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On approximation by Lüroth Series

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RÉSUMÉ. Pour $x \in]0,1]$, on note p_n/q_n la suite des convergents de la série de Lüroth associée, et on définit par $\theta_n = q_n x - p_n$, $n \geq 1$ ses coefficients d'approximation. Dans [BBDK], on détermine la fonction de répartition limite de la suite (θ_n), en utilisant l'extension naturelle du système ergodique sous-jacent au développement en série de Lüroth. Nous montrons ici que cela peut être fait sans cette considération . Plus précisément, nous démontrons que pour tout n, la répartition de θ_n coïncide avec la répartition limite. On étudiera aussi la répartition pour presque tout x de la suite $(\theta_n, \theta_{n+1})_{n\geq 1}$, ainsi que celles issues de suites telles que $(\theta_n + \theta_{n+1})_{n\geq 1}$. On obtiendra que pour presque tout x, la suite (θ_n, θ_{n+1}) possède une fonction de répartition continue et singulière. On observera de plus que θ_n et θ_{n+1} sont positivement corrélés.

ABSTRACT. Let $x \in (0,1]$ and p_n/q_n, $n \geq 1$ be its sequence of Lüroth Series convergents. Define the approximation coefficients $\theta_n = \theta_n(x)$ by $\theta_n = q_n x - p_n$, $n \geq 1$. In [BBDK] the limiting distribution of the sequence $(\theta_n)_{n\geq 1}$ was obtained for a.e. x using the natural extension of the ergodic system underlying the Lüroth Series expansion. Here we show that this can be done without the natural extension. In fact we will prove that for each n, θ_n is already distributed according to the limiting distribution. Using the natural extension we will study the distribution for a.e. x of the sequence $(\theta_n, \theta_{n+1})_{n\geq 1}$ and related sequences like $(\theta_n + \theta_{n+1})_{n\geq 1}$. It turns out that for a.e. x the sequence $(\theta_n, \theta_{n+1})_{n\geq 1}$ is distributed according to a continuous singular distribution function G. Furthermore we will see that two consecutive θ's are positively correlated.

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1. Introduction

Let $x \in (0,1]$, then

(1)
$$x = \frac{1}{a_1} + \frac{1}{a_1(a_1-1)a_2} + \ldots + \frac{1}{a_1(a_1-1)\cdots a_{n-1}(a_{n-1}-1)a_n} + \cdots ,$$

where $a_n \geq 2$, $n \geq 1$. J. Lüroth, who introduced the series expansion (1) in 1883, showed (among other things) that every irrational number x has a unique infinite expansion (1) and that each rational either has a finite or an infinite periodic expansion, see also [L] and [Pe]. The series expansion (1) of x is called the *Lüroth Series* of x.

Dynamically the Lüroth series expansion (1) of x is generated by the operator $T : [0,1] \to [0,1]$, defined by

$$Tx := \lfloor\frac{1}{x}\rfloor \left(\lfloor\frac{1}{x}\rfloor + 1\right) x - \lfloor\frac{1}{x}\rfloor\ ,\ x \neq 0; T0 := 0, \tag{2}$$

(see also figure 1), where $\lfloor\xi\rfloor$ denotes the greatest integer not exceeding ξ. For $x \in [0,1]$ we define $a(x) := \lfloor\frac{1}{x}\rfloor + 1$, $x \neq 0$; $a(0) := \infty$ and $a_n(x) = a(T^{n-1}x)$ for $n \geq 1$. From (2) it follows that $Tx = a_1(a_1-1)x - (a_1-1)$, and therefore

$$x = \frac{1}{a_1} + \frac{1}{a_1(a_1-1)}Tx = \frac{1}{a_1} + \frac{1}{a_1(a_1-1)a_2} + \cdots + \frac{T^n x}{a_1(a_1-1)\cdots a_n(a_n-1)}.$$

Putting

$$\frac{p_n}{q_n} = \frac{1}{a_1} + \sum_{k=1}^{n-1} \frac{1}{a_1(a_1-1)\cdots a_k(a_k-1)a_{k+1}}\ ,\ \ n \geq 1, \tag{3}$$

where $q_1 := a_1$; $q_n = a_1(a_1-1)\cdots a_{n-1}(a_{n-1}-1)a_n$, $n \geq 2$, it follows from (3) that

$$x - \frac{p_n}{q_n} = \frac{T^n x}{q_n(a_n-1)}\ ,\ \ n \geq 1. \tag{4}$$

From $a_n \geq 2$ and $0 \leq T^n x \leq 1$ it follows that the series from (1) converges to x. We will write

$$x = < a_1, a_2, \cdots, a_n, \cdots > \quad \text{and} \quad \frac{p_n}{q_n} = < a_1, a_2, \cdots, a_n > . \tag{5}$$

In [JdV], H. Jager and C. de Vroedt showed that the stochastic variables $a_1(x), \ldots, a_n(x), \ldots$ are independent with $\lambda_1(a_n = k) = \frac{1}{k(k-1)}$ for $k \geq 2$, and that T is measure preserving and ergodic with respect to Lebesgue measure. Here and in the following λ_n will denote Lebesgue measure on $\mathbf{R}^n$. From the ergodicity of T and Birkhoff's Individual Ergodic Theorem a number of results were obtained, analogous to classical results on continued fractions, e.g.

$$\lim_{n\to\infty} (a_1 a_2 \cdots a_n)^{1/n} = e^c, \text{ a.e. where } c \approx 1.25\,,$$

$$\lim_{n\to\infty} \frac{1}{n} \log(x - \frac{p_n}{q_n}) = -d, \text{ a.e., where } d \approx 2.03\,.$$

Here and in the following a.e. will be with respect to Lebesgue measure.

