Journal de Théorie des Nombres
de Bordeaux **8** *(1996), 377–386*

Boundedness of oriented walks generated by substitutions

par F.M. DEKKING ET Z.-Y. WEN

RÉSUMÉ. Soit $x = x_0x_1\ldots$ un point fixe de la substitution sur l'alphabet $\{a,b\}$, et soit $U_a = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$ et $U_b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. On donne une classification complète des substitutions $\sigma : \{a,b\}^* \to \{a,b\}^*$ selon que la suite de matrices $(U_{x_0}U_{x_1}\ldots U_{x_n})_{n=0}^{\infty}$ est bornée ou non. Cela correspond au fait que les chemins orientés engendrés par les substitutions sont bornés ou non.

ABSTRACT. Let $x = x_0x_1\ldots$ be a fixed point of a substitution on the alphabet $\{a,b\}$, and let $U_a = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$ and $U_b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. We give a complete classification of the substitutions $\sigma : \{a,b\}^* \to \{a,b\}^*$ according to whether the sequence of matrices $(U_{x_0}U_{x_1}\ldots U_{x_n})_{n=0}^{\infty}$ is bounded or unbounded. This corresponds to the boundedness or unboundedness of the oriented walks generated by the substitutions.

1. Introduction

Let A be the alphabet $\{a,b\}$, and let $x = x_0x_1\ldots$ be an infinite sequence over A. Any such sequence generates an *oriented walk* $(S_N) = (S_{N,f}(x))$ on the integers by the following rules:

$$\begin{aligned} S_{-1} &= -1, \quad S_0 = 0 \\ S_{N+1} &= \begin{cases} S_{N-1} & \text{if} \quad x_N = a, \\ 2S_N - S_{N-1} & \text{if} \quad x_N = b. \end{cases} \end{aligned} \tag{1}$$

In other words: we move one step in the same direction if $x_N = b$, and one step in the reversed direction if $x_N = a$. Another way to describe $(S_N)_{N=0}^{\infty}$ is by introducing the matrices

$$U_a = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, U_b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

then

$$S_N(x) = (1 \quad 0)U_{x_0}U_{x_1}\ldots U_{x_{N-1}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Mots-clés : Substitutions, self-similarity, walks.
Manuscrit reçu le 3 juin 1995.

In the probability literature $(S_N(x))$ is also known as persistent random walk or correlated random walk if x is obtained according to a product measure on $A^{\mathbb{N}}$. Here we shall consider the case where the sequence x is a fixed point of a primitive substitution σ on $\{a,b\}$.

Non-oriented walks $(S_{N,f}(x))$ on $\mathbb{R}$ are defined by $S_0(x) = 0$ and

$$S_{N,f}(x) = \sum_{k=0}^{N-1} f(x_k) \qquad \text{for } N \geq 1$$

where $f : A \to \mathbb{R}$, and A is now an arbitrary finite set. (It is convenient to extend f homomorphically to $A^* = \cup_{k\geq 0} A^k$, i.e. $f(w_1 \dots w_k) = f(w_1) + \dots + f(w_k)$ for $w_1 \dots w_k \in A^*$).

Non-oriented walks with x a fixed point of a substitution have been studied in [2], [3], [5], [6], [7], [10]. Here [10] contains a rather complete analysis of the behaviour of $(S_{N,f}(x))$ for two letter alphabets $A = \{a,b\}$.

It follows from a general result in [4] that by enlarging the alphabet A oriented walks generated by substitutions may be viewed as non-oriented walks generated by substitutions. Hence it might look as if the main result of [5],[6] - which admits alphabets of arbitrary sizes - would answer all questions on the two symbol oriented walk. Their result is that as $N \to \infty$

$$(2) \quad S_{N,f}(x) = (v,f)N + (\log_\theta(N))^\alpha \, N^{\log_\theta(\theta_2)} F(N) + o\left((\log(N))^\alpha \, N^{\log_\theta(\theta_2)}\right)$$

