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Note on the Jacobi sum $J(\chi, \chi)$

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Notation

$\zeta_l = \cos\frac{2\pi}{l} + i\sin\frac{2\pi}{l}$

$\zeta_p = \cos\frac{2\pi}{p} + i\sin\frac{2\pi}{p}$

$m = \frac{l-1}{2}$

χ - the Dirichlet character modulo p, $\chi(x) = \zeta_l^{\mathrm{ind}(x)}$

$J(\chi, \chi) = \sum_{x+y=1} \chi(x)\chi(y)$ - Jacobi sum

$\tau(\chi) = \sum_{x=1}^{p-1} \chi(x)\zeta_p^x$ - Gaussian sum

Recall that $\tau(\chi) \in K\mathbf{Q}(\zeta_l)$, where $K \subset \mathbf{Q}(\zeta_p)$ and $[K : \mathbf{Q}] = l$.

Introduction

Let $J(\chi, \chi)$ be the Jacobi sum, $J(\chi, \chi) \in \mathbf{Q}(\zeta_l)$. It is well known that $J(\chi, \chi)\overline{J(\chi, \chi)} = p$, and one easily proves that

$$J(\chi, \chi) \equiv \overline{\chi}(4) \pmod 2.$$

The main aim of this paper is to solve the problem: When is $J(\chi, \chi)$ up to association and conjugation uniquely determined by the solution of the equation

$$X\overline{X} = p,\ X \in \mathbf{Z}(\zeta_l),\ X \equiv 1 \pmod 2)?$$

We give a complete solution in cases $l = 11, 19$.

On the basis of this result, the following question is answered:

When is the prime 2 an 11-th resp. a 19-th power modulo p if p is not representable by the quadratic form $x^2 + 11y^2$, resp. $x^2 + 19y^2$.

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We shall now present a survey of results obtained by solving the problem when the prime 2 is an l-th power modulo p.

Jacobi has given necessary and sufficient conditions for primes $q < 37$ to be cubes modulo primes $p \equiv 1 \pmod 3$. For example, he proves the following

Proposition 1. *2 is a cube modulo p if and only if $L \equiv 0 \pmod 2$, where*

$$4p = L^2 + 27M^2,$$
$$L \equiv 1 \pmod 3.$$

Emma Lehmer [2] finds the following result:

Proposition 2. *Let $p \equiv 1 \pmod 5$ be a prime. Then 2 is a fifth power modulo p if and only if $x \equiv 0 \pmod 2$, where (x, u, v, w) is one of the exactly four solutions $(x, u, v, w), (x, -u, -v, w), (x, v, -u, -w), (x, -v, u, -w)$ of the diophantine system (Dickson):*

$$16p = x^2 + 50u^2 + 50v^2 + 125w^2,$$

$$xw = v^2 - 4uv - u^2,$$

$$x \equiv 1 \pmod 5.$$

P.A. Leonard and K.S. Williams [4] prove the following

Proposition 3. *Let $p \equiv 1 \pmod 7$ be a prime. Then 2 is a seventh power modulo p if and only if $x_1 \equiv 0 \pmod 2$, where $(x_1, x_2, \ldots, x_6)$ is one of the exactly six non-trivial solutions of the diophantine system of equations*

$$(i)\ 72p = 2x_1^2 + 42(x_2^2 + x_3^2 + x_4^2) + 343(x_5^2 + 3x_6^2),$$

$$(ii)\ 12x_2^2 - 12x_4^2 + 147x_5^2 - 441x_6^2 + 56x_1x_6 + 24x_2x_3 - 24x_2x_4 + 48x_3x_4 + 98x_5x_6 = 0,$$

$$(iii)\ 12x_3^2 - 12x_4^2 + 49x_5^2 - 147x_6^2 + 28x_1x_5 + 48x_2x_3 + 24x_2x_4 + 24x_3x_4 + 490x_5x_6 = 0,$$

$$(iv)\ x_1 \equiv 1 \pmod 7.$$

P.A. Leonard, B.C. Mortimer and K.S. Williams [3] prove the following

Proposition 4. *Let $p \equiv 1 \pmod{11}$ be a prime. Then 2 is an eleventh power modulo p if and only if a certain condition involving solutions of a very complicated diophantine system holds (the exact statement may be seen in [3]).*

