Journal de Théorie des Nombres
de Bordeaux **8** *(1996), 101–123*

Some problems on mean values of the Riemann zeta-function

par Aleksandar IVIĆ*

Résumé. On s'intéresse à des problèmes et des résultats relatifs aux valeurs moyennes de la fontion $\zeta(s)$. On étudie en particulier des valeurs moyennes de $|\zeta(\frac{1}{2}+it)|$, ainsi que le moment d'ordre 4 de $|\zeta(\sigma+it)|$ pour $1/2 < \sigma < 1$.

Abstract. Several problems and results on mean values of $\zeta(s)$ are discussed. These include mean values of $|\zeta(\frac{1}{2}+it)|$ and the fourth moment of $|\zeta(\sigma+it)|$ for $1/2 < \sigma < 1$.

1. Introduction

One of the fundamental problems in the theory of the Riemann zeta-function $\zeta(s)$ is the evaluation of power moments, namely integrals of $|\zeta(\sigma+ +it)|^k$, where $k > 0$ and σ are fixed real numbers. This topic is extensively discussed in [4], [5] and [16], where additional references to other works may be found. Of particular interest are the values of σ in the so-called "critical" strip $0 < \sigma < 1$, while the case $\sigma = 1$ is treated in [2] and [6]. In view of the functional equation $\zeta(s) = \chi(s)\zeta(1-s)$, where

$$\chi(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s) = \left(\frac{2\pi}{t}\right)^{\sigma+it-1/2} e^{i(t+\pi/4)}\left(1+O\left(\frac{1}{t}\right)\right)$$

for $s = \sigma + it, t \ge t_0 > 0$, it transpires that the relevant range for σ in the evaluation of power moments of $\zeta(s)$ is $1/2 \le \sigma < 1$.

The aim of this paper is to discuss several problems and results involving power moments of $|\zeta(\sigma+it)|$. Some of the problems that I have in mind are quite deep, and even partial solutions would be significant. In section 2 problems concerning mean values on the "critical line" $\sigma = 1/2$ are discussed. Section 3 is devoted to problems connected with the evaluation of

---

1991 *Mathematics Subject Classification.* Primary 11 M 06.
Key words and phrases. Riemann zeta-function, mean values, asymptotic formulas .
Manuscrit reçu le 28 avril 1993

*Research financed by the Mathematical Institute of Belgrade

$\int_0^T |\zeta(\sigma+it)|^4 dt$ for $1/2 < \sigma < 1$ fixed. This topic is a natural one, since problems involving the fourth moment on $\sigma = 1/2$ are extensively treated in several works of Y. Motohashi and the author (see Ch. 5 of [5] and [8], where additional references may be found). Motohashi found a way to apply the powerful methods of spectral theory to this problem, thereby opening the path to a thorough analysis of this topic. Thus it seems appropriate to complete the knowledge on the fourth moment of $\zeta(s)$ by considering the range $1/2 < \sigma < 1$ as well.

The notation used in the text is standard, whenever this is possible. ε denotes positive constants which may be arbitrarily small, but are not necessarily the same ones at each occurrence. $f(x) \ll g(x)$ and $f(x) = O(g(x))$ both mean that $|f(x)| \le cg(x)$ for $x \ge x_0$, some $c > 0$ and $g(x) > 0$. $f(x) = o(g(x))$ as $x \to \infty$ means that $\lim_{x\to\infty} f(x)/g(x) = 0$, while the Perelli symbol $f(x) = \infty(g(x))$ means that $\lim_{x\to\infty} f(x)/g(x) = +\infty$. $f(x) = \Omega(g(x))$ means that $\limsup_{x\to\infty} |f(x)|/g(x) > 0$, $f(x) = \Omega_+(g(x))$ (resp. $f(x) = \Omega_-(g(x))$) that $\limsup_{x\to\infty} f(x)/g(x) > 0$ (resp. $\limsup_{x\to\infty} f(x)/g(x) < 0$), provided that $g(x) > 0$ for $x \ge x_0$. Finally $f(x) = \Omega_\pm(g(x))$ means that both $f(x) = \Omega_+(g(x))$ and $f(x) = \Omega_-(g(x))$ hold.

2. Problems on the critical line $\sigma = 1/2$

In this section we shall investigate some mean value problems on the critical line $\sigma = 1/2$. The first problem is as follows. Let $0 \le H \le T$, $0 \le \alpha < \beta$ and $T \to \infty$. For which values of α, β and $H = o(T)$ does one have

$$(1) \qquad \int_T^{T+H} |\zeta(\tfrac{1}{2}+it)|^\alpha dt \;\le\; \int_T^{T+H} |\zeta(\tfrac{1}{2}+it|^\beta dt \ ?$$

Furthermore, can one find specific values of α and β (they may be constants or even fuctions of T) and $H = H(T)$ such that (1) fails to hold?

It turns out that this is not an easy problem, and what I can prove is certainly not the complete solution. It is contained in

Theorem 1. *Let $\beta_0 > 0$ be any fixed constant. If $0 \le \alpha < \beta$, then (1) holds for $\beta \ge \beta_0$ and $\log\log T \ll H \le T$. If $0 \le \alpha < \beta < \beta_0$ and*

$H = \infty(\log T)$, *then*

$$(1 + o(1)) \int_T^{T+H} |\zeta(\tfrac{1}{2} + it)|^\alpha dt \le \int_T^{T+H} |\zeta(\tfrac{1}{2} + it)|^\beta dt. \tag{2}$$

Proof. What Theorem 1 roughly says is that (1) holds for β not too small, while for small β only the weaker asymptotic inequality (2) can be established.

Assume first that $0 \le \alpha < \beta$ and $\beta \ge \beta_0 > 0$. By Hölder's inequality for integrals we have

$$\int_T^{T+H} |\zeta(\tfrac{1}{2}+it)|^\alpha dt \le (\int_T^{T+H} |\zeta(\tfrac{1}{2}+it)|^\beta \, dt)^{\alpha/\beta} H^{(\beta-\alpha)/\beta} \le \int_T^{T+H} |\zeta(\tfrac{1}{2}+it)|^\beta dt$$

provided that

$$\int_T^{T+H} |\zeta(\tfrac{1}{2} + it|^\beta dt \ge H. \tag{3}$$

It is enough to prove (3) for $\beta = \beta_0$, since for $\beta > \beta_0$ the inequality again follows by Hölder's inequality. Now (3) follows from the results on mean values of K. Ramachandra (see his monograph [14] for an extensive account). For example, by Corollary 1 to Th. 1 of Ramachandra [13] we have

$$\int_T^{T+H} |\zeta(\tfrac{1}{2} + it)|^{\beta_0} dt \gg H(\log H)^{\beta_0^2/4} \qquad (\log\log T \ll H \le T), \tag{4}$$

if we assume that β_0 is rational, which we may since β_0 is arbitrary (but fixed). Since

$$\lim_{T\to\infty} (\log H)^{\beta_0^2/4} = +\infty,$$

we obtain (3) with $\beta = \beta_0$ from (4).