where θ is the Perron-Frobenius eigenvalue of the matrix M_σ of the substitution σ (with entries $m_{st} = |\sigma(t)|_s$ = number of occurrences of the symbol s in the word $\sigma(t)$), θ_2 the second largest (in absolute value) eigenvalue of M_σ (which is required to be unique and larger than 1) and v is the vector satisfying $M_\sigma v = \theta v$ and $\sum_{s\in A} v_s = 1$. Furthermore $\alpha+1$ is the order of θ_2 in the minimal polynomial of M, and $F : [1,\infty) \to \mathbb{R}$ is a bounded continuous function which satisfies a self-similarity property:

$$F(\theta x) = F(x) \qquad x \geq 1.$$

However, even such a simple property as boundedness of $(S_{N,f}(x))_{N\geq 0}$ can often not be resolved with (2). Let us take for example $A = \{a,b,c\}, f(a) = f(b) = +1, f(c) = -1$ and σ such that the matrix of σ equals

$$M_\sigma = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 3 & 2 \\ 4 & 4 & 4 \end{pmatrix}.$$

We quickly see that M_σ has eigenvalues $0, \theta_2 = 2$ and $\theta = 8$, and that the Perron Frobenius eigenvector $v = (1,1,2)^T$ satisfies $(v,f) = 0$. We obtain from (2) that as $N \to \infty$

$$(3) \qquad S_{N,f}(x) = N^{1/3} F(N) + o(N^{1/3}).$$

But $f(\sigma a) = f(\sigma b) = f(\sigma c) = 0$, so $S_{8N,f}(x) = 0$ for all N, and $(S_{N,f}(x))$ is bounded, in spite of the behaviour suggested by (3). (Of course (3) and the self-similarity property of F imply that $F \equiv 0$.)

The goal of this study is to determine for any substitution σ on $\{a, b\}$ whether the oriented walk in (1) will be bounded or not.

Although not explicitly formulated, the analysis of the oriented two symbol case in [10] heavily relies on the fact that a substitution σ on a two symbol alphabet with $(v, f) = 0$ automatically admits a *representation* with the same f in $\mathbb{R}$ (terminology from [1]), i.e., there exists $\lambda \in \mathbb{R}$ such that $f(\sigma(s)) = \lambda f(s)$ for $s = a, b$. (In [9] such σ are called *geometric*, see also [8]). However this is no longer true for larger alphabets, and this is the main reason that our solution to the boundedness problem is rather delicate.

2. Four types of substitutions

Let σ be a substitution on $\{a, b\}$ such that the first letter of $\sigma(a)$ is a, and let u be the fixed point of σ with $u_0 = a$. Let

$$M_\sigma = \begin{pmatrix} |\sigma a|_a & |\sigma b|_a \\ |\sigma a|_b & |\sigma b|_b \end{pmatrix}$$

be the matrix of σ. Here, as usual, $|v|_w$ denotes the number of occurrences of a word w in a word v. It appears that the question of boundedness of $(S_N(u))$ depends crucially on the entries of M_σ reduced modulo two. Let $\overline{M}_\sigma$ be this matrix. Then there are $2^4 = 16$ of these matrices possible. However, since σ, σ^2 and σ^3 all generate the same fixed point u, we only have to consider four types, namely

$$\text{(I) } \overline{M}_\sigma = \begin{pmatrix} 0 & 0 \\ \cdot & \cdot \end{pmatrix}, \text{ (II) } \overline{M}_\sigma = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \text{ (III) } \overline{M}_\sigma = \begin{pmatrix} 1 & 0 \\ \cdot & \cdot \end{pmatrix}, \text{ (IV) } \overline{M}_\sigma = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Here the dots indicate that the corresponding entries are either 0 or 1. The four cases cover respectively 6,1,8 and 1 of the 16 possibilities. For example the Fibonacci substitution $a \to ab,\ b \to a$ belongs to Type III since $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)^3 \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