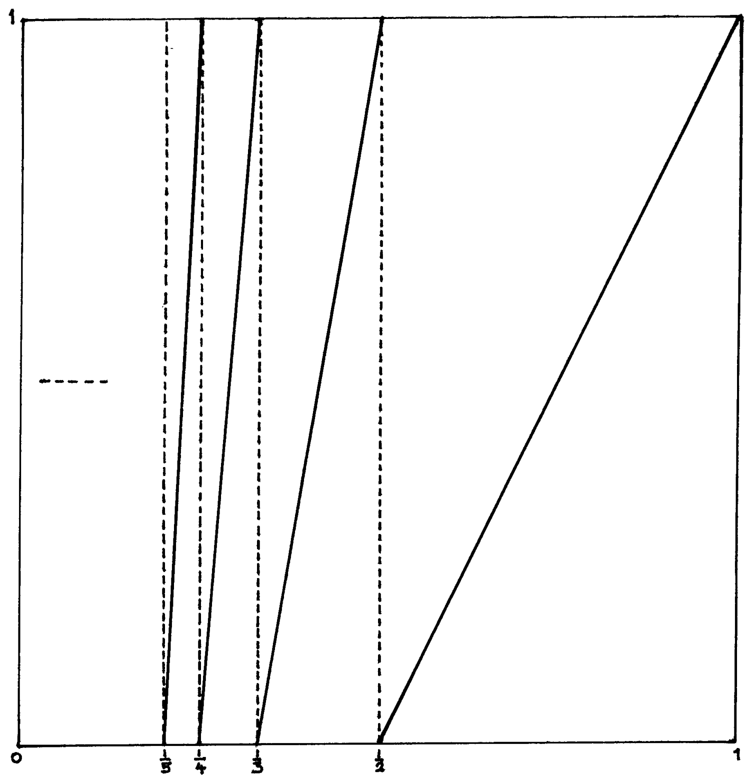

FIGURE 1. The map T

In view of (4) it is natural to define and study the so-called *approximation coefficients* $\theta_n = \theta_n(x)$, $n \geq 1$, defined by

$$\theta_n = \theta_n(x) := q_n \left| x - \frac{p_n}{q_n} \right|, \quad n \geq 1.$$

As in the case of the regular continued fraction these θ's give an indication of "the quality of approximation of x by its n-th convergent p_n/q_n", see also [JK]. (In case of the regular continued fraction one defines $\Theta_n :=$

$q_n|q_nx - p_n|$, $n \geq 1$, where p_n/q_n is the n-th regular convergent of x). Note that the absolute value signs are in fact superfluous here. In view of (4) one has

$$\theta_n \;=\; \frac{T^n x}{a_n - 1}\;,\;\; n \geq 1. \tag{6}$$

Putting $T_n := T^n x$ it follows from (2) and (5) that

$$T_n \;=<\; a_{n+1},\, a_{n+2}, \cdots >\;.$$

We say that T_n is *the future of* x *at time* n. Similarly is

$$\begin{aligned} V_n &= \;<\; a_n,\, a_{n-1}, \cdots a_1 > \\ &= \;\frac{1}{a_n} + \frac{1}{a_n(a_n-1)a_{n-1}} + \cdots + \frac{1}{a_n(a_n-1)\cdots a_2(a_2-1)a_1} \end{aligned}$$

the past of x *at time* n. Putting $V_0 := 0$, from (6) one sees that θ_n is expressed in terms of both the past (viz. a_n) and the future. Therefore, in order to obtain the distribution of the sequence $(\theta_n)_{n\geq 1}$ for a.e. x the natural extension of the ergodic system ((0,1], $\mathcal{B}_1$, λ_1, T) (here $\mathcal{B}_1$ is the collection of Borel sets of (0,1]) was constructed in [BBDK].

THEOREM 1. ([*BBDK*]) *Let* $\Omega := [0,1] \times [0,1]$ *and* $\mathcal{B}_2$ *be the collection of Borel sets of* Ω. *Let* $\mathcal{T} : \Omega \to \Omega$ *be defined by*

$$\mathcal{T}(x,y) := \left(\, Tx \,,\; \frac{1}{a(x)} + \frac{y}{a(x)(a(x)-1)} \,\right)\,,\;\; (x,y) \in \Omega,$$

then the system

$$([0,1] \times [0,1], \mathcal{B}_2,\, \lambda_2,\, \mathcal{T})$$

is the natural extension of $([0,1], \mathcal{B}_1,\, \lambda_1,\, T)$.
Moreover, $([0,1] \times [0,1], \mathcal{B}_2,\, \lambda_2,\, \mathcal{T})$ *is Bernoulli.*

From this theorem we have the following lemma.

