Journal de Théorie des Nombres
de Bordeaux **8** *(1996), 315–329*

Limit Theorem in the space of continuous functions for the Dirichlet polynomial related with the Riemann zeta-funtion

par ANTANAS LAURINČIKAS*

RÉSUMÉ. Dans cet article on prouve un théorème limite dans l'espace des fonctions continues pour le polynôme de Dirichlet

$$\sum_{m \leq T} \frac{d_{\kappa_T}(m)}{m^{\sigma_T + it}},$$

où $d_{\kappa_T}(m)$ sont les coefficients du développement en série de Dirichlet de la fonction $\zeta^{\kappa_T}(s)$ dans le demi–plan $\sigma > 1$, $\kappa_T = (2^{-1} \log l_T)^{-\frac{1}{2}}$, $\sigma_T = \frac{1}{2} + \frac{\log^2 l_T}{l_T}$, $l_T > 0$, $l_T \leq \log T$ et $l_T \to \infty$ lorsque $T \to \infty$.

ABSTRACT. A limit theorem in the space of continuous functions for the Dirichlet polynomial

$$\sum_{m \leq T} \frac{d_{\kappa_T}(m)}{m^{\sigma_T + it}},$$

where $d_{\kappa_T}(m)$ denote the coefficients of the Dirichlet series expansion of the function $\zeta^{\kappa_T}(s)$ in the half-plane $\sigma > 1, \kappa_T = (2^{-1} \ln l_T)^{-1/2}$, $\sigma_T = \frac{1}{2} + \frac{\ln^2 l_T}{l_T}$ and $l_T > 0$, $l_T \leq \ln T$ and $l_T \to \infty$ as $T \to \infty$, is proved.

Let s be a complex variable and $\zeta(s)$, as usual, denote the Riemann zeta-function. To study the distribution of values of the Riemann zeta-function the probabilistic methods can be used, and the obtained results usually are presented as the limit theorems of probability theory. The first theorems of this type were obtained in [1],[2], and they were proved in [3]-[5] using other methods. In modern terminology we can formulate it as follows. Let

---

Manuscrit reçu le 20 octobre 1993

*This paper has been written during the author's stay at the University Bordeaux I in the framework of the program "S&T Cooperation with Central and Eastern European Countries".

$\mathbf{C}$ be the complex space and let $\mathcal{B}(S)$ denote the class of Borel sets of the space S. Let $\operatorname{meas}\{A\}$ be the Lebesgue measure of the set A and

$$\nu_T^t(\ldots) = \frac{1}{T}\operatorname{meas}\{t \in [0,T], \ldots\}$$

where in place of dots we write the conditions which are satisfied by t. We define the probability measure

$$P_T(A) = \nu_T^t(\zeta(\sigma + it) \in A), \quad A \in \mathcal{B}(\mathbf{C})$$

Theorem A. *For $\sigma > \frac{1}{2}$ there exists a probability measure P on $(\mathbf{C}, \mathcal{B}(\mathbf{C}))$ such that P_T converges weakly to P as $T \to \infty$.*

More general results were obtained in [6]. Let M denote the space of functions meromorphic in the half-plane $\sigma > \frac{1}{2}$, equipped with the topology of uniform convergence on compacta. Define the probability measure

$$Q_T(A) = \nu_T^\tau(\zeta(s + i\tau) \in A), \quad A \in \mathcal{B}(M).$$

Theorem B. *There exists a probability measure Q on $(M, \mathcal{B}(M))$ such that Q_T converges weakly to Q as $T \to \infty$.*

Note that the explicit form of the measure Q can be indicated, and, obviously, Theorem A is a corollary of Theorem B.

The situation is more complicated when σ depends on T and tends to $\frac{1}{2}$ as $T \to \infty$, or $\sigma = \frac{1}{2}$. It turns out that in this case some power norming is necessary. Let $l_T > 0$ and let l_T tend to infinity as $T \to \infty$, or $l_T = \infty$. We take

$$\tilde{\sigma}_T = \frac{1}{2} + \frac{1}{l_T}, \quad \kappa = \kappa_T = \begin{cases} (2^{-1}\log l_T)^{-1/2}, & l_T \le \log T, \\ (2^{-1}\log\log T)^{-1/2}, & l_T \ge \log T. \end{cases}$$

The case $l_T = \infty$ corresponds to $\tilde{\sigma}_T = \frac{1}{2}$.