3. Type I substitutions

Here $\overline{M}_\sigma = \left(\begin{smallmatrix} 0 & 0 \\ \cdot & \cdot \end{smallmatrix}\right)$. The essential feature of this case is that the number of a's in both $\sigma(a)$ and $\sigma(b)$ being even, the orientation at the beginning is the same as at the end of these words. Hence if we consider $\sigma(a)$ and $\sigma(b)$ as new symbols we can obtain a non-oriented walk which behaves very much as the original oriented walk. Formally, define the homomorphism (w.r.t. concatenation)

$$\varphi : \{\sigma(a), \sigma(b)\}^* \to \{\alpha, \beta\}^*$$

by $\varphi(\sigma(a)) = \alpha, \varphi((\sigma(b)) = \beta$. Then define a substitution $\hat{\sigma}$ on $\{\alpha, \beta\}^*$ by

$$\hat{\sigma}(\alpha) = \varphi(\sigma^2 a), \quad \hat{\sigma}(\beta) = \varphi(\sigma^2 b)$$

(actually $\sigma = \hat{\sigma}$ but for a change of alphabet!), and define $\hat{f} : \{\alpha, \beta\} \to \mathbb{R}$ by

$$\hat{f}(\alpha) = S_{\ell_a}(\sigma(a)), \quad \hat{f}(\beta) = S_{\ell_b}(\sigma(b))$$

where $\ell_a = |\sigma(a)|$ (the length of $\sigma(a)$), and $\ell_b = |\sigma(b)|$.
Let $\hat{u}$ be the fixed point of $\hat{\sigma}$ with $\hat{u}_0 = \alpha$. Then the non-oriented walk $(\hat{S}_{N,\hat{f}}(\hat{u}))$ visits a subsequence of the original oriented walk $(S_N(u))$, the time instants not being further apart than $\max(\ell_a, \ell_b)$. Hence boundedness of $(S_N(u))$ is equivalent to boundedness of $(\hat{S}_{N,\hat{f}}(\hat{u}))$. The latter can be easily resolved with Theorem 1.27 of [10].

Example. Let σ be the Prouhet-Thue-Morse substitution $\sigma(a) = abba, \sigma(b) = baab$. Then $f(\alpha) = -2, f(\beta) = 2$ and $\hat{\sigma}(\alpha) = \alpha\beta\beta\alpha, \hat{\sigma}(\beta) = \beta\alpha\alpha\beta$. It is easy to see that $(\hat{S}_{N,\hat{f}}(\hat{u})) \in \{-2, 0, 2\}$, so the original oriented walk is also bounded (actually it is confined to the set $\{-3, -2, -1, 0, 1, 2\}$).

4. An equivalence relation

We call words $v, w \in A^* = \cup_{k\geq 0} A^k$ of length n and length m *equivalent*, and denote this by $v \sim w$ if

$$S_{n-1}(v) = S_{m-1}(w) \text{ and } S_n(v) = S_m(w),$$

i.e., the associated oriented walks end at the same integer with the same orientation. In terms of the matrices U_a and U_b introduced in Section 1 we have $v \sim w$ iff $(1\ \ 0)\ U_v = (1\, 0) U_w$ (here $U_{w_1 w_2 \ldots w_k} = U_{w_1} U_{w_2} \ldots U_{w_k}$ if $w \in A^k$). Note that concatenation preserves equivalence. We denote the empty word by ϵ. Typical examples are

$$a^2 \sim \epsilon \quad , \quad abab \sim \epsilon.$$

Since the orientation changes iff an a occurs we have

Lemma 1. *If $v \sim w$, then $|v|_a \equiv |w|_a$ modulo 2.*
The following lemma is important in the analysis of Type II and IV.