LEMMA 1. *For almost all* x *the two-dimensional sequence*

$$\mathcal{T}^n(x,0) \;=\; (T_n, V_n)\;,\;\; n \geq 1,$$

is uniformly distributed over $\Omega = [0,1] \times [0,1]$.

The distribution of the sequence $(\theta_n)_{n\geq 1}$ now follows from lemma 1.

THEOREM 2. *For almost all* x *and for every* $z \in (0,1]$ *the limit*

$$\lim_{N\to\infty} \frac{1}{N} \#\{\, 1 \leq j \leq N \;:\; \theta_j(x) < z \,\}$$

exists and equals $F(z)$, *where*

$$F(z) = \sum_{k=2}^{\lfloor\frac{1}{z}\rfloor+1} \frac{z}{k} + \frac{1}{\lfloor\frac{1}{z}\rfloor + 1}, \quad 0 < z \leq 1. \tag{7}$$

Taking the first moment, theorem 2 yields that for a.e. x

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \theta_n = \frac{\zeta(2)-1}{2} = 0.322467\cdots, \tag{8}$$

where $\zeta(s)$ is the zeta-function.

Remarks. In fact one needs not use the natural extension to study the distribution of the sequence $(\theta_n)_{n\geq 1}$. Since

$$T_n = \frac{1}{a_{n+1}} + \frac{T_{n+1}}{a_{n+1}(a_{n+1}-1)},$$

see also (2), it follows that

$$a_{n+1}T_n = 1 + \frac{T_{n+1}}{a_{n+1}-1}$$

and therefore (6) yields that

$$\theta_{n+1} = a_{n+1}T_n - 1, \quad n \geq 1, \tag{9}$$

i.e. the distribution of the sequence $(\theta_n)_{n\geq 1}$ can be obtained from $([0,1], \mathcal{B}, \lambda, T)$.

It was pointed out by one of the referees that Lemma 1 and Theorem 2 above can also be obtained as simple Corollaries of a strong result by J. Galambos, see [G], Theorem 6.2. Furthermore, using (6) and Galambos' Theorem 6.2, one can calculate the exact distribution of each of the θ_n's, not only the limiting distribution.

In fact, from (9) and the fact that T_n is uniformly distributed on the unit interval for each n, one has

$$\begin{aligned} P(\theta_{n+1} < z) &= \sum_{a=2}^{\infty} P(T_n < \tfrac{z+1}{a},\, T_n \in [\tfrac{1}{a}, \tfrac{1}{a-1})) \\ &= \sum_{k=2}^{\lfloor\frac{1}{z}\rfloor+1} \tfrac{z}{k} + \tfrac{1}{\lfloor\frac{1}{z}\rfloor+1}, \quad 0 < z \leq 1, \end{aligned}$$

thus we see that F from (7) is the distribution function for each θ_n. As a corollary we also have that $E(\theta_n) = \frac{1}{2}(\zeta(2)-1) = 0.322467\cdots$.

In this paper we will study the distribution for a.e. x of the sequence $(\theta_n, \theta_{n+1})_{n\geq 1}$ and related sequences like $(\theta_n + \theta_{n+1})_{n\geq 1}$. We will show that two consecutive θ's are positively correlated.

2. On the relation between θ_n and θ_{n+1}

From (6) and (9) it is natural to define the map $\Psi : \Omega \to \Omega$, given by

$$\Psi(x,y) := \left(\frac{x}{a(y)-1},\, a(x)x-1\right),\ (x,y)\in\Omega\,.$$

Obviously one has

$$\Psi(T_n,V_n) = (\theta_n,\theta_{n+1}),\ n\geq 1. \tag{10}$$

Putting

$$V_{A,B} := \{(x,y)\in\Omega : a(x)=A,\, a(y)=B\},\ A,B\geq 2,$$

one finds

$$V_{A,B} = (\frac{1}{A},\frac{1}{A-1}]\times(\frac{1}{B},\frac{1}{B-1}]\,.$$

For $(x,y)\in V_{A,B}$ one has $\Psi(x,y) = (\frac{x}{B-1},\, Ax-1)$ (where $1/A < x \leq 1/(A-1)$). Hence putting

$$\begin{cases} \alpha := \frac{x}{B-1} \Leftrightarrow x = (B-1)\alpha \\ \beta := Ax-1 \end{cases}$$

yields

$$\beta = A(B-1)\alpha - 1\,,\ \alpha\in\left(\frac{1}{A(B-1)},\frac{1}{(A-1)(B-1)}\right]\,. \tag{11}$$

Thus we see that Ψ maps the rectangle $V_{A,B}$ onto the line segment $L_{A,B}$, which has endpoints $(\frac{1}{A(B-1)},0)$ and $(\frac{1}{(A-1)(B-1)},\frac{1}{A-1})$. Notice that from (10) and (11) one has

$$\theta_{n+1} = a_{n+1}(a_n-1)\theta_n - 1,\ n\geq 1, \tag{12}$$

and $(\theta_n,\theta_{n+1})\in\Xi$, where

$$\Xi := \bigcup_{A,B\geq 2} L_{A,B}\,,$$

see also figure 2.