The function

$$w(\tau, k) \stackrel{\text{def}}{=} \int\limits_{\mathbf{C}\setminus\{0\}} |\, s\,|^{i\tau}\, e^{ik\arg s}\, dP \quad \tau \in \mathbb{R}, k \in \mathbf{Z},$$

is called the characteristic transform of the probability measure P on the space $(\mathbf{C}, \mathcal{B}(\mathbf{C}))$ [7]. The lognormal probability measure on $(\mathbf{C}, \mathcal{B}(\mathbf{C}))$ is defined by the characteristic transform

$$w(\tau, k) = \exp\left\{-\frac{\tau^2}{2} - \frac{k^2}{2}\right\}.$$

Theorem C. *The probability measure*

$$\nu_T^t(\zeta^{\kappa_T}(\tilde{\sigma_T}+it)\in A), \quad A\in\mathcal{B}(\mathbf{C}),$$

converges weakly to the lognormal probability measure as $T\to\infty$.

Here if $\zeta(s)\neq 0$, $a\in\mathbb{R}$, then $\zeta^a(s)$ is understood as $\exp\{a\log\zeta(s)\}$ where $\log\zeta(s)$ is defined by continuous displacement from the point $s=2$ along the path joining the points 2, $2+it$ and $\sigma+it$.

When $\tilde{\sigma}_T=\frac{1}{2}$ Theorem C was proved by A.Selberg (unpublished), see also [8], and for different form of l_T, it was obtained in [8]–[10], [5].

Now it arises the problem to obtain some results of the kind of Theorem C in the space of continuous functions.

Let $\mathbf{C}_\infty=\mathbf{C}\cup\{\infty\}$ be the Riemann sphere and let $d(s_1,s_2)$ be a metric on $\mathbf{C}_\infty$ given by the formulae

$$d(s_1,s_2)=\frac{2\mid s_1-s_2\mid}{\sqrt{1+\mid s_1\mid^2}\sqrt{1+\mid s_2\mid^2}}, \quad d(s,\infty)=\frac{2}{\sqrt{1+\mid s\mid^2}}, \quad d(\infty,\infty)=0.$$

Here $s,s_1,s_2\in\mathbf{C}$. This metric is compatible with the topology of $\mathbf{C}_\infty$. Let $C(\mathbb{R})=C(\mathbb{R},\mathbf{C}\infty)$ denote the space of continuous functions $f:\mathbb{R}\to\mathbf{C}\infty$ equipped with the topology of uniform convergence on compacta. In this topology, sequence $\{f_n,f_n\in C(\mathbb{R})\}$ converges to the function $f\in C(\mathbb{R})$ if

$$d(f_{n(t)},f(t))\to 0$$

as $n\to\infty$ uniformly in t on compact subsets of $\mathbb{R}$.

The functional analogue of the probability measure in Theorem C is the measure

$$\nu_T^\tau(\zeta^{\kappa_T}(\tilde{\sigma}_T+it+i\tau)\in A), \quad A\in\mathcal{B}(C(\mathbb{R})). \tag{1}$$

Does this measure converge weakly as $T\to\infty$ to some probability measure on $(C(\mathbb{R}),\mathcal{B}\ (C(\mathbb{R})))$? At this moment this question is open and it seems to be very difficult.

In the proof of Theorem C an inportant role is played by the Dirichlet polynomial

$$S_u(s)=\sum_{m\leq u}\frac{d_\kappa(m)}{m^s}$$

where $d_\kappa(m)$ denote the coefficients of the Dirichlet series expansion of the function $\zeta^\kappa(s)$ in the half-plane $\sigma > 1$ (see [11], [12]). Therefore the aim of this paper is to prove the limit theorem in the space of continuous functions for $S_u(s)$. This theorem will be the first step to study the weak convergence of the probability measure (1).

Now let $l_T \le \log T$, $\sigma_T = \frac{1}{2} + \frac{\log^2 l_T}{l_T}$, and let

$$P_{T,S_u}(A) = \nu_T^\tau(S_u(\sigma_T + it + i\tau) \in A), \quad A \in \mathcal{B}(C(\mathbb{R})).$$

Moreover we suppose that

$$l_{T+U} - l_T = \frac{BU}{T} \tag{2}$$

for all $U > 0$ as $T \to \infty$. Here B denotes a number (not always the same) which is bounded by a constant.

Theorem *There exists a probability measure P on $(C(\mathbb{R}), \mathcal{B}(C(\mathbb{R})))$ such that P_{T,S_T} converges weakly to P as $T \to \infty$.*

Proof of the theorem is based on the following probability result. Let S_1 and S_2 be two metric spaces, and let $h : S_1 \to S_2$ be a measurable function. Then every probability measure P on $(S_1, \mathcal{B}(S_1))$ induces on $(S_2, \mathcal{B}(S_2))$ the unique probability measure Ph^{-1} defined by the equality $Ph^{-1}(A) = P(h^{-1}A)$, $A \in \mathcal{B}(S_2)$.