Lemma 2. *For all $w \in A^*$ there exist $r, \ell \in \{0, 1\}$ and $n \in \mathbb{N}$ such that $w \sim a^\ell b^n a^r$.*
Proof. Apply $a^2 \sim \epsilon$ until $w \sim a^\ell b^{n_1} a b^{n_2} a \ldots a b^{n_k} a^r$ for some k. Then apply $bab \sim a$ $\min(n_{k-1}, n_k)$ times on $b^{n_{k-1}} a b^{n_k}$. The result is that w is equivalent to a word of the form above with k one smaller. The lemma then follows by induction. □

Lemma 3 (Squaring Lemma). *If $|w|_a$ is odd then $w^2 \sim \epsilon$.*

Proof. By Lemma 2, $w^2 \sim a^\ell b^n a^{r+\ell} b^n a^r$, and by Lemma 1, $r+\ell$ is odd. So $w^2 \sim a^\ell b^n a b^n a^r$. But since $b^n a b^n \sim a$, we obtain $w^2 \sim a^{\ell+r+1} \sim \epsilon$. □

Warning: note that in general $u \sim v$ does not imply $\sigma(u) \sim \sigma(v)$.

5. Type II substitutions

Here $\overline{M}_\sigma = \left(\begin{smallmatrix}1 & 1\\ 0 & 0\end{smallmatrix}\right)$. Note that also $\overline{M}_\sigma^n = \left(\begin{smallmatrix}1 & 1\\ 0 & 0\end{smallmatrix}\right)$. Hence the Squaring Lemma implies

$$\sigma^n(a^2) \sim \epsilon \quad , \quad \sigma^n(b^2) \sim \epsilon \quad \text{for all } n \geq 1. \tag{4}$$

Proposition 1. *If σ is of Type II, then $(S_N(u))$ is bounded iff $\sigma^2(ab) \sim \epsilon$.*

Proof. Note first that $\sigma^2(ab) \sim \epsilon$ iff $\sigma^2(ba) \sim \epsilon$. Since if for example $\sigma^2(ab) \sim \epsilon$, then

$$\sigma^2(ba) \sim \sigma^2(a^2)\sigma^2(ba) = \sigma^2(a)\sigma^2(ab)\sigma^2(a) \sim \sigma^2(a^2) \sim \epsilon.$$

$\Leftarrow$) Now $u = \sigma^2(u) = \sigma^2(u_0)\sigma^2(u_1)\ldots$, where each $\sigma^2(u_{2k}u_{2k+1}) \sim \epsilon$. This obviously implies that $(S_N(u))$ is bounded.

$\Rightarrow$) We will show that $\sigma^2(ab) \not\sim \varepsilon$ implies that $(S_N(u))$ is unbounded. Because of (4) any word $\sigma^n(w)$ is equivalent to one of the following words for some $k \geq 0$

$$[\sigma^n(ab)]^k, [\sigma^n(ba)]^k, \sigma^n(b)[\sigma^n(ab)]^k \text{ or } \sigma^n(a)[\sigma^n(ba)]^k.$$

Now we take for w the word $\sigma(ab)$. Since the numbers of a's and b's in $\sigma(ab)$ are both even, only the first two possibilities above remain, and moreover, k is even. Let us consider the first possibility, i.e., $\sigma^{n+1}(ab) \sim [\sigma^n(ab)]^k$. Then also $\sigma^n(ab) = \sigma^{n-1}(\sigma(ab)) = [\sigma^{n-1}(ab)]^k$, hence

$$\sigma^{n+1}(ab) \sim [\sigma^{n-1}(ab)]^{k^2}.$$

Continuing in this fashion we obtain

$$\sigma^{n+1}(ab) \sim [\sigma^2(ab)]^{k^{n-1}}. \tag{5}$$

Since we assume that $\sigma^2(ab) \not\sim \varepsilon$, and since $\sigma^2(ab) \sim [\sigma(ab)]^k$, we have $k > 0$, so $k \geq 2$. Since ab, and hence $\sigma^{n+1}(ab)$ has to occur (5) implies that $(S_N(u))$ is unbounded, because $\sigma^2(ab)$ contains an even number of a's which implies that the walk corresponding to $\sigma^2(ab)$ does not change orientation. In case $\sigma^{n+1}(ab) \sim [\sigma^n(ba)]^k$ the same argument applies with a and b interchanged. □

Example. We consider 2 substitutions with matrix $\left(\begin{smallmatrix}3 & 1\\ 2 & 2\end{smallmatrix}\right)$.

