrak{p}$, $\bar{k} = \bar{R}/\bar{\mathfrak{p}}$, and we view k as a subfield of $\bar{k}$. Since $\bar{R}$ is a finitely generated R-module, we obtain $[\bar{k} : k] < \infty$; therefore **b)** and **c)** are equivalent.

For the following, recall that there are (canonical) isomorphisms $\bar{R}^\times/(1+\bar{\mathfrak{p}}) \simeq \bar{k}^\times$ and $(1+\bar{\mathfrak{p}}^a)/(1+\bar{\mathfrak{p}}^{a+1}) \simeq \bar{k}$ for any $a \geq 1$; see [6; ch. 1, Prop. 4].

c) $\Longrightarrow$ a): If $[R : \bar{R}] = \bar{\mathfrak{p}}^c$ is the conductor of R, then $1 + \bar{\mathfrak{p}}^c \subset R^\times$, and consequently there is an epimorphism

$$\bar{R}^\times/(1+\bar{\mathfrak{p}}^c) \longrightarrow \bar{R}^\times/R^\times .$$

Since $\bar{k}$ is finite, the same is true for $\bar{R}^\times/(1+\bar{\mathfrak{p}}^c)$ and hence for $\bar{R}^\times/R^\times$.

a) $\Longrightarrow$ c): The canonical mapping $\bar{R} \to \bar{k}$ induces an epimorphism $\bar{R}^\times/R^\times \to \bar{k}^\times/k^\times$, showing that $\bar{k}^\times/k^\times$ is finite. If $\bar{k} \neq k$, this implies that both, k and $\bar{k}$, are finite. If $\bar{k} = k$, then $R + \bar{\mathfrak{p}} = \bar{R}$; by Nakayama's Lemma, we have $R + \mathfrak{p}\bar{R} \neq \bar{R}$ and hence $\mathfrak{p}\bar{R} = \bar{\mathfrak{p}}^e$ for some $e \geq 2$. Since $\bar{R} = R + \bar{\mathfrak{p}}$, we obtain $\bar{R}^\times = R^\times(1+\bar{\mathfrak{p}})$, and therefore

$$\bar{R}^\times/R^\times \simeq (1+\bar{\mathfrak{p}})/(1+\bar{\mathfrak{p}}) \cap R^\times = (1+\bar{\mathfrak{p}})/(1+\mathfrak{p}) .$$

is finite. But $1 + \mathfrak{p} \subset 1 + \mathfrak{p}\bar{R} = 1 + \bar{\mathfrak{p}}^e \subset 1 + \bar{\mathfrak{p}}$, and consequently $(1+\bar{\mathfrak{p}})/(1+\bar{\mathfrak{p}}^e)$ is also finite, which implies that $\bar{k}$ is finite. □

Theorem 5. *Let R be a one-dimensional noetherian domain with class group $G = \mathrm{Pic}(R)$. Suppose that the integral closure $\bar{R}$ of R is a finitely generated R-module, and let $\mathfrak{F} = [R : \bar{R}]$ be the conductor of R. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_m \in \max(R)$ be the prime ideals of R lying above $\mathfrak{F}$, and set*

$$\rho^*(R) = \max\{\rho(R_{\mathfrak{p}_j}) \mid 1 \leq j \leq m\} .$$

Let G_0 *be the set of all classes* $g \in G$ *containing some prime ideal* $\mathfrak{p} \in \max(R)$ *different from* $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ *.*

i) *We have the estimate*

$$\rho\big(\mathcal{B}(G_0)\big) \le \rho(R) \le D(G)\rho^*(R) \ .$$

ii) *If* $G = [G_0]$ *, then*

$$\rho^*(R) \le \rho(R)D(G) \ .$$

iii) *Suppose that* $G = [G_0]$ *; then we have* $\rho^*(R) < \infty$ *if and only if, for each* $j \in \{1, \dots, m\}$ *, there is exactly one prime ideal* $\bar{\mathfrak{p}}_j$ *of* $\bar{R}$ *lying above* $\mathfrak{p}_j$ *; if this is the case and* $\mathfrak{F} = \bar{\mathfrak{p}}_1^{\alpha_1} \cdot \ldots \cdot \bar{\mathfrak{p}}_m^{\alpha_m}$ *then*

$$\rho^*(R) \le 2\max\{\alpha_1, \dots, \alpha_m\} - 1 \ .$$

iv) *Suppose that* $\rho^*(R) < \infty$, G_0 *is finite and* $(R : \mathfrak{p}_j) < \infty$ *for* $1 \le j \le m$ *. Then* R *has accepted elasticity.*