We suppose now that $0 \le \alpha \le \beta \le \beta_0$ and $H = \infty(\log T)$. From Ivić-Perelli [7] (or (6.38) of [5]) one has

$$0 \le \int_{\frac{1}{2}}^{1} (N(\sigma, T+H) - N(\sigma, T))\, d\sigma = \int_T^{T+H} \log|\zeta(\tfrac{1}{2} + it)|\, dt + O(\log T),$$

which implies with a suitable $C > 0$ that

$$\int_T^{T+H} \log|\zeta(\tfrac12 + it)|\,dt \ge -C\log T. \tag{5}$$

We obtain

$$\int_T^{T+H} \log|\zeta(\tfrac12 + it)|\,dt = \frac{1}{\beta}\int_T^{T+H} \log|\zeta(\tfrac12 + it)|^\beta\,dt$$

$$\le \frac{H}{\beta}\log\Bigl(\frac{1}{H}\int_T^{T+H} |\zeta(\tfrac12 + it)|^\beta dt\Bigr)$$

similarly as in [2]. By using (5) it follows that

$$\frac{1}{H}\int_T^{T+H} |\zeta(\tfrac12 + it)|^\beta\,dt \ge e^{-\beta CH^{-1}\log T} \ge e^{-\beta_0 CH^{-1}\log T} \ge 1 - \frac{\beta_0 C\log T}{H}. \tag{6}$$

Thus we have from (6) (with $D = \beta_0 C$) and Hölder's inequality

$$\int_T^{T+H} |\zeta(\tfrac12 + it)|^\beta dt = \Bigl(\int_T^{T+H} |\zeta(\tfrac12 + it)|^\beta dt\Bigr)^{\alpha/\beta}\Bigl(\int_T^{T+H} |\zeta(\tfrac12 + it)|^\beta dt\Bigr)^{1-\alpha/\beta}$$

$$\ge \Bigl(\int_T^{T+H} |\zeta(\tfrac12 + it)|^\beta\,dt\Bigr)^{\alpha/\beta} H^{1-\alpha/\beta}\Bigl(1 - \frac{D\log T}{H}\Bigr)^{1-\alpha/\beta}$$

$$\ge \int_T^{T+H} |\zeta(\tfrac12 + it)|^\alpha\,dt\,\Bigl(1 - \frac{D\log T}{H}\Bigr)^{1-\alpha/\beta}$$

$$\ge \Bigl(1 - \frac{D\log T}{H}\Bigr)\int_T^{T+H} |\zeta(\tfrac12 + it)|^\alpha dt = (1 + o(1))\int_T^{T+H} |\zeta(\tfrac12 + it)|^\alpha dt$$

since $H = \infty(\log T)$. This completes the proof of Theorem 1.

Concerning the values of α, β and H for which (1) fais to hold I wish to make the following remark: For many $0 \le \alpha < \beta$ there exists arbitrarily large values of T such that

$$\int_T^{T+T^{-1/6}} |\zeta(\tfrac{1}{2}+it)|^\alpha dt > \int_T^{T+T^{-1/6}} |\zeta(\tfrac{1}{2}+it)|^\beta dt. \tag{7}$$

For T we may simply take the points for which $\zeta(\frac{1}{2}+iT) = 0$, and there are $\gg \tau$ such points in $[0, \tau]$. If, as usual, for any real σ we define

$$\mu(\sigma) = \limsup_{t\to\infty} \frac{\log|\zeta(\sigma+it)}{\log t},$$

then

$$\zeta'(\tfrac{1}{2}+it) \ll_\varepsilon |t|^{\mu(\frac{1}{2})+\varepsilon},$$

which follows by applying Cauchy's theorem to a circle of radius $1/\log t$ with center at $\frac{1}{2}+it$. Since $\mu(\frac{1}{2}) < \frac{1}{6}$, it follows that for $T \le t \le T+H$, $H = T^{-1/6}$,

$$\zeta(\tfrac{1}{2}+it) \ll H \max_{T\le t\le T+H} |\zeta'(\tfrac{1}{2}+it)| \ll HT^{\mu(\frac{1}{2})+\varepsilon} \le \tfrac{1}{2}$$

for sufficiently small $\varepsilon > 0$ and $T \ge T_0(\varepsilon)$. Hence

$$|\zeta(\tfrac{1}{2}+it)|^{\beta-\alpha} \le 2^{\alpha-\beta} < 1,$$

$$|\zeta(\tfrac{1}{2}+it)|^{\alpha} > |\zeta(\tfrac{1}{2}+it)|^{\beta},$$

and (7) readily follows. Under the Riemann hypothesis one can in (7) replace $H = T^{-1/6}$ by $H = \exp(-\frac{A\log T}{\log\log T})$ with some $A > 0$. However, the largest such H is difficult to determine. Perhaps $H = \exp(-A\sqrt{\log\log T})$ can be taken unconditionally, or even larger H is permissible? This is certainly an open and difficult question.

The construction leading to (7) was basically simple: one finds and interval which is not too small, and where $|\zeta(\frac{1}{2}+it)| \le \frac{1}{2}$ holds. Points arounds zeros of $\zeta(\frac{1}{2}+iT)$ are of course likely candidates for such intervals, only we can estimate (unconditionally) $\zeta(\frac{1}{2}+it)$ rather crudely near these zeros. This accounts for the rather poor value $H = T^{-1/6}$ in (7). The following problems then naturally may be posed: What is the measure $\mu(A_T)$ of the set

$$A_T = \left\{t : T \le t \le 2T,\ |\zeta(\tfrac{1}{2}+it)| \le \tfrac{1}{2}\right\} ?$$

Clearly A_T consists of disjoint intervals $[T_1, T_1+H_1], \cdots, [T_R, T_R+H_R]$ where $R = R(T)$ and $H_r > 0$ for $r = 1, \ldots, R$. What is the order of magnitude of the function

$$H(T) \ := \ \max_{1\le j\le R(T)} H_j \ ?$$

Many results are known on the problems involving *large* values of $|\zeta(\frac12+it)|$, but here is a problem involving *small* values of $|\zeta(\frac12+it)|$. The significance of $H(T)$ is that obviously

$$\int\limits_T^{T+H(T)} |\zeta(\tfrac12+it)|^\alpha dt > \int\limits_T^{T+H(T)} |\zeta(\tfrac12+it)|^\beta dt \quad (0\le\alpha<\beta).$$

I recall that, by results of A. Selberg (see D. Joyner [9]) and A. Laurinčikas [10], for a given real y one has ($\mu(\cdot)$ again denotes the measure of a set)
(8)

$$\lim_{T\to\infty}\frac1T\,\mu(0\le t\le T: |\zeta(\tfrac12+it)| \le e^{y\sqrt{\frac12\log\log T}}) = (2\pi)^{-1/2}\int\limits_{-\infty}^{y} e^{-u^2/2}\,du,$$

but determining the true order of magnitude of $\mu(A_T)$ and $H(T)$ is a different (and perhaps even harder) problem. Presumably $R(T) \ll T\log T$, so that in view of

$$\mu(A_T) \le R(T)H(T) \ll H(T)T\log T$$

we would need a lower bound for $\mu(A_T)$ in order to improve (7).