J.C. Parnami, M.K. Agrawal and A.R. Rajwade [5] have the following

Proposition 5. *Let $p \equiv 1 \pmod{l}$. Then 2 is an l-th power modulo p if and only if*

$$a_1 + a_2 + \cdots + a_{l-1} \equiv 0 \pmod{2},$$

where $(a_1, a_2, \ldots, a_{l-1})$ is one of the exactly $l-1$ solutions of the diophantine system of equations

$$(i)\ p = \sum_{i=1}^{l-1} a_i^2 - \sum_{i=1}^{l-1} a_i a_{i+1},$$

$$(ii)\ \sum_{i=1}^{l-1} a_i a_{i+1} = \sum_{i=1}^{l-1} a_i a_{i+2} = \cdots = \sum_{i=1}^{l-1} a_i a_{i+l-1},$$

$$(iii)\ p \text{ does not divide } \prod_{\lambda(2k)>k} \sigma_k \left(\sum_{i=1}^{l-1} a_i \zeta_l^i \right),$$

where $\lambda(n)$ is the least non-negative residue of n modulo l, and σ_k is the automorphism $\zeta_l \to \zeta_l^k$,

$$(iv)\ 1 + a_1 + \cdots + a_{l-1} \equiv 0 \pmod{l},$$

$$(v)\ a_1 + 2a_2 + \cdots + (l-1)a_{l-1} \equiv 0 \pmod{l}.$$

Now let $X\overline{X} = p$, and let $J(\chi, \chi)$ be associated with the number X, i.e. $J(\chi, \chi) = \varepsilon X$, where ε is a unit of the field $\mathbf{Q}(\zeta_l)$. Then

$$J(\chi, \chi)\overline{J(\chi, \chi)} = p = \varepsilon\overline{\varepsilon}X\overline{X}$$

implies $\varepsilon\overline{\varepsilon} = 1$ and hence $\varepsilon = (-\zeta_l)^n$. So we have

$$J(\chi, \chi) = (-\zeta_l)^n X.$$

Let 2 be a primitive root modulo l. Consider a residue class field $\mathbf{Z}(\zeta_l)/(2)$ of the degree $f = l-1$ over $\mathbf{Z}/2\mathbf{Z}$. Let g be a generator of the multiplicative group $(\mathbf{Z}(\zeta_l)/(2))^*$ of the field $\mathbf{Z}(\zeta_l)/(2)$ such that there holds $\varphi(g) = g^2$, where φ is a generator of the Galois group $G(\mathbf{Q}(\zeta_l)/\mathbf{Q})$.

LEMMA 1. *Every unit ε of the field $\mathbf{Q}(\zeta_l)$ is a $\frac{2^m+1}{l}-th$ power in the group $(\mathbf{Z}(\zeta_l)/(2))^*$, $m=\frac{l-1}{2}$.*

Proof. Let $\varepsilon \equiv g^n \pmod 2$. It is necessary to prove that $\frac{2^m+1}{l}|n$. Consider the unit $\varepsilon_1=\prod_{i=0}^{m-1}\varphi^i(\varepsilon)$. Then $\varepsilon_1\cdot\overline{\varepsilon}_1=N(\varepsilon)=1$, hence ε_1 must be a root of 1, therefore $\varepsilon_1^{2l}=1$. Further,

$$1=\varepsilon_1^{2l}=\prod_{i=0}^{m-1}\varphi^i(\varepsilon)^{2l}\equiv\prod_{i=0}^{m-1}\varphi^i(g^n)^{2l}\equiv\prod_{i=0}^{m-1}g^{2nl\cdot 2^i}\equiv g^{2ln(2^m-1)}\pmod 2.$$

It follows that $2ln(2^m-1)\equiv 0 \pmod{2^{l-1}-1}$, and therefore $n\equiv 0 \pmod{\frac{2^m+1}{l}}$. □

LEMMA 2. *Let 2 be a primitive root modulo l. For a natural number a, $0<a\leq l-1$ the following identity holds*

$$a\frac{2^{l-1}-1}{l}=\sum_{n=1}^{l-1}\left[\frac{2n}{l}\right]2^{r_n}$$

(the decomposition into binary system), where $r_n\equiv l-2-ind(n)+ind(a) \pmod{l-1}$, $0\leq r_n<l-1$, and $ind(x)$ is the index of the element x in the group $(\mathbf{Z}/l\mathbf{Z})^*$ under the base 2, i.e. $2^{\mathrm{ind}(x)}\equiv x \pmod l$.