Proof. Let

$$\widetilde{\varphi} : R^\bullet \longrightarrow \coprod_{\mathfrak{p} \in \max(R)} R_\mathfrak{p}^\bullet / R_\mathfrak{p}^\times$$

be the divisor homomorphism associated with the finite character representation

$$R = \bigcap_{\mathfrak{p} \in \max(R)} R_\mathfrak{p} \ .$$

We set $P = \max(R) \setminus \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$, $T_j = R_{\mathfrak{p}_j}^\bullet / R_{\mathfrak{p}_j}^\times$ and $T = T_1 \times \dots \times T_m$. If $\mathfrak{p} \in P$, then $R_\mathfrak{p}$ is a discrete valuation ring, and we denote by $\pi_\mathfrak{p} \in R_\mathfrak{p}$ a prime element. The mapping

$$\psi : \begin{cases} \mathcal{F}(P) & \to \quad \coprod_{\mathfrak{p} \in P} R_\mathfrak{p}^\bullet / R_\mathfrak{p}^\times \\ \prod_{\mathfrak{p} \in P} \mathfrak{p}^{n_\mathfrak{p}} & \mapsto \quad (\pi_\mathfrak{p}^{n_\mathfrak{p}} R_\mathfrak{p}^\times)_{\mathfrak{p} \in P} \end{cases}$$

is an isomorphism, and

$$\varphi : R^\bullet \xrightarrow{\tilde{\varphi}} \coprod_{\mathfrak{p} \in \max(R)} R_\mathfrak{p}^\bullet / R_\mathfrak{p}^\times \xrightarrow{\psi^{-1} \times id} \mathcal{F}(P) \times T$$

is a divisor homomorphism whose class group coincides with the class group of R . If $\iota : \mathcal{F}(P) \times T \to G$ is the canonical homomorphism, then $\iota(P) = G_0$. By Proposition 5,

$$\rho(R) = \rho\big(\mathcal{B}(G_0, T, \iota)\big) \ ,$$

and if G_0 is finite and T is finitely generated, then R has accepted elasticity. Now **i)** and **ii)** follow from Theorem 3, since

$$\rho(T) = \max\{\rho(T_j) \mid 1 \le j \le m\} = \max\{\rho(R_{\mathfrak{p}_j}) \mid 1 \le j \le m\} = \rho^*(R)$$

by Proposition 4.

For $j \in \{1, \ldots, m\}$, let s_j be the number of prime ideals of $\bar{R}$ lying above $\mathfrak{p}_j$, and let $\overline{R_{\mathfrak{p}_j}})$ be the integral closure of $R_{\mathfrak{p}_j}$; then $\#\max(\overline{R_{\mathfrak{p}_j}}) = s_j$ and, by Proposition 6 $R^\bullet_{\mathfrak{p}_j}$ is finitely primary of rank s_j. If $s_j \ge 2$ for some $j \in \{1, \ldots, m\}$, then $\rho(R^\bullet_{\mathfrak{p}_j}) = \infty$ by Theorem 4 and consequently $\rho^*(R) = \infty$. If $s_j = 1$, then $[R_{\mathfrak{p}_j} : \overline{R_{\mathfrak{p}_j}}] = [R : \bar{R}]\, \overline{R_{\mathfrak{p}_j}} = (\bar{\mathfrak{p}}_j \overline{R_{\mathfrak{p}_j}})^{\alpha_j}$, and Theorem 4 implies $\rho(R^\bullet_{\mathfrak{p}_j}) \le 2\alpha_j - 1$, whence $\rho^*(R) \le 2\max\{\alpha_1, \ldots, \alpha_m\} - 1$.

It remains to prove **iv)**. We suppose that $s_1 = \cdots = s_m = 1$ and that $(R : \mathfrak{p}_j) < \infty$ for $1 \le j \le m$. Then we obtain $(R_{\mathfrak{p}_j} : \mathfrak{p}_j R_{\mathfrak{p}_j}) = (R : \mathfrak{p}_j) < \infty$ and hence

$$(\bar{R}^\times_{\bar{\mathfrak{p}}_j} : R^\times_{\mathfrak{p}_j}) < \infty$$

by Proposition 7. Hence all T_j are finitely generated by Theorem 4. Consequently, T is finitely generated and R has accepted elasticity by Proposition 5. □

COROLLARY 4. *Let R be an order in an algebraic number field and $\bar{R}$ its integral closure.*

i) *If for some prime ideal $\mathfrak{p}$ of R there is more than one prime ideal of $\bar{R}$ lying above $\mathfrak{p}$, then $\rho(R) = \infty$.*

ii) *If for every prime ideal $\mathfrak{p}$ of R there is exactly one prime ideal of $\bar{R}$ lying above $\mathfrak{p}$, then R has accepted elasticity.*

Proof. If R is an order in an algebraic number field, then its class group is finite and every class contains infinitely many prime ideals; $\bar{R}$ is a finitely generated R-module, and all residue fields are finite. Now the assertion follows from Theorem 5.

EXAMPLE. We consider the ring $R = \mathbb{Z}[3i]$ $(i = \sqrt{-1})$. Then we have $\bar{R} = \mathbb{Z}[i], [R : \bar{R}] = 3R$, and the class group of R is of order 2. Theorem 5 implies $\rho^*(R) = 1$, and consequently $\rho(R) \le 2$. The conjecture stated on p. 231 in [1] would imply $\rho(R) = 1$, but this is not the case.

If $\beta = 1 + 2i$, $\beta' = 1 - 2i$, then 3β, $3\beta'$, 3 and 5 are irreducible elements of R satisfying $(3\beta)(3\beta') = 3^2 \cdot 5$. In fact, it is not difficult to see that $\rho(R) = \frac{3}{2}$.

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