Now let h and h_n be the measurable functions from S_1 into S_2 and

$$E = \{x \in S_1 : h_n(x_n) \underset{n\to\infty}{\not\to} h(x) \quad \text{for some} \quad x_n \underset{n\to\infty}{\to} x\}.$$

Lemma 1. *Let P and P_n be the probability measures on $(S_1, \mathcal{B}(S_1))$. Suppose that P_n converges weakly to P as $n \to \infty$ and that $P(E) = 0$. Then the measure $P_n h_n^{-1}$ converges weakly to Ph^{-1} as $n \to \infty$.*

Proof. This lemma is Theorem 5.5 from [13].

Let γ denote the unit circle on complex plane, that is $\gamma = \{s \in \mathbf{C} : \mid s \mid = 1\}$.We put

$$\Omega = \prod_p \gamma_p$$

where $\gamma_p = \gamma$ for each prime p. With the product topology and pointwise multiplication the infinite-dimentional torus Ω is a compact Abelian topological group. Let P be a probability measure on $(\Omega, \mathcal{B}(\Omega))$.

The Fourier transform $g(\underline{k})$ of the measure P is defined by the formula

$$g(\underline{k}) = \int\limits_{\Omega} \prod_{p} x_p^{k_p}\, dP.$$

Here $\underline{k} = (k_2, k_3, ...)$ where only a finite number of integers k_p are distinct of zero, and $x_p \in \gamma$.

LEMMA 2. *Let $\{P_n\}$ be a sequence of probability measures on $(\Omega, \mathcal{B}(\Omega))$ and let $\{g_n(\underline{k})\}$ be a sequence of corresponding Fourier transforms. Suppose that for every vector $\underline{k}$ the limit $g(\underline{k}) = \lim_{n\to\infty} g_n(\underline{k})$ exists. Then there exists a probability measure P on $(\Omega, \mathcal{B}(\Omega))$ such that P_n converges weakly to P as $n \to \infty$. Moreover, $g(\underline{k})$ is the Fourier transform of P.*

Proof. The lemma is the special case of the continuity theorem for compact Abelian group, see, for example, [14].

Let

$$Q_T(A) = \nu_T^{\tau}((p_1^{i\tau}, p_2^{i\tau}, ...) \in A), \quad A \in \mathcal{B}(\Omega).$$

LEMMA 3. *The probability measure Q_T converges weakly to the Haar measure m on $(\Omega, \mathcal{B}(\Omega))$ as $T \to \infty$.*

Proof. The Fourier transform $g_T(\underline{k})$ of the measure Q_T is given by

$$g_T(\underline{k}) = \int\limits_{\Omega} \prod_{p} x_p^{k_p}\, dQ_T = \frac{1}{T}\int\limits_{0}^{T} \prod_{j=1}^{\infty} p_j^{ik_j\tau}\, d\tau =$$

$$= \begin{cases} 1 & \text{if} \quad \underline{k} = \underline{0}, \\ \dfrac{\exp\{iT \sum\limits_{j=1}^{\infty} k_j \log p_j\} - 1}{iT \sum\limits_{j=1}^{\infty} k_j \log p_j} & \text{if} \quad \underline{k} \neq \underline{0}. \end{cases}$$

Here $x_p \in \gamma, \underline{k} = (k_1, k_2, ...)$. By definition of the Fourier transform of probability measure on $(\Omega, \mathcal{B}(\Omega))$, only a finite number of k_j are distinct from zero. Since the logarithms of prime numbers are linearly independent over the field of rational numbers, we find that

$$g_T(\underline{k}) \to \begin{cases} 1 & if \underline{k} = \underline{0}, \\ 0 & if \underline{k} \neq \underline{0} \end{cases}$$

as $T \to \infty$. In view of Lemma 2, this proves the lemma.