Notice, that from (6) it follows that always

$$0\leq\theta_n<1\,,\ n\geq 1.$$

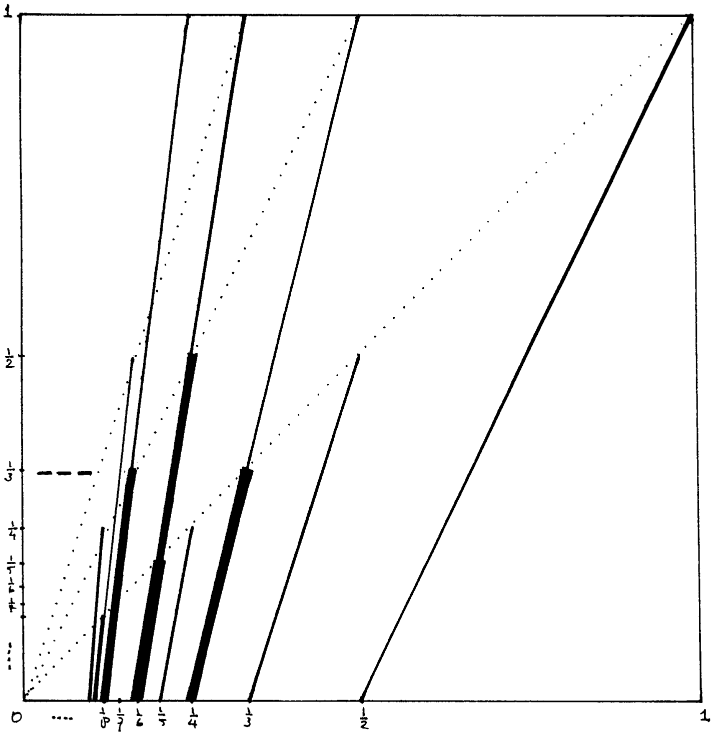

FIGURE 2. The set Ξ

Note that figure 2 shows that a Vahlen-type theorem as one has for the continued fraction (see [JK]) is not possible for Lüroth Series. That is, there does not exist a constant $c < 1$, such that for every x one has

$$\min(\theta_n(x), \theta_{n+1}(x)) < c$$

(recall that for continued fractions always $0 \leq \Theta_n(x) < 1$ and $\min(\Theta_n(x), \Theta_{n+1}(x)) < 1/2$). However, it is also clear from figure 2, that

$$(T_n, V_n) \notin V_{2,2} \Leftrightarrow \theta_n < \frac{1}{2}$$

and

$$(T_n, V_n) \in (\frac{1}{2}, \frac{3}{4}) \times (\frac{1}{2}, 1] \Rightarrow \theta_{n+1} < \frac{1}{2}.$$

We have the following proposition, which follows directly from lemma 1 (see also theorem 2 with $z = 1/2$).

PROPOSITION 1. *For almost all x one has with probability $3/4$ that $\theta_n < \frac{1}{2}$ and with probability $7/8$ that*

$$\min(\theta_n(x), \theta_{n+1}(x)) < \frac{1}{2}.$$

Furthermore, given that $\theta_n < 1/2$ one has with probability $5/6$ that $\theta_{n+1} < 1/2$. The same holds when θ_n and θ_{n+1} are interchanged.
Remarks. In view of (12) it is obvious that for a.e. x two consecutive θ's are NOT independent. In fact proposition 1 suggests that two consecutive θ's are positively correlated. That this is the case almost surely is shown in section 3.2. The situation here is similar to that for the regular continued fraction; there Vahlen's theorem suggests that two consecutive Θ's are negatively correlated. This is indeed the case as was shown by Vincent Nolte in an unpublished document, see also [N].

3. On the distribution of $(\theta_n, \theta_{n+1})_{n\geq 1}$

In this section we will show in 3.1 that for almost all x the sequence $(\theta_n, \theta_{n+1})_{n\geq 1}$ is distributed according to a continuous singular distribution function G. Before stating the result we first recall the definition of a continuous distribution function, see also [T], p. 20. In 3.2 we will study for a.e. x the distribution of the sequence $(\theta_n + \theta_{n+1})_{n\geq 1}$, which will then be used to show that two consecutive θ's are positively correlated.

3.1. A continuous singular distribution function.

DEFINITION 1. *A distribution function G is said to be continuous singular if it is continuous and if there exists a Borel set S with Lebesgue measure zero such that $\mu_G(S) = 1$. Here μ_G denotes the Lebesgue-Stieltjes measure determined by G.*
We have the following theorem.