Let $\delta > 0$ be a given constant and define

$$K(T) = \{\inf k : k = k(T) \text{ and } \int\limits_0^T |\zeta(\tfrac12+it)|^{2k}dt > T^{1+\delta} \text{ for } T \ge T_0(\delta)\}.$$

The problem is to bound, as accurately as possible, the function $K(T)$. Certainly we have

$$K(T) \ll_\delta \frac{\log T}{\sqrt{\log\log T}}, \tag{9}$$

which easily follows from the limit law (8). The significance of $K(T)$ comes from the fact that from

$$T^{1+\delta} < \int_0^T |\zeta(\tfrac12 + it)|^{2k} dt \le T \max_{0\le t\le T} |\zeta(\tfrac12 + it)|^{3K(T)}$$

one obtains

$$\max_{0\le t\le T} |\zeta(\tfrac12 + it)| \ge \exp\Bigl(\frac{\delta \log T}{3K(T)}\Bigr). \tag{10}$$

One can substantially improve (9) by using R. Balasubramanian's bound [1]

$$\max_{0\le t\le T} |\zeta(\tfrac12 + it)| \ge \exp\biggl(\frac34\Bigl(\frac{\log H}{\log\log H}\Bigr)^{1/2}\biggr), \tag{11}$$

which is valid for $100 \log\log T \le H \le T, T \ge T_0$. Actually (11) is proved with $3/4 + \eta$ for some $\eta > 0$ as the constant in the exponential. Hence with $H = T$ it follows that

$$|\zeta(\tfrac12 + iT')| = \max_{\frac14 T\le t\le \frac12 T} |\zeta(\tfrac12 + it)| \ge \exp\Bigl(\Bigl(\frac34 + \frac{\eta}{2}\Bigr)\Bigl(\frac{\log T}{\log\log T}\Bigr)^{1/2}\Bigr).$$

Thus if $|t - T'| \le T^{-1/6}$, then

$$|\zeta(\tfrac12 + it)| = |\zeta(\tfrac12 + iT')| + O\bigl(|T' - t| \max_{|v-T'|\le T^{-1/6}} |\zeta'(\tfrac12 + iv)|\bigr)$$

$$\ge \exp\Bigl(\Bigl(\frac34 + \frac{\eta}{3}\Bigr)\Bigl(\frac{\log T}{\log\log T}\Bigr)^{1/2}\Bigr).$$

This gives

$$\int_0^T |\zeta(\tfrac12 + it)|^{2k}\, dt \ge \int_{T'-T^{-1/6}}^{T'+T^{-1/6}} |\zeta(\tfrac12 + it)^{2k}\, dt$$

$$\ge 2T^{-1/6} \exp\Bigl(\frac{3k\sqrt{\log T}}{2\sqrt{\log\log T}}\Bigr) \; > \; T^{1+\delta}$$

certainly for

$$k = (2 + \delta)\sqrt{\log T \log\log T}. \tag{12}$$

Hence (12) gives trivially

$$K(T) \le (2+\delta)\sqrt{\log T \log\log T}. \tag{13}$$

Naturally, any improvement of (13) would be of great interest, since in view of (10) it would mean, improvement of (11) in the most interesting case when $H = T$. Perhaps even

$$K(T) \ll_\delta \log\log T$$

holds. In the other direction

$$K(T) = o(\log\log T) \qquad\qquad (T \to \infty)$$

would, in view of (10), contradict the Riemann hypothesis which gives (see Ch. XIV of [16]), for some $A > 0$,

$$\zeta(\tfrac{1}{2} + it) \ll \exp\Big(\frac{A\log t}{\log\log t}\Big).$$

Hence it is reasonable to conjecture that

$$K(T) = \Omega(\log\log T)$$

holds unconditionally.

3. The fourth moment for $1/2 < \sigma < 1$

In this section problems involving the fourth moment of $|\zeta(\sigma+it)|$ $(1/2 < \sigma < 1)$ will be discussed. To this end we define, for fixed σ satisfying $1/2 < \sigma < 1$,

$$E_1(T,\sigma) = \int_0^T |\zeta(\sigma+it)|^2\,dt - \zeta(2\sigma)T - (2\pi)^{2\sigma-1}\frac{\zeta(2-2\sigma)}{2-2\sigma}T^{2-2\sigma} \tag{14}$$

and

$$E_2(T,\sigma) = \tag{15}$$

$$= \int_0^T |\zeta(\sigma+it)|^4\,dt - \frac{\zeta^4(2\sigma)}{\zeta(4\sigma)}T - (A_1(\sigma)\log T + A_2(\sigma))T^{2-2\sigma} - A_3(\sigma)T^{3-4\sigma}$$

as the error terms for the second and fourth moment in the critical strip, respectively. The constants $A_j(\sigma)$ $(j = 1, 2, 3)$ are such that both

$$\lim_{\sigma\to 1/2+0} E_1(T,\sigma) = E_1(T) \equiv E(T), \quad \lim_{\sigma\to 1/2+0} E_2(T,\sigma) = E_2(T)$$

hold, where

$$E(T) = \int_0^T |\zeta(\tfrac{1}{2} + it)|^2 dt - T(\log \frac{T}{2\pi} + 2\gamma - 1),$$

$$E_2(T) = \int_0^T |\zeta(\tfrac{1}{2} + it)|^4 dt - TP_4(\log T).$$

Here γ is Euler's constant and $P_4(y)$ is a polynomial of degree four in y with suitable coefficients, of which the leading one equal $1/(2\pi^2)$. A detailed account on $E(T)$ and $E_2(T)$ is to be found in [5].

Prof. Y. Motohashi kindly informed me in correspondence (Jan. 7, 1991) that he evaluated, for $0 < \Delta \le T/\log T$ and $1/2 < \sigma < 1$,

$$I_4(T,\sigma;\Delta) := (\Delta\sqrt{\pi})^{-1} \int_{-\infty}^{\infty} |\zeta(\sigma + iT + it)|^4 e^{-(t/\Delta)^2}\, dt \tag{16}$$

by means of spectral theory of automorphic forms. The method of proof is similar to the one that he used for evaluating $I_4(T, \frac{1}{2}; \Delta)$ (see e.g. Ch. 5 of [5]). Motohashi notes that the expressions for $A_j(\sigma)$ in (15) turn out to be quite complicated, and in particular $A_1(\sigma) = 0$ cannot be ruled out. He also stated that he can obtain

$$E_2(T,\sigma) \ll T^{2/(1+4\sigma)} \log^C T \qquad (\tfrac{1}{2} < \sigma < 1). \tag{17}$$

It will be sketched a little later how by taking $\Delta = T^{2/(1+4\sigma)}$ in the integrated version of (16) one can obtain (17) for $1/2 < \sigma < 3/4$. This is because $\Delta > T^{1/2}$ has to be observed, and $2/(1+4\sigma) \ge 1/2$ for $\sigma \ge 3/4$. But for $\sigma \ge 3/4$ we have $2/(1+4\sigma) \ge 2 - 2\sigma$, so that the right-hand side of (17) is larger than the second main term in (15) for $\int_0^T |\zeta(\sigma + it)|^4 dt$. Therefore for $3/4 < \sigma < 1$ (17) is superseded by

Theorem 2. *For fixed σ satisfying $1/2 < \sigma < 1$ we have*

$$\int_0^T |\zeta(\sigma+it)|^4\,dt = \frac{\zeta^4(2\sigma)}{\zeta(4\sigma)}T + O(T^{2-2\sigma}\log^3 T). \tag{18}$$

Proof. Note first that the only published result heretofore on the integral in (18) is contained in Th. 8.5 of [4]. This is

$$\int_0^T |\zeta(\sigma+it)|^4\,dt = \frac{\zeta^4(2\sigma)}{\zeta(4\sigma)}T + O(T^{\frac12(3-2\sigma)+\varepsilon}) \quad (1/2 < \sigma < 1), \tag{19}$$

so that (18) sharpens (19).