Proof. The lemma can be readily proved when the rational number $\frac{a}{l}$ is expressed in the binary system, $\frac{a}{l}=\sum_{n=1}^{\infty}a_n2^{-n}$.

LEMMA 3. *The factorisation of the Jacobi sum $J(\chi,\chi)$ into prime divisors of the field $\mathbf{Q}(\zeta_l)$ is $J(\chi,\chi)\approx\prod_{n=1}^{l-1}\sigma_{\frac{1}{n}}(\mathfrak{p})^{[\frac{2n}{l}]}$, where $\mathfrak{p}$ is a prime divisor of the field $\mathbf{Q}(\zeta_l)$, $\mathfrak{p}|p$, and $\sigma_{\frac{1}{n}}$ is an automorphism $\sigma_{\frac{1}{n}}(\zeta_l)=\zeta_l^{\frac{1}{n}}$.*

Proof. According to [1],

$$J(\chi,\chi)=\frac{\tau(\chi)\tau(\chi)}{\tau(\chi^2)}.$$

The factorisation $J(\chi,\chi)\approx\prod_{n=1}^{l-1}\sigma_{\frac{1}{n}}(\mathfrak{p})^{[\frac{2n}{l}]}$ is obtained using the factorisation of the Gaussian sum into prime divisors of the field $K\mathbf{Q}(\zeta_l)$. □

Consider a divisor $A = \prod_{i=0}^{l-2} \varphi^i(\mathfrak{p})^{j_i}$, where $j_i = 0; 1$, and define

$$\Psi(\prod_{i=0}^{l-2} \varphi^i(\mathfrak{p})^{j_i}) = \sum_{i=0}^{l-2} j_i 2^i.$$

Lemma 4. *The factorisation* $A = \prod_{i=0}^{l-2} \varphi^i(\mathfrak{p})^{j_i}$ *is a conjugation of the factorisation* $J(\chi,\chi) = \prod_{n=1}^{l-1} \sigma_{\frac{1}{n}}(\mathfrak{p}) \left[\frac{2n}{l}\right]$ *if and only if* $\Psi\left(\prod_{i=0}^{l-2} \varphi^i(\mathfrak{p})^{j_i}\right) = a\frac{2^{l-1}-1}{l}$, *where* $0 < a \le l-1$.

Proof. Let A be a conjugation of $J(\chi,\chi)$. Then $A = J(\chi^s,\chi^s)$ for some s. If $s \equiv \frac{k}{2} \pmod{l}$, then we can write

$$J_s = J_{\frac{k}{2}} \approx \prod_{n=1}^{l-1} \sigma_{\frac{1}{n}}(\mathfrak{p})^{\left[\frac{2n}{l}\right]} = \prod_{n=1}^{l-1} \varphi^{\mathrm{ind}\left(\frac{1}{n}\cdot\frac{k}{2}\right)} \mathfrak{p}^{\left[\frac{2n}{l}\right]}$$

$$\Psi\left(\prod_{n=1}^{l-1} \varphi^{\mathrm{ind}\left(\frac{1}{n}\cdot\frac{k}{2}\right)}(\mathfrak{p})^{\left[\frac{2n}{l}\right]}\right) = \sum_{n=1}^{l-1} \left[\frac{2n}{l}\right] 2^{\mathrm{ind}\left(\frac{1}{n}\cdot\frac{k}{2}\right)} =$$

$$= \sum_{n=1}^{l-1} \left[\frac{2n}{l}\right] \cdot 2^{l-1-\mathrm{ind}(n)+\mathrm{ind}(k)-1} = k\frac{2^{l-1}-1}{l} \quad \text{(by Lemma 2).}$$

2. Conversely, let $\Psi\left(\prod_{i=0}^{l-2} \varphi^i(\mathfrak{p})^{j_i}\right) = a\frac{2^{l-1}-1}{l}$, $0 < a \le l-1$.