THEOREM 3. *For almost all x and for all $(z_1, z_2) \in [0,1] \times [0,1]$ the limit*

$$\lim_{N\to\infty} \frac{1}{N} \#\{\, 1 \leq j \leq N \,:\, \theta_j(x) < z_1 \,,\, \theta_{j+1} < z_2 \,\}$$

exists and equals $G(z_1, z_2)$, where G is given by

$$G(\xi, \eta) := \sum_{A,B\geq 2} \lambda_2(V^*_{A,B}(\xi, \eta))\,, \qquad (\xi, \eta) \in \Omega, \tag{13}$$

where

$$V^*_{A,B}(\xi, \eta) := \{(\alpha, \beta) \in V_{A,B} : \alpha < \min((B-1)\xi, \frac{1+\eta}{A})\}\,. \tag{14}$$

Finally, G is a continuous singular distribution function with support Ξ.

Proof. The first assertion follows from (6), (9) and lemma 1. In order to show that G is a continuous distribution function we have to show, see also [T], section 2.2 :

(i) $G(x_1, x_2) \to 1$ as $\min(x_1, x_2) \to \infty$.
(ii) For each $i \in \{1, 2\}$, $G(x_1, x_2) \to 0$ as $x_i \to -\infty$.
(iii) $G(x_1, x_2)$ is continuous.
(iv) Let $a = (a_1, a_2)$, $b = (b_1, b_2)$, where $a_i < b_i$, $i \in \{1, 2\}$ and put

$$(a,\, b] \,:=\, \{x \,=\, (x_1, x_2) \in \mathbf{R}^2 : \, a_i < x_i \leq b_i, \, i \in \{1, 2\}\, \}.$$

Then for each cell $(a,\, b] \subset \mathbf{R}^2$ we must have

$$\Delta_a^b G \,\geq\, 0\,,$$

where

$$\Delta_a^b G \,=\, G(b_1, b_2) - G(a_1, b_2) - G(b_1, a_2) + G(a_1, a_2)\,.$$

Notice that (i) and (ii) follow from the definition of G; clearly G is monotone in each of its coordinates, and in case $x_i < 0$ (for $i \in \{1, 2\}$) one has that $G(x_1, x_2) = 0$. In case $\min(x_1, x_2) \geq 1$ it follows that $G(x_1, x_2) = 1$. That G is continuous clearly follows from (13). In order to prove (iv) we introduce for $A, B \geq 2$ a function $G_{A,B} : \Omega \to \mathbf{R}$, given by

$$G_{A,B}(\xi, \eta) \,:=\, \lambda_2(V^*_{A,B}(\xi, \eta))\,, \qquad (\xi, \eta) \in \Omega,$$

where $V^*_{A,B}(\xi, \eta)$ is as in (14). Notice that

$$G(\xi, \eta) \,:=\, \sum_{A,B \geq 2} G_{A,B}(\xi, \eta)\,, \qquad (\xi, \eta) \in \Omega.$$

It is now sufficient to show that for all $A, B \geq 2$ and each cell $(a,\, b] \subset \Omega$ one has

$$\Delta_a^b G_{A,B} \,\geq\, 0\,.$$

Fix $A, B \geq 2$ and let

$$m(\xi, \eta) \,=\, m_{A,B}(\xi, \eta) \,:=\, \min\left((B-1)\xi,\, \frac{1+\eta}{A}\right)$$

and $\pi_1(\xi, \eta) = \pi_{(A,B),1} := (B-1)\xi$, $\pi_2(\xi, \eta) = \pi_{(A,B),2} := \frac{1+\eta}{A}$, one has the following, possibly overlapping, cases.

(I) $m(a_1, b_2) < m(b_1, b_2)$ and

(Ia) $m(a_1,a_2) < m(b_1,a_2)$.
Notice that the monotonicity of π_2 as a function of its first coordinate yields that

$$\pi_1(a_1,a_2) < \pi_2(a_1,a_2)$$

and therefore $m(a_1,b_2) = m(a_1,a_2)$, from which it follows, by definition of $G_{A,B}$:

$$\Delta_a^b G_{A,B} \;=\; G_{A,B}(b_1,b_2) - G_{A,B}(b_1,a_2) \;\geq\; 0\,.$$

(Ib) $m(a_1,a_2) = m(b_1,a_2)$.
In this case one has

$$\Delta_a^b G_{A,B} \;=\; G_{A,B}(b_1,b_2) - G_{A,B}(a_1,b_2) \;>\; 0\,.$$

(II) $m(a_1,b_2) = m(b_1,b_2)$, which implies that $\pi_2(a_1,b_2) \leq \pi_1(a_1,b_2)$, which in turn yields that

$$\pi_2(a_1,a_2) \leq \pi_1(a_1,a_2)\,.$$

But then we only can have that

$$m(a_1,a_2) = m(b_1,a_2)\,,$$

from which it at once follows that

$$\Delta_a^b G_{A,B} \;=\; 0\,.$$

(III) $m(b_1,a_2) < m(b_1,b_2)$: see case **(I)**.