We define the function $h_T : \Omega \to C(\mathbb{R})$ by the formula

$$h_T(t; e^{i\eta_1}, e^{i\eta_2}, \dots) = \sum_{k \le T} \frac{d_\kappa(k)}{k^{\sigma_T + it} \prod\limits_{p_j^{\alpha_j} \| k} e^{i\alpha_j \eta_j}}. \tag{3}$$

Here $p^\alpha \| k$ means that $p^\alpha \mid k$ but $p^{\alpha+1} \not| k$. Then, clearly,

$$S_T(\sigma_T + it + i\tau) = h_T(t; p_1^{i\tau}, p_2^{i\tau}, \dots). \tag{4}$$

Let, for brevity,

$$h_T(t; e^{i\tau_1}, e^{i\tau_2}, \dots) = S_T(\sigma_T + it + i\underline{\tau}),$$

and let

$$Z_{nk}(it, \underline{\tau}) = S_{n+k}(\sigma_{n+k} + it + i\underline{\tau}) - S_n(\sigma_n + it + i\underline{\tau}).$$

Let K be a compact subset of $\mathbb{R}$. For every $\epsilon > 0$ we define the set A_{nk}^ϵ by

$$A_{nk}^\epsilon(K) = \{(e^{i\tau_1}, e^{i\tau_2}, \dots) : \sup_{t \in K} \mid Z_{nk}(it, \underline{\tau}) \mid \ge \epsilon\}$$

and we put

$$A_k^\epsilon(K) = \bigcap_{l=1}^{\infty} \bigcup_{n > l} A_{nk}^\epsilon.$$

Lemma 4. *$m(A_k^\epsilon(K)) = 0$ for every $\epsilon > 0, K$, and $k \in \mathbb{N}$.*

Proof. By the Chebyshev inequality

$$m(A_{nk}^\epsilon(K)) \le \frac{1}{\epsilon^2} \int\limits_\Omega \sup_{t \in K} \mid Z_{nk}(it, \underline{\tau}) \mid^2 dm. \tag{5}$$

Using the Cauchy formula, we have that

$$Z_{nk}^2(it, \underline{\tau}) = \frac{1}{2\pi i} \int\limits_L \frac{Z_{nk}^2(z, \underline{\tau})}{z - it} \, dz$$

where L denotes the restangle, enclosing the set $iK = \{ia,\, a \in K\}$, with the sides $\sigma = -\frac{1}{l_{n+k}} + it$, $\sigma = \frac{1}{l_{n+k}} + it$, and with two other sides parallel to the real axis. Moreover, we suppose that the distance of L from the set iK is $\geq \frac{1}{l_{n+k}}$. From this equality it follows that

$$\sup_{t\in K} \mid Z_{nk}(it,\underline{\tau}) \mid^2 = Bl_{n+k} \int\limits_L \mid Z_{nk}(z,\underline{\tau}) \mid^2 \mid dz \mid .$$

Hence, having in mind the inequality (5), we obtain that

$$\begin{aligned} m(A^{\epsilon}_{nk}) =& \frac{Bl_{n+k}}{\epsilon^2} \int\limits_L \mid dz \mid \int\limits_\Omega \mid Z_{nk}(z,\underline{\tau}) \mid^2 dm = \\ =& \frac{Bl_{n+k} \mid L \mid}{\epsilon^2} \sup_{z\in L} \int\limits_\Omega \mid Z_{nk}(z,\underline{\tau}) \mid^2 dm \end{aligned} \tag{6}$$

where $\mid L \mid$ is the length of L. From the definitions of $Z_{nk}(z,\underline{\tau})$ and $S_n(\sigma_n + z + i\tau)$ we have that, for $z = u + iv$,

$$\begin{aligned} Z_{nk}(z,\underline{\tau}) =& \sum_{l\leq n+k} \frac{d_{\kappa_{n+k}}(l)}{l^{\sigma_{n+k}+u+iv} \prod\limits_{p_j^{\alpha_j}\| l} e^{i\alpha_j\tau_j}} - \sum_{l\leq n} \frac{d_{\kappa_n}(l)}{l^{\sigma_n+u+iv} \prod\limits_{p_j^{\alpha}\| l} e^{i\alpha\tau_j}} = \\ =& \sum_{n<l\leq n+k} \frac{d_{\kappa_{n+k}}(l)}{l^{\sigma_{n+k}+u+iv} \prod\limits_{p_j^{\alpha_j}\| l} e^{i\alpha_j\tau_j}} + \\ &+ \sum_{l\leq n} \Big(\frac{d_{\kappa_{n+k}}(l)}{l^{\sigma_{n+k}+u+iv} \prod\limits_{p_j^{\alpha_j}\| l} e^{i\alpha_j\tau_j}} - \frac{d_{\kappa_n}(l)}{l^{\sigma_n+u+iv} \prod\limits_{p_j^{\alpha_j}\| l} e^{i\alpha_j\tau_j}} \Big) \stackrel{def}{=} V + W. \end{aligned}$$