By Lemma 2,

$$a\frac{2^{l-1}-1}{l} = \sum_{n=1}^{l-1} \left[\frac{2n}{l}\right] 2^{r_n} = \Psi\left(\prod_{n=1}^{l-1} \varphi^{\mathrm{ind}\left(\frac{1}{n}\cdot\frac{a}{2}\right)}(\mathfrak{p})^{\left[\frac{2n}{l}\right]}\right) = \sum_{i=0}^{l-2} j_i \cdot 2^i.$$

Since the expansion of a number in the binary system is uniquely determined, Lemma 4 is proved. □

For $\mathfrak{p}|p$, denote by $h_{\mathfrak{p}}$ the least natural number such that a principal divisor $\mathfrak{p}^{h_{\mathfrak{p}}} = (\alpha)$.

THEOREM 1. *Let 2 be a primitive root modulo l, and let $\alpha \equiv g^M \pmod 2$, where $(M, \frac{2^m+1}{l}) = 1$.*

If

$$X \in \mathbf{Z}(\zeta_l),\ X\overline{X} = p,\ X \equiv 1 \pmod 2,$$

then X is, up to association and conjugation, equal to the Jacobi sum $J(\chi, \chi)$.

Proof. According to Lemma 1, the choice of a generator α of the principal divisor $\mathfrak{p}^{h_\mathfrak{p}} = (\alpha)$ is not substantial.

Suppose the factorisation of X into prime divisors of the field $\mathbf{Q}(\zeta_l)$ is

$$X \approx \prod_{i=0}^{l-2} \varphi^i(\mathfrak{p})^{j_i}.$$

It is necessary to prove that this factorisation is a conjugate of the factorisation $J(\chi, \chi)$.

Clearly

$$X^{h_\mathfrak{p}} \approx \prod_{i=0}^{l-2} \varphi^i(\mathfrak{p}^{h_\mathfrak{p}})^{j_i} \approx \prod_{i=0}^{l-2} \varphi^i(\alpha)^{j_i},$$

hence

$$X^{h_\mathfrak{p}} = \varepsilon \prod_{i=0}^{l-2} \varphi^i(\alpha)^{j_i},$$

where ε is a unit of the field $\mathbf{Q}(\zeta_l)$.

But $\varepsilon\overline{\varepsilon} = 1$ implies $\varepsilon = (-\zeta_l)^s$, and therefore

$$X^{h_\mathfrak{p}} = (-\zeta_l)^s \prod_{i=0}^{l-2} \varphi^i(\alpha)^{j_i}.$$

From $X \equiv 1 \pmod 2$ we obtain

$$1 \equiv (-\zeta_l)^s \prod_{i=0}^{l-2} \varphi^i(\alpha)^{j_i} \equiv (-\zeta_l)^s \prod_{i=0}^{l-2} \varphi^i(g^M)^{j_i} \equiv (-\zeta_l)^s \prod_{i=0}^{l-2} g^{Mj_i.2^i} \equiv$$

$$\equiv (-\zeta_l)^s g^{M\sum_{i=0}^{l-2} j_i 2^i} \pmod 2.$$

Since $\zeta_l \equiv -\zeta_l \pmod 2$, we have

$$g^{Ml\sum_{i=0}^{l-2} j_i 2^i} \equiv 1 \pmod 2,$$

hence

$$lM \sum_{i=0}^{l-2} j_i 2^i \equiv 0 \pmod{2^{l-1}-1}.$$

Consequently

$$M \sum_{i=0}^{l-2} j_i 2^i \equiv 0 \pmod{\frac{2^{l-1}-1}{l}}. \tag{1}$$

It is easy to prove that the condition $X\overline{X} = p$ gives

$$\sum_{i=0}^{l-2} j_i 2^i \equiv 0 \pmod{2^m-1}. \tag{2}$$

From the congruences (1) and (2), using the assumption $(M, \frac{(2^m+1)}{l}) = 1$ and the fact that $(2^m-1, 2^m+1) = 1$, we obtain the congruences

$$\sum_{i=0}^{l-2} j_i 2^i \equiv 0 \pmod{\frac{2^{l-1}-1}{l}}, \tag{3}$$

hence

$$\sum_{i=0}^{l-2} j_i 2^i = a\frac{2^{l-1}-1}{l}.$$

From $X\overline{X} = p$, it follows that $j_i \leq 1$, and this implies $a \leq l-1$. Due to Lemma 4, Theorem 1 is proved. $\square$