(IV) $m(b_1,a_2) = m(b_1,b_2)$: see case **(II)**.

In order to show that $\mu_G(\Xi) = 1$, or equivalently that $\mu_G(\Xi^c) = 0$, it is sufficient to show that for each cell $(a,\,b] \subset \Omega$, for which

$$\mathrm{card}(\,(a,\,b] \cap \Xi\,) \;\leq\; 2,$$

one has that $\mu_G((a,\,b]) = 0$, which is equivalent with

$$\Delta_a^b G \;=\; 0\,.$$

Notice that we may assume that $(a,\,b]$ is contained in $\mathcal{S}_k$ for some $k \geq 2$, where

$$\mathcal{S}_k \;:=\; (\frac{1}{k},\,\frac{1}{k-1}] \times [0,1]\,, \qquad \text{for } k \geq 2.$$

Obviously there are only finitely many values of A and B such that $L_{A,B} \cap \mathcal{S}_k \neq \emptyset$. Let A and B two such values, then $(a,\,b]$ either "lies above" $L_{A,B}$ or "below" $L_{A,B}$. Let $\mathcal{U} = \mathcal{U}(a,b)$ be the collection of all pairs (A,B) for which $(a,\,b]$ "lies above" $L_{A,B}$.

Clearly one has

$$\mu_G((a,\, b]) \;=\; \mu_G((a^*,\, b]) - \mu_G((a^*,\, b^*])\,,$$

where $a^* := (a_1, 0)$ and $b^* := (b_1, a_2)$. For $(A, B) \in \mathcal{U}$ we now define $L^*_{A,B}$ by

$$L^*_{A,B} \;:=\; \{(x,y) \in L_{A,B} \,:\, a_1 \le x \le b_1\}\,,$$

then

$$\mu_G((a^*,\, b]) \;=\; G(b_1, b_2) - G(a_1, b_2) \;=\; \lambda_2\Big(\bigcup_{(A,B)\in\mathcal{U}} \Psi^{-1} L^*_{A,B} \Big)$$

$$=\; G(b_1, a_2) - G(a_1, a_2) \;=\; \mu_G((a^*,\, b^*])\,,$$

from which the theorem follows.□

3.2. On the correlation between θ_n and θ_{n+1}. In section 2 we saw that it is likely that θ_n and θ_{n+1} are positively correlated. In order to show this, we first give some definitions.

DEFINITION 2. *The correlation-coefficient $\rho(\theta_n, \theta_{n+1})$ of θ_n and θ_{n+1} is defined by*

$$\rho(\theta_n, \theta_{n+1}) \;:=\; \frac{E(\theta_n\theta_{n+1}) - E(\theta_n)E(\theta_{n+1})}{\sqrt{V(\theta_n)}\sqrt{V(\theta_{n+1})}}\;,$$

where $E(\theta_n)$ is the expectation of θ_n, as given in (8) and $V(\theta_n)$ is the variance of θ_n, defined by

$$V(\theta_n) \;:=\; E(\theta_n^2) - (E(\theta_n))^2\,.$$

The numerator of $\rho(\theta_n, \theta_{n+1})$ equals the covariance $\mathrm{C}(\theta_n, \theta_{n+1})$ of θ_n and θ_{n+1}.

DEFINITION 3. *Let X and Y be two stochastic variables. Then the covariance $C(X,Y)$ of X and Y is given by*

$$C(X,Y) \;:=\; E(\,(X - E(X))(Y - E(Y))\,)\,.$$

Notice that $\mathrm{C}(X,Y) > 0$ indicates that whenever X (resp. Y) is bigger or smaller than its mean $\mathrm{E}(X)$ (resp. $\mathrm{E}(Y)$), the same is likely to hold for Y (resp. X), i.e. X and Y are positively correlated. Similarly $\mathrm{C}(X,Y) < 0$ tells us that X and Y are negatively correlated. In case $\mathrm{C}(X,Y) = 0$ we say that X and Y are uncorrelated. One has that

$$X \text{ and } Y \text{ are independent } \;\Rightarrow X \text{ and } Y \text{ are uncorrelated,}$$

but the converse does not hold in general.

THEOREM 4. *For almost all x one has that*

$$\rho(\theta_n, \theta_{n+1}) = \frac{(\zeta(2)-1)((1-3\zeta(2)+4\zeta(3))}{4\zeta(3)-(\zeta(2)-1)(1+3\zeta(2))} = 0.5744202\ldots$$

and therefore θ_n and θ_{n+1} are positively correlated a.s.