A. Let $\sigma(a) = aabab$, $\sigma(b) = bba$.
Then $\sigma^2(ab) \sim [\sigma(ba)]^2 \sim b^4$, so the walk is unbounded.

B. Let $\sigma(a) = abbaa$, $\sigma(b) = abb$.
Then $\sigma^2(ab) \sim \epsilon$, so the walk is bounded.

(In case B also $\sigma(ab) \sim \epsilon$. The substitution given by $\sigma(a) = abb, \sigma(b) = a$ provides an example where $\sigma^2(ab) \sim \epsilon$, but $\sigma(ab) \not\sim \epsilon$).

6. Type III substitutions

Here $\overline{M}_\sigma = \left(\begin{smallmatrix} 1 & 0 \\ \cdot & \cdot \end{smallmatrix}\right)$. Now the orientation has not changed after occurrence of $\sigma(b)$. The idea is then to keep track of the parity of the number of a's that have occurred until occurrence of u_n in u, and obtain $(S_N(u))$ as *non-oriented* walk $(S_{N,\dot{f}}(\dot{u}))$. To this end we consider a four symbol alphabet $\dot{A} = \{a^+, a^-, b^+, b^-\}$ with a substitution $\dot{\sigma}$ with fixed point $\dot{u}$. E.g. $\dot{u}_n = a^+$ will mean that $u_n = a$ and that an even number of a's have occurred in $u_0 \dots u_{n-1}$. The substitution $\dot{\sigma}$ is defined by exponentiating the symbols a and b of $\sigma(a)$ and $\sigma(b)$, by $+$'s and $-$'s according to the rules: (i) the first symbol obtains a $+$, (ii) if a symbol follows an a the exponent is reversed if it follows a b it remains equal to that of its predecessor. The $\dot{\sigma}(a^-)$ and $\dot{\sigma}(b^-)$ are obtained by reversing the signs in $\dot{\sigma}(a^+)$, respectively $\dot{\sigma}(b^+)$. Now we define $\dot{f} : A \to \mathbb{R}$ by

$$\dot{f}(a^+) = \dot{f}(b^+) = 1, \quad \dot{f}(a^-) = \dot{f}(b^-) = -1.$$

Then it maybe verified (this is a special case of the construction in [4]) that for $N = -1, 0, 1, \dots$

$$S_N(u) = S_{N+1,\dot{f}}(\dot{u}) - 1$$

where $\dot{u}$ is the fixed point of $\dot{\sigma}$ with $\dot{u}_0 = a^+$.

Example. Let σ be given by $\sigma(a) = aabab, \sigma(b) = ababb$. This induces a substitution $\dot{\sigma}$ on the alphabet $\{a^+, a^-, b^+, b^-\}$ by

$$\begin{aligned} \dot{\sigma}(a^+) &= a^+a^-b^+a^+b^-, \qquad & \dot{\sigma}(a^-) &= a^-a^+b^-a^-b^+ \\ \dot{\sigma}(b^+) &= a^+b^-a^-b^+b^+, \qquad & \dot{\sigma}(b^-) &= a^-b^+a^+b^-b^- \end{aligned}$$

By definition sign changes (a^+ followed by a^- or b^-, etc.) occur and only occur in the word $\dot{\sigma}(\cdot)$ directly following a symbol a^+ or a^-. Since $\overline{M}_\sigma = \left(\begin{smallmatrix} 1 & 0 \\ \cdot & \cdot \end{smallmatrix}\right)$, this implies that in $\dot{\sigma}(a^+)$ there is exactly one more a^+, say $y+1$, than a^- (note that $\dot{\sigma}(a^+)$ always starts with a^+), and in $\dot{\sigma}(b^+)$ there are equal numbers of a^+ and a^-, say x. Moreover, one obtains $\dot{\sigma}(a^-)$ from $\dot{\sigma}(a^+)$ by reversing signs, and similarly for $\dot{\sigma}(b^-)$. Combining these constraints, we see that the matrix $M_{\dot{\sigma}}$ of $\dot{\sigma}$ has the form