The proof of (18) follows the method of §4.3 of my book [5], with $k = 2$, and with some modifications that will be now indicated. All the notation will be as in Ch. 4 of [5]. From Theorem 4.2 with

$$\Sigma_1(t) = \sum_{n=1}^{\infty} d(n)\nu(\frac{t}{2\pi n})n^{-\sigma-it}, \quad \Sigma_2(t) = \sum_{n=1}^{\infty} d(n)\nu(\frac{t}{2\pi n})n^{\sigma-1+it},$$

where $d(n)$ is the number of divisors of n and $\nu(\cdot)$ is the smoothing function, we have

$$\zeta^2(\sigma+it) = \Sigma_1(t) + \chi^2(\sigma+it)\Sigma_2(t) + O(R_2(t)),$$

hence

$$\zeta^2(\sigma-it) = \overline{\Sigma_1}(t) + \chi^2(\sigma-it)\overline{\Sigma_2}(t) + O(R_2(t)).$$

Here $R_2(t)$ is the error term in the smoothed approximate functional equation for $\zeta^2(s)$. By (4.39) and the bound at the bottom of p. 179 of [5] we have, for $T \ll t \ll T$,

$$R_2(t) = t^{\varepsilon-1-2\sigma}\log t + t^{2\varepsilon-1}\int_{-T^\varepsilon}^{T^\varepsilon} |\zeta(\sigma+it+\delta+iv)|^2 dv, \tag{20}$$

where δ is any constant such that $0 < \delta < 1$. If $f(t)$ is the appropriate smoothing function that majorizes or minorizes the characteristic function of $[T, 2T]$, then from the expressions for $\zeta^2(\sigma \pm it)$ we obtain

$$\int_0^\infty f(t)|\zeta(\sigma+it)|^4 dt = \int_0^\infty f(t)|\Sigma_1(t)|^2 dt$$

$$+ 2\,\mathrm{Re}\Big\{\int_0^\infty f(t)\chi^2(\sigma-it)\Sigma_1(t)\overline{\Sigma_2}(t)\,dt\Big\} + \int_0^\infty f(t)|\chi^2(\sigma+it)\Sigma_2(t)|^2\,dt$$

$$+ O\Big(\int_{T/2}^{5T/2} R_2(t)(|\Sigma_1(t)| + T^{1-2\sigma}|\Sigma_2(t)|)\,dt\Big) + O\Big(\int_{T/2}^{5T/2} R_2^2(t)\,dt\Big).$$

By using (20) and taking δ sufficiently small it follows that

$$\int_{T/2}^{5T/2} R_2^2(t)dt \ \ll\ T^{\varepsilon-1}. \tag{21}$$

The mean value theorem for Dirichlet polynomials (see Th. 5.2 of [4]) cannot be used directly for the evaluation of mean values of $\Sigma_1(t)$ and $\Sigma_2(t)$, because the sums in question contain the ν-function. However, this is not an essential difficulty, since this function is smooth. Thus we can square out the sums, perform integration by parts on non-diagonal terms, use inequality (5.5) of [4] and the first derivative test (Lemma 2.1 of [4]). In this way we obtain

$$\int_0^\infty f(t)|\Sigma_1(t)|^2 dt \le \int_{T/2}^{5T/2} |\Sigma_1(t)|^2 dt \ll T$$

and

$$\int_0^\infty f(t)|\chi^2(\sigma+it)\Sigma_2(t)|^2\,dt \ll T^{2-4\sigma}\int_{T/2}^{5T/2} |\Sigma_2(t)|^2 dt$$

$$\ll T^{2-4\sigma}\Big(T\sum_{n\le T} d^2(n)n^{2\sigma-2} + \sum_{n\le T} d^2(n)n^{2\sigma-1}$$

$$+ \sum_{m\ne n\le T} d(m)d(n)(mn)^{\sigma-1}\frac{1}{m\log^2(\frac{m}{n})}\Big) \ll T^{2-2\sigma}\log^3 T.$$

To estimate the last double sum above one sets $|m-n|=r$ and uses the bound

$$\sum_{n\le x} d(n)d(n+r) \;\ll\; \Bigl(\sum_{d|r}\frac{1}{d}\Bigr)\, x\log^2 x. \tag{22}$$

This is uniform for $1\le r\le x$ and follows from the work of P. Shiu [16] on multiplicative functions. Hence from the above estimates we obtain

$$\int_0^\infty f(t)|\zeta(\sigma+it)|^4\,dt =$$

$$\int_0^\infty f(t)|\Sigma_1(t)|^2\,dt + 2\mathrm{Re}\,\Bigl\{\int_0^\infty f(t)\chi^2(\sigma-it)\Sigma_1(t)\overline{\Sigma_2}(t)\,dt\Bigr\} + O(T^{2-2\sigma}\log^3 T).$$

Note that the argument in [5] that precedes (4.58) gives

$$\int_0^\infty f(t)\chi^2(\sigma-it)\Sigma_1(t)\overline{\Sigma_2}(t)dt \;\ll\; T_0^{-1}T^{1-2\sigma}\sum_{m\ll T} d(m)m^{-\sigma}\sum_{n\ll T} d(n)n^{\sigma-1}$$

$$\ll T_0^{-1}T^{1-2\sigma}T^{1-\sigma}\log T\cdot T^\sigma\log T \;=\; T^{2-2\sigma}T_0^{-1}\log^2 T$$

for a parameter T_0 satisfying $T^\varepsilon \ll T_0 \ll T^{\varepsilon-1}$, so that we further have

$$\int_0^\infty f(t)|\zeta(\sigma+it)|^4\,dt = \sum_{n=1}^\infty d^2(n)n^{-2\sigma}\mathrm{Re}\Bigl\{\int_0^\infty f(t)\nu\Bigl(\frac{t}{2\pi n}\Bigr)\,dt\Bigr\} \tag{23}$$

$$+ \sum_{m,n=1;m\ne n,1-\delta\le m/n\le 1+\delta}^\infty d(m)d(n)(mn)^{-\sigma}\mathrm{Re}\Bigl\{\int_0^\infty f(t)\nu\Bigl(\frac{t}{2\pi m}\Bigr)\Bigl(\frac{m}{n}\Bigr)^{it}\,dt\Bigr\}$$

$$+ O(T^{2-2\sigma}\log^3 T),$$

by the argument leading to Theorem 4.3 of [5]. In fact, (23) is a weak analogue of Th. 4.3, since the error term in (23) actually contributes to the second main term in the asymptotic formula for $\int_0^T |\zeta(\sigma+it)|^4 dt$, but for our purposes (23) is sufficient. By the first derivative test we have