Since $\mathrm{E}(\theta_n) = \mathrm{E}(\theta_{n+1})$ and $\mathrm{E}(\theta_n^2) = \mathrm{E}(\theta_{n+1}^2)$ one has

$$\rho(\theta_n, \theta_{n+1}) = \frac{\mathrm{E}(\theta_n\theta_{n+1}) - \mathrm{E}(\theta_n)^2}{\mathrm{V}(\theta_n)} .$$

Notice, that from theorem 2 one has that F has density f, where

$$f(x) = \sum_{\ell=2}^{k} \frac{1}{\ell}, \quad \text{for } x \in (\frac{1}{k}, \frac{1}{k-1}], \ k \geq 2.$$

Taking second moments thus yields

$$\begin{aligned} \mathrm{E}(\theta_n^2) &= \sum_{k=2}^{\infty} \int_{\frac{1}{k}}^{\frac{1}{k-1}} z^2 f(z)dz = \sum_{k=2}^{\infty} \frac{1}{3k} \frac{1}{(k-1)^3} = \frac{1-\zeta(2)+\zeta(3)}{3} \\ &= 0.185708\ldots . \end{aligned}$$

But then it follows from (8) that

$$\mathrm{V}(\theta_n) = \frac{4\zeta(3)-(\zeta(2)-1)(1+3\zeta(2))}{12} = 0.0817226\ldots .$$

In order to find $\mathrm{E}(\theta_n\theta_{n+1})$ we will determine $\mathrm{E}((\theta_n+\theta_{n+1})^2)$, since

$$\mathrm{E}((\theta_n+\theta_{n+1})^2) = 2\mathrm{E}(\theta_n^2) + 2\mathrm{E}(\theta_n\theta_{n+1}) .$$

Hence we find for Lüroth series that

$$\mathrm{E}(\theta_n\theta_{n+1}) = \frac{1}{2}\mathrm{E}((\theta_n+\theta_{n+1})^2) - \mathrm{E}(\theta_n^2) .$$

From (6) and (9) one has

$$\theta_n + \theta_{n+1} = (a_{n+1} + \frac{1}{a_n - 1})T_n - 1 , \ n \geq 1, \tag{15}$$

and from the definition of $L_{A,B}$ it follows that

$$\frac{1}{a_{n+1}(a_n-1)} < \theta_n + \theta_{n+1} \leq \frac{a_n}{(a_{n+1}-1)(a_n-1)} , \ n \geq 1. \tag{16}$$

From lemma 1, (16) and (17) we have the following theorem.

Theorem 5. *For almost all x and for every $z \in (0,2]$ the limit*

$$\lim_{N\to\infty} \frac{1}{N} \#\{\, 1 \le j \le N \,:\, \theta_j(x) + \theta_{j+1} < z \,\}$$

exists and equals $S(z)$, where S is a continuous distribution function with density s, given by

$$s(z) \;=\; \sum_{A,B\ge 2} \frac{B-1}{A(B-1)+1} 1_{(\frac{1}{A(B-1)},\, \frac{B}{(A-1)(B-1)}]}(z)\,.$$

Here $1_{(\alpha,\beta]}(z)$ is the indicator function of the interval $(\alpha,\beta]$, i.e.

$$1_{(\alpha,\beta]}(z) \;:=\; \begin{cases} 1 & , z \in (\alpha,\beta], \\ 0 & , z \notin (\alpha,\beta]. \end{cases}$$

Theorem 6 now at once yields that

$$\mathrm{E}((\theta_n + \theta_{n+1})^2) \;=\; \sum_{B=2}^{\infty}\sum_{A=2}^{\infty} \frac{1}{B} \int_{\frac{1}{A(B-1)}}^{\frac{B}{(A-1)(B-1)}} \frac{1}{A(B-1)+1}\, z^2 dz$$

$$= \sum_{A,B\ge 2} \frac{A^2B^2 + AB(A-1) + (A-1)^2}{3BA^3(A-1)^3(B-1)^3} \;=\; \frac{3 - 3\zeta(2) + 2\zeta(2)\zeta(3)}{3}$$

$$= \; 0.6737\ldots .$$

But then

$$\mathrm{E}(\theta_n\theta_{n+1}) - \mathrm{E}(\theta_n)\mathrm{E}(\theta_{n+1}) \;=\; \frac{(\zeta(2)-1)((1-3\zeta(2)+4\zeta(3))}{12}\,,$$

and therefore the first assertion of theorem 5 is immediate. It follows that θ_n and θ_{n+1} are indeed positively correlated.□

Notice that the proof of theorem 6 can easily be adapted to derive the distribution for a.e. x of the sequence $(\theta_n - \theta_{n+1})_{n\ge 1}$. We leave this to the reader, but mention one - surprising - case : one has that the probability $\mathrm{P}(\theta_n < \theta_{n+1})$ that θ_n is smaller than θ_{n+1} is $0.391\cdots$, i.e. θ_{n+1} has the tendency to be smaller than its predecessor. To be more precise, we have the following proposition.