$$M_{\dot{\sigma}} = \begin{pmatrix} y+1 & y & x & x \\ y & y+1 & x & x \\ s & t & p & q \\ t & s & q & p \end{pmatrix}$$

where all entries are non-negative. It turns out that the crucial parameters are α and β defined by

$$\alpha := s - t = |\dot\sigma(a^+)|_{b^+} - |\dot\sigma(a^+)|_{b^-}$$
$$\beta := p - q = |\dot\sigma(b^+)|_{b^+} - |\dot\sigma(b^+)|_{b^-}.$$

PROPOSITION 2. *If σ if of Type III, then $(S_N(u))$ is bounded iff $\beta = 0$, or if there exist even numbers m_a and m_b such that*

$$\sigma(a) = (ab)^{m_a} a, \quad \sigma(b) = (ba)^{m_b} b.$$

Proof. For real numbers K, L let

$$w_{K,L} = (K, -K, L, -L).$$

Then

$$w_{K,L} M_{\dot\sigma} = (K + \alpha L, -K - \alpha L, \beta L, -\beta L). \tag{6}$$

Therefore one has for all $n \geq 1$

$$w_{K,L} M_{\dot\sigma}^n = (K + \alpha L(1 + \beta + \cdots + \beta^{n-1}), \ldots, \beta^n L, -\beta^n L). \tag{7}$$

Note that

$$(\dot f(a^+), \dot f(a^-), \dot f(b^+), \dot f(b^-)) = (1, -1, 1, -1) = w_{1,1}. \tag{8}$$

So, if $\ell_n = |\dot\sigma^n(a^+)|$, then

$$S_{\ell_n, \dot f}(\dot u) = \sum_{k=0}^{\ell_n - 1} \dot f(\dot u_k) = w_{1,1} M_{\dot\sigma}^n \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We shall rather concentrate on the occurrences of b^+ in $\dot u$. These will certainly take place, and thus $\dot\sigma^n(b^+)$ will also occur for each n. But from the beginning to the end of such an occurrence the walk will travel

$$w_{1,1} M_{\dot\sigma}^n \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \beta^n \tag{9}$$

steps (by (7)). Hence if $|\beta| > 1$, then $(S_N(u))$ is unbounded.
There remain 3 possibilities: $\beta = 0$ or $\beta = \pm 1$.
In case $\beta = 0$, $\dot f(\dot\sigma(b^+)) = \dot f(\dot\sigma(b^-)) = 0$. But also $\dot f(\dot\sigma(a^+a^-)) = 0$. Since symbols a^+ and a^- alternate in $\dot u$, the sequence $\dot u$ has a decomposition in words from the set $V = \{(b^+)^k, a^+(b^-)^k a^- : k \geq 0\}$. Moreover, we may assume this set to be finite, since (by almost periodicity) the distance between the occurrence of two

a's in u is bounded. Each word v in the set V has $\dot{f}(\dot{\sigma}(v)) = 0$. This clearly implies that $(\dot{S}_{N,\dot{f}}(\dot{u}))$, and hence $(S_N(u))$ is bounded.
Now the case $\beta = \pm 1$. We see that β is an eigenvalue of $M_{\dot{\sigma}}$ (with right eigenvector $(0,0,1,-1)^T$). Passing from σ to σ^2 we may therefore assume (again by idempotency of $\overline{M}_\sigma$) that $\beta = +1$. In that case

$$S_{\ell_n,\dot{f}}(\dot{u}) = w_{1,1} M_{\dot{\sigma}}^n \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 1 + (n-1)\alpha.$$