$$\sum_{m,n=1;m\ne n,1-\delta\le m/n\le 1+\delta}^\infty d(m)d(n)(mn)^{-\sigma}\mathrm{Re}\Bigl\{\int_0^\infty f(t)\nu\Bigl(\frac{t}{2\pi m}\Bigr)\Bigl(\frac{m}{n}\Bigr)^{it}\,dt\Bigr\}$$

$$\begin{aligned}&\ll \sum_{m\neq n\le 3T, 1-\delta\le m/n\le 1+\delta} d(m)d(n)(mn)^{-\sigma}|\log\frac{m}{n}|^{-1} \qquad (\delta = 1/2)\\ &\ll \sum_{n\le 3T} d(n)n^{1-2\sigma}\sum_{n/2\le m\le 3n/2, m\neq n} d(m)|m-n|^{-1}\\ &\ll \sum_{1\le r\le 3T}\frac{1}{r}\sum_{n\le 3T} d(n)d(n+r)n^{1-2\sigma} \ll T^{2-2\sigma}\log^3 T,\end{aligned}$$

where we used partial summation and (22). Finally with

$$\hat{f}(s) = \int_0^\infty f(x)x^{s-1}dx, \quad P(s) = \int_0^\infty \nu(x)x^{-s}dx$$

we have, similarly as in the proof of (4.62) in [5],

$$\sum_{n=1}^\infty d^2(n)n^{-2\sigma}\mathrm{Re}\Big\{\int_0^\infty f(t)\nu(\frac{t}{2\pi n})\,dt\Big\}$$

$$= \mathrm{Re}\Big\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\hat{f}(s)(2\pi)^{1-s}\frac{\zeta^4(2\sigma+s-1)}{\zeta(4\sigma+2s-2)}P(s)\,ds\Big\},$$

where $c > 0$. For $c > \frac{3}{2} - 2\sigma$ the poles of the integrand are $s = 1$ with residue

$$\hat{f}(1)\frac{\zeta^4(2\sigma)}{\zeta(4\sigma)} = \Big(\int_T^{2T} dx + O(T_0)\Big)\frac{\zeta^4(2\sigma)}{\zeta(4\sigma)} = (T + O(T_0))\frac{\zeta^4(2\sigma)}{\zeta(4\sigma)},$$

and at $s = 2-2\sigma$ with residue

$$T^{2-2\sigma}(D_1(\sigma)\log^3 T + D_2(\sigma)\log^2 T + D_3(\sigma)\log T + D_4(\sigma)) + O(T_0T^\varepsilon).$$

Hence shifting the line of integration to $\mathrm{Re}\, s = 2-2\sigma-\delta$ for small $\delta > 0$ we obtain, in view of

$$\hat{f}(s) \ll T^{\mathrm{Re}\, s}, \quad P(s) \ll_A |\mathrm{Im}\ s|^{-A} \quad (A > 0 \text{ fixed}),$$

that

$$\sum_{n=1}^\infty d^2(n)n^{-2\sigma}\mathrm{Re}\ \Big\{\int_0^\infty f(t)\nu(\frac{t}{2\pi n})\ dt\Big\} = \frac{\zeta^4(2\sigma)}{\zeta(4\sigma)}T + O(T^{2-2\sigma}\log^3 T)$$

with a suitable choice of T_0. This completes the proof of (18), with the remark that the above proof clearly shows that by further elaboration one could obtain a more exact estimation of $E_2(T,\sigma)$. However, any improvements of (18) that could be obtained in this way would not improve (17) for σ close to 1/2.

We note that (17) is analogous to

$$E_1(T,\sigma) \ll T^{1/(1+4\sigma)}\log^c T \qquad (c>0, 1/2<\sigma<3/4), \tag{24}$$

proved by K. Matsumoto [11]. Bounds for $E_1(T,\sigma)$ when $3/4 \le \sigma < 1$ are given in Ch. 2 of [5] by using the theory of exponent pairs. In particular, it is proved that $E_1(T,\sigma) \ll T^{1-\sigma}$ holds for $1/2<\sigma<1$, which supersedes (24) for $3/4\le\sigma<1$, so that the analogy with (17) is complete.

The evaluation of (16) may be obtained by going carefully through Motohashi's evaluation of $I_4(T,\frac{1}{2};\Delta)$ with the appropriate modifications. The latter is extensively expounded in Ch. 5 of [5]. In particular, in (5.90) of [5] the expressions over the discrete and continuous spectrum provide analytic continuation for $u=v=w=z=\sigma$. Hence eventually one obtains

$$\begin{aligned} I_4(T,\sigma;\Delta) &= F_0(T,\sigma;\Delta) + \\ &+ \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|\zeta(\frac12+i\xi)|^4|\zeta(2\sigma-\frac12+i\xi)|^2}{|\zeta(1+2i\xi)|^2}\Theta(\xi;T,\sigma,\Delta)\,d\xi \\ &+ \sum_{j=1}^{\infty}\alpha_j H_j^2(1/2)H_j(2\sigma-\tfrac12)\Theta(\varkappa_j;T,\sigma,\Delta) + \\ &+ \sum_{k=6}^{\infty}\sum_{j=1}^{d_{2k}}\alpha_{j,2k}H_{j,2k}^2(2\sigma-\tfrac12)\Lambda(k;T,\sigma,\Delta) + O\Big(\frac{\log^2 T}{T}\Big), \end{aligned} \tag{25}$$

where the functions F_0,Θ,Λ appearing in (25) are the appropriate modifications of the functions $F_0(T,\Delta),\Theta(\xi;T,\Delta),\Lambda(k;T,\Delta)$ appearing in (5.10) of [5], and the remaining notation from spectral theory is the same as in [5]. To see quickly what will be the shape of the asymptotic expression for $I_4(T,\sigma;\Delta)$ when $T^{1/2}\le\Delta\le T^{1-\varepsilon}$, note first that the contribution of everything except the discrete spectrum over non-holomorphic cusp forms will be $O(\log^C T)$. This is the same as for the case $\sigma=1/2$, and moreover the discrete spectrum may be truncated at $\varkappa_j = T\Delta^{-1}\log^2 T$ with negligible error. This is essentially due to the presence of the exponential factor $\exp(-(t/\Delta)^2)$ in the definition of I_4, which is in one way or another reproduced through all the transformations leading to (25). Further note that the function Θ in (25) will itself contain the function

$$M(s,w;\Delta)\ :=\int\limits_0^\infty y^{s-1}(1+y)^{-w}\exp(-\frac{\Delta^2}{4}\log^2(1+y))\,dy\quad(\mathrm{Re}s>0).$$