Proposition 2. *For almost all x one has that*

$$\lim_{N\to\infty} \frac{1}{N} \#\{\, 1 \le j \le N \,:\, \theta_j(x) < \theta_{j+1} \,\} \;=\; 0.391\cdots .$$

Proof. From (6) and (9) it follows that $\theta_n < \theta_{n+1}$ is equivalent with

$$T_n > \frac{a_n - 1}{a_{n+1}(a_n - 1) - 1} .$$

But then lemma 1 yields that $\lim_{N\to\infty} \frac{1}{N}\#\{1 \le j \le N \,:\, \theta_j(x) < \theta_{j+1}\}$ exists for a.e. x, and equals $\lambda(\mathcal{D})$, where $\mathcal{D}$ is given by

$$\mathcal{D} = \bigcup_{A,B\ge 2} \left[\frac{B}{AB-1}, \frac{B}{A(B-1)-1}\right) \times \left[0, \frac{1}{B}\right] ,$$

see also figure 3. The proposition now follows from

$$\lambda(\mathcal{D}) = \frac{\gamma}{2} + \sum_{k\ge 2} \frac{\psi(2-1/k)}{k(k+1)} = 0.39234\cdots ,$$

where γ is Euler's constant and $\psi(x) = \Gamma'(x)/\Gamma(x)$.□

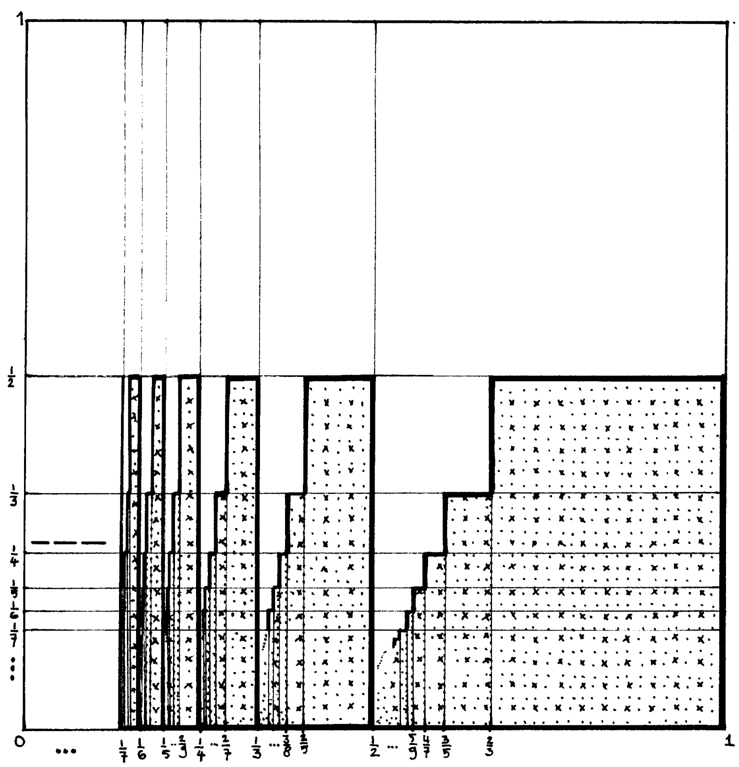

FIGURE 3. The set $\mathcal{D}$

Final remarks. Recently Jose Barrionuevo, Bob Burton and the present authors generalized the whole concept of Lüroth Series, see also [BBDK]. A new class of series expansions, the so-called *Generalized Lüroth Series* (or GLS), was introduced and their ergodic properties were studied. Examples of these GLS are the recent *alternating Lüroth Series*, as introduced by S. Kalpazidou and A. and J. Knopfmacher, but also familiar expansions like r-adic expansions (for $r \in \mathbf{Z}$, $r \geq 2$). Although β-expansions are not in this class, it turned out that many important ergodic properties of these expansions can be obtained using the appropriate GLS-expansion, see also [DKS].

Here we only want to mention that the approach of this paper can be carried over to GLS-expansions. In order to keep the exposition clear and easy we only dealt with the "classical" Lüroth expansion. Details are left to the reader, see also section 3.1 of [BBDK].

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References

[BBDK] Barrionuevo, Jose, Robert M. Burton, Karma Dajani and Cor Kraaikamp – *Ergodic Properties of Generalized Lüroth Series,* Acta Arithm., **LXXIV** (4) (1996), 311-327.

[DKS] Dajani, Karma, Cor Kraaikamp and Boris Solomyak – *The natural extension of the β-transformation,* Acta Math. Hungar., **73** (1-2) (1996), 97-109.

[G] Galambos, J. – *Representations of Real numbers by Infinite Series,* Springer LNM **502**, Springer-Verlag, Berlin, Heidelberg, New York , 1976.

[JK] Jager, H. and C. Kraaikamp – *On the approximation by continued fractions,* Indag. Math., **51** (1989), 289-307.

[JdV] Jager, H. and C. de Vroedt – *Lüroth series and their ergodic properties,* Indag. Math. **31** (1968), 31-42.

[L] Lüroth, J. – *Ueber eine eindeutige Entwickelung von Zahlen in eine unendliche Reihe,* Math. Annalen **21** (1883), 411-423.

[N] Nolte, Vincent N. – *Some probabilistic results on continued fractions,* Doktoraal scriptie Universiteit van Amsterdam, Amsterdam, August 1989.

[Pe] Perron, O. – *Irrationalzahlen,* Walter de Gruyter & Co., Berlin, 1960.

[T] Tucker, H. G. – *A Graduate Course in Probability,* Academic Press, New York, 1967.

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