So if $\alpha \neq 0$, then the walk is unbounded. We are left with the case $\alpha = 0, \beta = 1$. Then we see from (6) that $w_{1,1}$ is a left eigenvector of $M_{\dot{\sigma}}$. Because of (8) this is a necessary and sufficient condition for $\dot{\sigma}$ to admit a representation by $\dot{f}$ in $\mathbb{R}$. (Cf. the remarks in the last paragraphs of the introduction). This implies that the walk is renormalizable in the following sense: if the 4 steps corresponding to the symbols a^+, a^-, b^+ and b^- are replaced by the sequences of steps corresponding to $\dot{\sigma}(a^+), \dot{\sigma}(a^-), \dot{\sigma}(b^+)$, respectively $\dot{\sigma}(b^-)$, then this new walk is equal to the original walk. From this one deduces that if the $S_{N,\dot{f}}(\dot{\sigma}(a^+)),\ 1 \leq N \leq |\dot{\sigma}(a^+)|$ crosses level 1, then $S_{N,\dot{f}}(\dot{\sigma}^2(a^+)),\ 1 \leq N \leq |\dot{\sigma}^2(a^+)|$ will cross level 2. More generally $(S_{N,\dot{f}}(\dot{\sigma}^{2^n}(a^+)))$ will cross level n and the walk will be unbounded. So suppose $(S_{N,\dot{f}}(\dot{u}))_{N=0}^{\infty}$ remains between the levels -1 and $+1$. Then b^2 cannot occur in u, as this would lead to three $+'s$ or three $-'s$ in $\dot{u}$. Also since $\dot{f}(\dot{\sigma}(a^+)) = 1$, and $(S_{N,\dot{f}}(\dot{\sigma}(a^-)))_{N=0}^{|\dot{\sigma}(a^-)|-1}$ equals $S_{N,\dot{f}}(\dot{\sigma}(a^+))_{N=0}^{|\dot{\sigma}(a^+)|-1}$ mirrored around zero, a^2 cannot occur, unless $(S_{N,\dot{f}}(\dot{\sigma}(a^+)))$ stays between 0 and 1, which would only be possible if a^+ and a^- alternate in $\dot{\sigma}(a^+)$, what contradicts the primitivity of $M_{\dot{\sigma}}$. We have shown that $\sigma(a)$ contains neither a^2 nor b^2, but then $\sigma(a) = (ab)^m a$, where m is even because σ is of type III. Since the same arguments apply to $\sigma^n(a)$, and $\sigma(b)$ has to appear in some $\sigma^n(a), \sigma(b)$ will also neither contain a^2 nor b^2. Since $\sigma(ab)$ and $\sigma(ba)$ will occur, it follows likewise that the first and the last letter of $\sigma(b)$ are equal to b. Hence $\sigma(b)$ has the claimed form. $\square$.

Example. Let τ be the Fibonacci substitution defined by $\tau(a) = ab, \tau(b) = a$. Then $\tau^3(a) = abaab, \tau^3(b) = aba$, and $\sigma = \tau^3$ if of Type III. We have $\dot{\sigma}(a^+) = a^+b^-a^-a^+b^-, \dot{\sigma}(b^+) = a^+b^-a^-$. hence $\alpha = -2$ and $\beta = -1$, so $(S_N(u))$ is unbounded. (The substitution σ given by $\sigma(a) = abb, \sigma(b) = abab$ gives a (nonperiodic) example of a bounded walk.)

7. Type IV Substitutions

Here $\overline{M}_\sigma = \binom{0\ 1}{0\ 1}$. Let us write $\nu = \sigma(a), \mu = \sigma(b)$. Then, because σ is of Type IV, the Squaring Lemma implies

$$\mu^2 \sim \epsilon \text{ and } (\mu\nu)^2 \sim \epsilon.$$

So if we consider ν and μ as symbols, we have that the two groups

$$G =< \overline{a}, \overline{b} | \overline{a}^2 = (\overline{a}\overline{b})^2 = \overline{\epsilon} > \text{ and } H =< \overline{\mu}, \overline{\mu\nu} | \overline{\mu}^2 = (\overline{\mu\nu})^2 = \overline{\epsilon} >$$

are isomorphic. Here $\overline{w}$ denotes the equivalence class of a word w under the equivalence relation introduced in Section 4.