In the case of $I_4(T,\frac12;\Delta)$ that above function essentially had to be evaluated at $s=1/2\pm i\varkappa_j, w=1/2\pm iT$, but in the case of the general $I_4(T,\sigma;\Delta)$ it has to be evaluated at $s=2\sigma-1/2\pm i\varkappa_j, w=\sigma\pm iT$. In both cases this may be achieved by the saddle-point method (this is where the condition $\Delta>T^{1/2}$ becomes useful) and, for $1\ll\varkappa_j\le T\Delta^{-1}\log^2 T$, the functions

$$M(\tfrac12-i\varkappa_j,\tfrac12+iT;\Delta),\quad M(2\sigma-\tfrac12-i\varkappa_j,\sigma+iT;\Delta)$$

have the same saddle point $y_0\sim\varkappa_j/T$. Since

$$y_0^{2\sigma-1/2-1}\sim y_0^{-1/2}(\varkappa_jT^{-1})^{2\sigma-1}\qquad\qquad(T\to\infty),$$

one will obtain in $I_4(T,\sigma;\Delta)$ essentially the same expression as for $\sigma=1/2$, only each term will be multiplied by a factor which is asymptotic to $(\varkappa_j/T)^{2\sigma-1}$, and $H_j(2\sigma-1/2)$ will appear instead of $H_j(1/2)$ at one place. Therefore one should obtain, for $T^{1/2}\le\Delta\le T^{1-\varepsilon}$ and a suitable constant $C(\sigma)$,

$$\begin{aligned}I_4(T,\sigma,\Delta)\ &\sim C(\sigma)T^{1/2-2\sigma}\times\\ \times&\sum_{\varkappa_j\le\Delta^{-1}\log^2T}\alpha_j\varkappa_j^{2\sigma-3/2}H_j^2(\tfrac12)H_j(2\sigma-\tfrac12)e^{-(\frac{\Delta\varkappa_j}{2T})^2}\sin(\varkappa_j\log\frac{\varkappa_j}{4eT})\\ &\qquad+\ O(\log^CT).\end{aligned}\tag{26}$$

By the same principles the integrated version of (25) shoud read, for $V^{1/2}\le\Delta\le V^{1-\varepsilon}$,

(27)

$$\begin{aligned}&\int\limits_V^{2V}I_4(T,\sigma;\Delta)\,dT\sim O(\Delta)+O(V^{1/2}\log V)\ +\ C(\sigma)V^{(3-4\sigma)/2}\times\\ &\times\sum_{\varkappa_j\le T\Delta^{-1}\log^2T}\alpha_j\varkappa_j^{2\sigma-5/2}H_j^2(\frac12)H_j(2\sigma-\frac12)e^{-(\frac{\Delta\varkappa_j}{2V})^2}\cos(\varkappa_j\log\frac{\varkappa_j}{4eV}).\end{aligned}$$

One can obtain without difficulty an analogue of Lemma 5.1 of [5] for $E_2(T,\sigma)$, which enables one to obtain upper bounds for $E_2(T,\sigma)$. In conjunction with (27) we shall therefore obtain, for $T^{1/2} \le \Delta \le T^{-\varepsilon}$ and $1/2 < \sigma < 3/4$,

$$\begin{aligned} E_2(2T,\sigma) - E_2(T,\sigma) \quad &\ll \quad \Delta \log T + T^{1/2}\log^C T + \\ &\max_{T/3\le t\le 3T} t^{\frac12(3-4\sigma)} \times \\ \times \sum_{\varkappa_j \le T\Delta^{-1}\log^2 T} &\alpha_j \varkappa_j^{2\sigma-5/2} H^2(\tfrac12)|H_j(2\sigma-\tfrac12)e^{-(\frac{\Delta\varkappa_j}{2t})^2}\cos(\varkappa_j \log\frac{\varkappa_j}{4et})| \\ &\ll \Delta\log T + T^{1/2}\log^C T + T^{(3-4\sigma)/2}(T\Delta^{-1})^{2\sigma-1/2}\log^C T \\ &\ll T^{2/(1+4\sigma)}\log^C T \end{aligned}$$

for $\Delta = T^{2/(1+4\sigma)}$, thereby establishing Motohashi's result (17). Here we used the bound

$$\begin{aligned} \sum_{\varkappa_j\le K} H_j^2(\tfrac12)|H_j(2\sigma-\tfrac12)| &\le \Big(\sum_{\varkappa_j\le K} H_j^4(\tfrac12)\Big)^{1/2}\Big(\sum_{\varkappa_j\le K} H_j^2(2\sigma-\tfrac12)\Big)^{1/2} \\ &\ll K^2\log^C K, \end{aligned}$$

since $\sum_{\varkappa_j\le K} H_j^4(\frac12) \ll K^2\log^C K$, and also

$$\sum_{\varkappa_j\le K} H_j^2(2\sigma-\tfrac12) \ll K^2\log^C K \qquad (1/2\le\sigma\le 1). \tag{28}$$

The bound in (28) is proved analogously as in the well-known case $\sigma = 1/2$, only it is less difficult since $2\sigma - 1/2 \ge 1/2$ for $\sigma \ge 1/2$, and in general $H_j(s)$ (like many other functions defined by analytic continuation of Dirichlet series) is less difficult to handle as $\mathrm{Re}\, s$ increases.

The first open problem I have in mind concerning $E_2(T,\sigma)$ is the conjecture pertaining to its true order of magnitude, namely

$$E_2(T,\sigma) = O(T^{\frac12(3-4\sigma)+\varepsilon}),\ E_2(T,\sigma) = \Omega_\pm(T^{1/2(3-4\sigma)}) \quad (1/2<\sigma<3/4). \tag{29}$$

Since $3/2 - 2\sigma > 0$ only for $\sigma < 3/4$, the line $\sigma = 3/4$ appears to be a sort of a boundary both for $E_2(T,\sigma)$ and $E_1(T,\sigma)$. For the latter function this phenomenon was mentioned already by K. Matsumoto [11]. The O–bound in (29) is certainly very difficult, while the omega-results may be within reach. For $\sigma = 1/2$ it is known that $E_2(T) = E_2(T, \frac{1}{2}) = \Omega(T^{1/2})$ (see Ch. 5 of [5]), although I am certain that the sharper result

$$E_2(T) = \Omega_{\pm}(T^{1/2})$$

must hold. Another reason for the fact that very likely "something" happens with $E_2(T, \sigma)$ at $\sigma = 3/4$ is that, for $\sigma > 3/4$, we have (see (26) and (27))

$$H_j(2\sigma - \tfrac{1}{2}) = \sum_{n=1}^{\infty} t_j(n) n^{1/2-2\sigma} \ll_j \quad 1,$$

while for $\sigma \le 3/4$ the above series representation is not valid.