Remark. If $(M, \frac{2^m+1}{l}) = d > 1$, then instead of the congruence (3) we get the congruence

$$\sum_{i=0}^{l-2} j_i 2^i \equiv 0 \pmod{\frac{2^{l-1}-1}{ld}}.$$

It can be proved that this congruence has always the solution $(j_0, j_1, \ldots, j_{l-2})$ which is not corresponding to the conjugates of the Jacobi sum $J(\chi,\chi)$.

The question of whether for p with $d > 1$ the Jacobi sum $J(\chi,\chi)$ can be uniquely determined transforms in the question, for which j_i the divisor

$$\prod_{i=0}^{l-2} \varphi^i(\mathfrak{p})^{j_i},$$

is principal.

COROLLARY 1. *Let 2 be a primitive root modulo l, and let the class number of the field $\mathbf{Q}(\zeta_l)$ be equal to 1. Then the Jacobi sum $J(\chi,\chi)$ is uniquely determined, up to association and conjugation, from the solution of the equation $X\overline{X} = p, X \equiv 1 \pmod 2$ $X \in \mathbf{Z}(\zeta_l)$ if and only if $(M, \frac{2^m+1}{l}) = 1$.*

Proof. This follows from the preceding Remark, because in such a case, every divisor is principal.

EXAMPLE 1. $l = 5$, $\frac{2^m+1}{l} = \frac{2^2+1}{5} = 1$. *It follows that for all p one has $d = 1$, hence the Jacobi sum $J(\chi,\chi)$ is uniquely determined, up to association and conjugation, by the solution of the equation $X\overline{X} = p$, $X \equiv 1 \pmod 2$, $X \in \mathbf{Z}(\zeta_l)$.*

EXAMPLE 2. $l = 11$, $\frac{2^m+1}{l} = \frac{2^5+1}{11} = 3$; $l = 19$, $\frac{2^m+1}{l} = \frac{2^9+1}{19} = 27$.

It is easy to see, that if we want to answer the question, for which primes p is the Jacobi sum $J(\chi,\chi)$ uniquely determined, up to association and conjugation, by the solution of the equation $X\overline{X} = p$ $X \equiv 1 \pmod 2$, $X \in \mathbf{Z}(\zeta_l)$, then we must know for which primes p the number α, where $N(\alpha) = p$, is a third power in the group $(\mathbf{Z}(\zeta_l)/(2))^*$. This question is solved in the following lemma.

LEMMA 5. *Let 2 be a primitive root modulo $l \equiv 3 \pmod 4$, $\mathfrak{p}|p$, and $\mathfrak{p}^{h_\mathfrak{p}} = \alpha$.*

Then α is a third power in the group $(\mathbf{Z}(\zeta_l)/(2))^*$ if and only if $p^{h_\mathfrak{p}} = x^2 + ly^2$, where x, y are not simultaneously divisible by p.

Proof. By Lemma 1, the choice of a generator α of the principal divisor $\mathfrak{p}^{h_\mathfrak{p}}$ is not substantial.

Consider the product

$$\beta = \prod_{(\frac{z}{l})=1} \sigma_z(\alpha), \text{ hence } \beta \in \mathbf{Q}(\sqrt{-l}).$$

Let

$$\beta = a' \sum_{(\frac{z}{l})=1} \zeta_l^z + b' \sum_{(\frac{z}{l})=1} \zeta_l^{-z},\ a', b' \in \mathbf{Z},$$

and let $\alpha \equiv g^r \pmod 2$, where $r \equiv 0 \pmod 3$,

$$\beta = \prod_{(\frac{z}{l})=1} \sigma_z(\alpha) \equiv \prod_{i=0}^{\frac{l-3}{2}} g^{r.2^i} \equiv g^{r\frac{2^{l-1}-1}{3}} \equiv 1 \pmod 2.$$

So we have $a' \equiv b' \equiv 1 \pmod 2$.