But the matrix $\left(\begin{smallmatrix}0 & 1\\ 0 & 1\end{smallmatrix}\right)$ of σ over $\{a, b\}$ transforms to the matrix $\left(\begin{smallmatrix}1 & 0\\ 1 & 0\end{smallmatrix}\right)$ of $\hat{\sigma}$ over $\{\mu, \nu\}$, where the substitution $\hat{\sigma}$ on $\{\mu, \nu\}$ is defined as in Section 3. We then use the analysis of Type III, where at an occurrence of μ respectively ν we move $K := S_{\ell_a}(\sigma(a))$, respectively $L := S_{\ell_b}(\sigma(b))$ steps ($\ell_a = |\sigma(a)|$, $\ell_b = |\sigma(b)|$). Because $\overline{M}_{\hat{\sigma}} = \left(\begin{smallmatrix}1 & 0\\ 1 & 0\end{smallmatrix}\right)$, there are an odd number of ν's in $\hat{\sigma}(\mu)$. But then the parameter $\alpha = s - t$ of the associated $\dot{\hat{\sigma}}$ matrix has to be odd, i.e., the renormalizable case $\alpha = 0, \beta = 1$ of the Type III analysis can not occur for these matrices. Furthermore, using the vector $w_{K,L}$ instead of $w_{1,1}$ in (9), we see from (7) that the walk is bounded iff $\beta = 0$, which occurs iff $\dot{\hat{\sigma}}(\nu^+)$ has an equal number of ν^+ and ν^-. Since $\dot{\hat{\sigma}}(\nu^+)$ already has an equal number of μ^+ and μ^- we finally obtain

Proposition 3. *If σ if of Type IV, then $(S_N(u))$ is bounded iff $\tau(b) \sim \epsilon$, where τ is the substitution obtained from σ by interchanging a and b.*

Example. Let σ be defined by $\sigma(a) = aabb, \sigma(b) = ab$. Then τ is given by $\tau(a) = ba, \tau(b) = bbaa$. Since $\tau(b) \not\sim \epsilon$, σ generates an unbounded oriented walk.

Acknowledgement We thank the referee for his remarks, which have lead to improvements of the exposition of this work.

References

[1] F.M. Dekking, *Recurrent sets*, Advances in Math. **44** (1982), 78–104.

[2] F.M. Dekking, *On transience and recurrence of generalized random walks*, Z. Wahrsch. verw. Geb. **61** (1982), 459–465.

[3] F.M. Dekking, *Marches automatiques*, J. Théor. Nombres Bordeaux **5** (1993), 93–100.

[4] F.M. Dekking, *Iteration of maps by an automaton*, Discrete Math. **126** (1994), 81–86.

[5] J.-M. Dumont et A. Thomas, *Systèmes de numération et fonctions fractales relatifs aux substitutions*, Theor. Comp. Science **65** (1989), 153–169.

[6] J.-M Dumont, *Summation formulae for substitutions on a finite alphabet*, Number Theory and Physics (Eds: J.-M. Luck, P. Moussa, M. Waldschmidt). Springer Lect. Notes Physics **47** (1990), 185–194.

[7] M. Mendès France and J. Shallit, *Wirebending and continued fractions*, J. Combinatorial Theory Ser. A **50** (1989), 1-23.

[8] D. Levine and P.J. Steinhardt, *Quasicrystals (I). Definition and structure.* Physical Review B, vol. (2) **34**, 1986, 596–615.

[9] P.A.B. Pleasants, *Quasicrystallography: some interesting new patterns.* Banach center publications, vol. **17**, 1985, 439–461.

[10] Z.-X Wem and Z.-Y Wen, *Marches sur les arbres homogènes suivant une suite substitutive*, J. Théor. Nombres Bordeaux, **4** (1992), 155–186.

F.M. DEKKING
Delft University of Technology

Z.-Y. WEN
Wuhan University