For $1/2 \le \sigma < 3/4$ fixed I also conjecture that

$$\int_0^T E_2^2(t, \sigma)\, dt \sim C_2(\sigma) T^{4-4\sigma} \qquad (T \to \infty) \tag{30}$$

holds with a suitable $C_2(\sigma) > 0$. However, I have no ideas what the explicit value of $C_2(\sigma)$ ought to be. For the less difficult problem of the mean square of $E_1(t, \sigma)$ the situation is different. Namely K. Matsumoto and T. Meurman [12] proved

$$\int_0^T E_1^2(t, \sigma)\, dt = C_1(\sigma) T^{(5-4\sigma)/2} + O(T) \qquad (1/2 < \sigma < 3/4) \tag{31}$$

with

$$C_1(\sigma) = \frac{2}{5 - 4\sigma}(2\pi)^{(4\sigma-3)/2} \frac{\zeta^2(3/2)}{\zeta(3)} \zeta(\tfrac{5}{2} - 2\sigma)\zeta(\tfrac{1}{2} + 2\sigma)$$

and

$$\int_0^T E_1^2(t, \tfrac{3}{4})\, dt = \frac{\zeta^2(3/2)\zeta(2)}{\zeta(3)} T \log T + O(T \log^{1/2} T). \tag{32}$$

Matsumoto and Meurman also proved

$$E_1(T, \sigma) = \Omega_{+}(T^{3/4-\sigma}(\log T)^{\sigma-1/4}) \qquad (1/2 < \sigma < 3/4), \tag{33}$$

while (32) yields $E_1(T, \frac{3}{4}) = \Omega(\log^{1/2} T)$. Note that (33) is the (strong) analogue of the conjectural Ω_+–result (29) for $E_2(T,\sigma)$. I am convinced that the technique used for proving the Ω_--result for $E(T) = E_1(T, \frac{1}{2})$ (see Th. 3.4 of [5]) can be used to obtain $E_1(T,\sigma) = \Omega_-(T^{3/4-\sigma})$ for $1/2 < \sigma < 3/4$, and maybe even a slightly stronger result (i.e. $T^{3/4-\sigma}$ multiplied by a log log-factor, or even by a log-factor).

In view of (29), its analogue (unproved yet)

$$E_1(T,\sigma) = O(T^{3/4-\sigma+\varepsilon}) \qquad (1/2 < \sigma < 3/4),$$

and

$$|\zeta(\sigma + iT)|^{2k} \ll \log T \Big(\int_{T-1}^{T+1} |\zeta(\sigma+it)|^{2k}\, dt + 1 \Big) \qquad (k \in \mathbb{N}),$$

an unsettling possibility comes to my mind. It involves the function

$$\mu(\sigma) = \limsup_{t\to\infty} \frac{\log|\zeta(\sigma+it)|}{\log t},$$

for which one has trivially $\mu(\sigma) = \frac{1}{2} - \sigma$ for $\sigma \le 0$ and $\mu(\sigma) = 0$ for $\sigma \ge 1$. If the Lindelöf hypothesis that $\zeta(\frac{1}{2}+it) \ll t^{\varepsilon}$ is true, then the graph of $\mu(\sigma)$ consists of the line segments $1/2 - \sigma$ for $\sigma \le 1/2$ and 0 for $\sigma \ge 1/2$. But the above discussion prompts me to think that it is not unlikely that perhaps one has even

$$\mu(\sigma) = \begin{cases} 1/2 - \sigma & \sigma \le 1/4, \\ 3/8 - \sigma/2 & 1/4 \le \sigma \le 3/4, \\ 0 & \sigma \ge 3/4. \end{cases}$$

This is much weaker than the Lindelöf hypothesis, as it implies $\zeta(\frac{1}{2}+it) = \Omega(t^{1/8-\delta})$ for any given $\delta > 0$. Thus it would a fortiori contradict the Riemann hypothesis, since it is well-known that the Riemann hypothesis implies the Lindelöf hypothesis.

If the problem of establishing (30) is perhaps intractable, perhaps it is possible to prove

$$\int_0^T E_2^2(t,\sigma)\, dt \ll T^{4-4\sigma} \log^C T \qquad (C > 0,\ 1/2 < \sigma < 3/4), \tag{34}$$

since (34) for $E_2(T) = E_2(T, \frac{1}{2})$ was proved in [8]. By the method used there it follows that

$$(35) \qquad \int_{t_r}^{t_r+\Delta} |\zeta(\sigma + it)|^4 dt \ll \Delta + \Delta^{-1} \int_{t_r-2\Delta}^{t_r+2\Delta} |E_2(t,\sigma)|\, dt,$$

and we are going to impose the spacing condition
(36)
$T < t_1 < \cdots < t_R \le 2T, t_{r+1} - t_r \ge \Delta(r = 1, \cdots, R-1), T^\varepsilon \le \Delta \le T^{1-\varepsilon}.$

Then if (34) holds we obtain from (35) that

$$(37) \qquad \sum_{r\le R}\Bigl(\int_{t_r}^{t_r+\Delta} |\zeta(\sigma + it)|^4 dt\Bigr)^2 \ll R\Delta^2 + \Delta^{-1}\sum_{r\le R}\int_{t_r-2\Delta}^{t_r+2\Delta} E_2^2(t,\sigma)\, dt$$

$$\ll R\Delta^2 + \Delta^{-1}\int_{T/3}^{3T} E_2^2(t,\sigma)\, dt \ll R\Delta^2 + T^{4-4\sigma}\Delta^{-1}\log^C T.$$

Hence by the Cauchy-Schwarz inequality one obtains from (37), for $1/2 < \sigma < 3/4$,

$$(38) \qquad \sum_{r\le R}\int_{t_r}^{t_r+\Delta} |\zeta(\sigma + it)|^4 dt \ll R\Delta + R^{1/2}T^{2-2\sigma}\Delta^{-1/2}\log^C T$$

provided that (36) holds. I note that Motohashi and I proved in [8] that, for $1/2 < \sigma < 3/4$,

$$(39) \qquad \sum_{r\le R}\int_{t_r}^{t_r+\Delta} |\zeta(\sigma + it)|^4 dt \ll R\Delta + R^{\sigma}T^{2-2\sigma}\Delta^{\sigma-1}\log^C T$$

again if (36) holds. Since

$$R^{1/2}T^{2-2\sigma}\Delta^{-1/2} < R^{\sigma}T^{2-2\sigma}\Delta^{\sigma-1} \qquad (\sigma > 1/2),$$

it follows that (38) improves (39) for $1/2 < \sigma < 3/4$. This shows the importance of establishing (34).

The discussion leading to (26) indicates that, crudely speaking, the spectral part appearing in the expression for $I_4(T,\sigma;\Delta)$ is by a factor of $\Delta^{1-2\sigma}$ smaller than the corresponding sum in the expression for $I_4(T,\frac{1}{2};\Delta)$. If one follows the proof of

$$\sum_{r\le R}\int\limits_{t_r}^{t_r+\Delta}|\zeta(\tfrac{1}{2}+it)|^4dt \ll (R\Delta + R^{1/2}T\Delta^{-1/2})\log^C T \tag{40}$$

(see Ch. 5 of [5]) with the appropriate modifications, one obtains

$$\sum_{r\le R}\int\limits_{t_r}^{t_r+\Delta}|\zeta(\sigma+it)|^4dt \ll (R\Delta + R^{1/2}T\Delta^{1/2-2\sigma})\log^C T \tag{41}$$

for $1/2<\sigma<3/4$ and $T^{1/2}\le\Delta\le T$. However, once (40) is known for $\Delta\ge T^{1/2}$, it can be essily established for $\Delta<T^{1/2}$, since $R\Delta\ll R^{1/2}T\Delta^{-1/2}$ for $\Delta\le T^{1/2}$ and each interval $[t_r,t_r+\Delta]$ lies in at most two intervals of the form $[T+(n-1)T^{1/2},T+nT^{1/2}]$, $(n=1,2,\dots)$. This procedure does not carry over to (41), in the sense that we cannot deduce in an obvious way the validity of (41) for $\Delta<T^{1/2}$ once it is known for $\Delta\ge T^{1/2}$. Note that (39) improves (41) for $R\le T^2\Delta^{-3}$, while (38) improves (41) in the whole range $1/2<\sigma<3/4$.