Hence

$$\beta = a' \sum_{(\frac{z}{l})=1} \zeta_l^z + b' \sum_{(\frac{z}{l})=1} \zeta_l^{-z} = a'\frac{-1+\sqrt{-l}}{2} + b'\frac{-1-\sqrt{-l}}{2}$$

$$= a + b\sqrt{-l},\ a, b \in \mathbf{Z},$$

therefore

$$\beta\overline{\beta} = p^{h_\mathfrak{p}} = a^2 + lb^2.$$

Let conversely $p^{h_\mathfrak{p}} = a^2 + lb^2$. Put

$$a + b\sqrt{-l} = a'\frac{-1+\sqrt{-l}}{2} + b'\frac{-1-\sqrt{-l}}{2} =$$

$$= (b-a)\frac{-1+\sqrt{-l}}{2} + (-a-b)\frac{-1-\sqrt{-l}}{2}.$$

This implies $a' \equiv b' \equiv 1 \pmod 2$, hence

$$\beta = a' \sum_{(\frac{z}{l})=1} \zeta_l^z + b' \sum_{(\frac{z}{l})=1} \zeta_l^{-z} \equiv 1 \pmod 2.$$

Let $\mathfrak{p}|\beta$. Then since β is invariant on σ_z, where $\left(\frac{z}{l}\right) = 1$, we set that $\sigma_z(\mathfrak{p})|\beta$.

But if $\left(\frac{n}{l}\right) = -1$, then $\sigma_n(\mathfrak{p})$ does not divide β (in the opposite case we would get $p|\beta$, hence a contradiction).

It implies

$$\beta \approx \prod_{(\frac{z}{l})=1} \sigma_z(\mathfrak{p})^{h_\mathfrak{p}},$$

therefore

$$1 \equiv \beta \equiv (-\zeta_l)^s \prod_{(\frac{z}{l})=1} \sigma_z(\mathfrak{p})^{h_\mathfrak{p}} \equiv (-\zeta_l)^s g^{r\frac{2^{l-1}-1}{3}} \pmod 2,$$

and

$$g^{lr\frac{2^{l-1}-1}{3}} \equiv 1 \pmod 2.$$

From this we finally obtain

$$lr\frac{2^{l-1}-1}{3} \equiv 0 \pmod{2^{l-1}-1} \text{ which implies } r \equiv 0 \pmod 3.$$

□

THEOREM 2. *Let $l = 11; 19$, and let $p \equiv 1 \pmod l$, $4p = A^2 + lB^2$. The Jacobi sum $J(\chi, \chi)$ is uniquely determined, up to conjugation and association, by the solution of*

$$X\overline{X} = p,\ X \in \mathbf{Z}(\zeta_l),\ X \equiv 1 \pmod 2,$$

if and only if $A \equiv B \equiv 1 \pmod 2$.

Proof. The proof follows from Lemma 5 and Corollary 1. □

By Proposition 5 and Theorem 2, we come to the following

THEOREM 3. *Let $l = 11; 19$, and let $p \equiv 1 \pmod l$, $4p = A^2 + lB^2$, $A \equiv B \equiv 1 \pmod 2$. Then 2 is an l-th power modulo p if and only if*

$$a_1 \equiv a_2 \equiv \cdots \equiv a_{l-1} \equiv 1 \pmod 2,$$

where $(a_1, a_2, \ldots, a_{l-1})$ is one of the exactly $l-1$ solutions of the diophantine system of equations

$$(i)\ p = \sum_{i=1}^{l-1} a_i^2 - \sum_{i=1}^{l-1} a_i a_{i+1},$$

$$(ii)\ \sum_{i=1}^{l-1} a_i a_{i+1} = \sum_{i=1}^{l-1} a_i a_{i+2} = \cdots = \sum_{i=1}^{l-1} a_i a_{i+l-1},$$

$$(iv)\ 1 + a_1 + \cdots + a_{l-1} \equiv 0 \pmod l,$$

$$(v)\ a_1 + 2a_2 + \cdots + (l-1)a_{l-1} \equiv 0 \pmod l.$$

Remark. As we can see, Theorem 2 enables us to remove condition (iii) of Proposition 5.

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