Another problem involving $E_2(T,\sigma)$ is to prove that $E_2(T,\sigma)$ has arbitrarily large zeros for a fixed σ satisfying $1/2<\sigma<3/4$. This a trivial consequence (since $E_2(T,\sigma)$ is a continuous function of T) of the conjectural $\Omega_\pm$-result in (29), but perhaps a direct a proof of this result might be within reach. The corresponding problem for $E_2(T)=E_2(T,\frac{1}{2})$ was mentioned in Ch. 5 of [5], where it was also noted how one obtains

$$\limsup_{T\to\infty}|E_2(T)|T^{-1/2} \;=\; +\infty$$

if a certain linear independence of spectral values can be established. The problem of the existence of arbitrarily large zeros of $E_2(T)$ still remains open. Closely related to the above topic is problem of sign changes of $E_2(T,\sigma)$. For $E_1(T,\sigma)$ I have proved (see Th. 3.3 of [5]) that every interval $[T,T+DT^{1/2}]$ for suitable $D>0$ and $T\ge T_0$ contains points τ_1,τ_2 such that

$$E_1(\tau_1,\sigma)>B\tau_1^{3/4-\sigma},\, E_1(\tau_2,\sigma)<-B\tau_2^{3/4-\sigma}$$

for $1/2 < \sigma < 3/4$ and a suitable constant $B > 0$. In my opinion the analogue of this result, which is an open problem, for $E_2(T,\sigma)$ would be the following assertion:

Every interval $[T, DT]$ for suitable $D > 1$ and $T \ge T_0$ contains points t_1, t_2 such that

$$E_2(t_1,\sigma) > Bt_1^{(3-4\sigma)/2},\, E_2(t_2,\sigma) < -Bt_2^{(3-4\sigma)/2}$$

for $1/2 < \sigma < 3/4$ and suitable $B > 0$.

Of course the latter result by continuity trivially implies both $E_2(T,\sigma) = \Omega_\pm(T^{(3-4\sigma)/2})$ and the existence of arbitrarily large zeros of $E_2(T,\sigma)$.

Finally, what is the smallest σ such that

$$\int_0^T |\zeta(\tfrac12 + it)|^4 |\zeta(\sigma + it)|^2 dt \ll T^{1+\varepsilon}\ ? \tag{42}$$

Trivially (42) holds for $\sigma = 1$, and its truth for $\sigma = 1/2$ (for $\sigma < 1/2$ it is false if ε is small enough) is the hitherto unproved sixth moment of $|\zeta(\frac12 + it)|$. In fact, at present it seems difficult to find any σ satisfying $\sigma < 1$ such that (42) holds. The bound

$$\int_0^T |\zeta(\tfrac12 + it)|^4 |\sum_{n\le N} a_n n^{it}|^2 dt \ll T^\varepsilon (T + T^{1/2}N^2 + T^{3/4}N^{5/4}) \sum_{n\le N} |a_n|^2,$$

where the $a_n{}'$s are arbitrary complex numbers, appears to be a natural tool for attacking this problem. This result is due to J.-M. Deshouillers and H. Iwaniec [3], but the term $T^{1/2}N^2$ is too large to give any $\sigma < 1$ in (42) when we approximate $\zeta(\sigma + it)$ by Dirichlet polynomials of length $\ll t^{1/2}$.

The similar problem of finding σ such that

$$\int_0^T |\zeta(\tfrac12 + it)|^2 |\zeta(\sigma + it)|^4 dt \ll T^{1+\varepsilon} \tag{43}$$

holds is certainly much less difficult. By using the bound

$$\int_0^T |\zeta(\tfrac58 + it)|^8 dt \ll T^{1+\varepsilon}$$

(see Ch. 8 of [4]) and the Cauchy-Schwarz inequality it follows that (43) holds for $\sigma = 5/8$. Again I ask whether one can find a value of σ less than $5/8$ for which (43) holds. Similarly as for (42), (43) also cannot hold for $\sigma < 1/2$ if ε is small enough, and its truth for $\sigma = 1/2$ is the sixth moment of $\zeta(s)$ on the critical line.

References

[1] R. Balasubramanian, *On the frequency of Titchmarsh's phenomenon for $\zeta(s)$ IV*, Hardy-Ramanujan **J.9** (1986), 1–10.

[2] R. Balasubramanian, A. Ivić and K. Ramachandra, *The mean square of the Riemann zeta-function on the line $\sigma = 1$*, L'Enseignement Mathématique **38** (1992), 13–25.

[3] J.-M. Deshouillers and H. Iwaniec, *Power mean values of the Riemann zeta-function*, Mathematika **29** (1982), 202–212.

[4] A. Ivić, *The Riemann zeta-function*, John Wiley & Sons, New York, (1985).

[5] A. Ivić, *The mean values of the Riemann zeta-function*, Tata Institute for Fundamental Research **LN's 82**, Bombay 1991 (distr. by Springer Verlag, Berlin etc., 1992).

[6] A. Ivić, *The moments of the zeta-function on the line $\sigma = 1$*, Nagoya Math. J. **135** (1994), 113–129.

[7] A. Ivić and A. Perelli, *Mean values of certain zeta-functions on the critical line*, Litovskij Mat. Sbornik **29** (1989), 701–714.

[8] A. Ivić and Y. Motohashi, *The mean square of the error term for the fourth moment of the zeta-function*, Proc. London Math. Soc. (3) **69** (1994), 309–329.

[9] D. Joyner, *Distribution theorems for L-functions*, Longman Scientific & Technical, Essex (1986).

[10] A. Laurinčinkas, *The limit theorem for the Riemann zeta-function on the critical line I*, (Russian), Litovskij Mat. Sbornik **27** (1987), 113–132 and II ibid. **27** (1987), 459–500.

[11] K. Matsumoto, *The mean square of the Riemann zeta-function in the critical strip*, Japan. J. Math. **13** (1989), 1–13.

[12] K. Matsumoto and T. Meurman, *The mean square of the Riemann zeta-function in the critical strip* II, Acta Arith. **68** (1994), 369–382; III, Acta Arith. **64** (1993), 357–382.

[13] K. Ramachandra, *Some remarks on the mean value of the Riemann zeta-function and other Dirichlet series IV*, J. Indian Math. Soc. **60** (1994), 107–122.

[14] K. Ramachandra, *Lectures on the mean-value and omega-theorems for the Riemann zeta-function*, LNs **85**, Tata Institute of Fundamental Research, Bombay 1995 (distr. by Springer Verlag, Berlin etc.).

[15] P. Shiu, *A Brun-Titchmarsh theorem for multiplicative functions*, J. Reine Angew. Math. **31** (1980), 161–170.

[16] E.C. Titchmarsh, *The theory of the Riemann zeta-function (2nd ed.)*, Oxford, Clarendon Press, (1986).

Aleksandar Ivić
Katedra Matematike RGF-a
Universiteta u Beogradu
Djušina 7, 11000 Beograd,
Serbia (Yugoslavia)