Since

$$\mid a + b \mid^2 \leq 2(\mid a \mid^2 + \mid b \mid^2),$$

hence we find that

$$\mid Z_{nk}(z,\tau) \mid^2 \leq 2(\mid V \mid^2 + \mid W \mid^2). \tag{7}$$

The properties of the Haar measure m imply the equality

$$\begin{aligned} \int\limits_\Omega \mid V \mid^2 dm =& \sum_{n<l\leq n+k} \frac{d^2_{\kappa_{n+k}}(l)}{l^{2(\sigma_{n+k}+u)}} + \sum\sum_{\substack{n<l_1\leq n+k \\ n<l_2\leq n+k \\ l_1\neq l_2}} \frac{d_{\kappa_{n+k}}(l_1) d_{\kappa_{n+k}}(l_2)}{l_1^{\sigma_{n+k}+u+iv} l_2^{\sigma_{n+k}+u-iv}} \times \\ & \times \int\limits_\Omega \frac{\prod\limits_{p_j^{\alpha_j}\| l_2} e^{i\alpha_j\tau_j}}{\prod\limits_{p_j^{\alpha_j}\| l_1} e^{i\alpha_j\tau_j}} dm = \sum_{n<l\leq n+k} \frac{d^2_{\kappa_{n+k}}(l)}{l^{2(\sigma_{n+k}+u)}}. \end{aligned} \tag{8}$$

By a similar manner we find that

$$\int\limits_{\Omega} |\, W \,|^2 \, dm = \sum_{l \le n} \left(\frac{d_{\kappa_{n+k}}(l)}{l^{\sigma_{n+k}+u}} - \frac{d_{\kappa_n}(l)}{l^{\sigma_n+u}} \right)^2 . \tag{9}$$

From the definition of the contour L it follows that

$$-\frac{1}{l_{n+k}} \le u \le \frac{1}{l_{n+k}} \tag{10}$$

for $z = u + iv \in L$.Then (8) together with (2) and the well-known estimate

$$\sum_{m \le x} \frac{1}{m} = \log x + \gamma_0 + \frac{B}{x},$$

where γ_0 is the Euler constant, yields

$$\int\limits_{\Omega} |\, V \,|^2 \, dm = Bn^{-\dfrac{2\log^2 l_{n+k} - 1}{l_{n+k}}} \sum_{n < l \le n+k} \frac{1}{l} = Be^{\dfrac{-\log n \log^2 l_n}{l_n}} \left(1 + \frac{Bk}{n}\right) \times$$

$$\times \left(\log \frac{n+k}{n} + \frac{B}{n} \right) = \frac{Bk}{n} e^{-c_1 \dfrac{\log n \log^2 l_n}{l_n}} \tag{11}$$

for $n \ge n_0$. Here we have used the inequality $0 < d_{\kappa_{n+k}}(l) < 1,\ n \ge n_0$, which follows trivially from the multiplicativity of $d_{\kappa_{n+k}}(m)$ and from the inequality $0 < d_{\kappa_{n+k}}(p^\alpha) < 1,\ n \ge n_0$, implied by the formula [11], [12]

$$d_\kappa(p^\alpha) = \frac{\kappa(\kappa+1)\ldots(\kappa+\alpha-1)}{\alpha!}. \tag{12}$$

From the assumption on l_T we deduce that, for $n \ge n_0$,

$$\sigma_{n+k} = \sigma_n(1 + \frac{Bk}{n}), \tag{13}$$

$$\log\, l_{n+k} = \log \left(l_n + \frac{Bk}{n} \right) = \log\, l_n \left(1 + \frac{Bk}{nl_n} \right) =$$

$$= \log\, l_n + \frac{Bk}{nl_n} = \log\, l_n \left(1 + \frac{Bk}{nl_n \log\, l_n} \right).$$

Thus,

$$\kappa_{n+k} = (2^{-1} \log\, l_{n+k})^{-\frac{1}{2}} = \kappa_n \left(1 + \frac{Bk}{nl_n \log\, l_n}\right)^{-\frac{1}{2}} =$$

$$= \kappa_n \left(1 + \frac{Bk}{nl_n \log\, l_n}\right) \stackrel{\text{def}}{=} \kappa_n(1 + r_{nk}).$$

Consequently, in view of (12), for $n \geq n_0$,

$$d_{\kappa_{n+k}}(p^\alpha) = \frac{\kappa_n(1 + r_{nk})(\kappa_n(1 + r_{nk}) + 1) \ldots (\kappa_n(1 + r_{nk}) + \alpha - 1)}{\alpha!} =$$

$$= \frac{\kappa_n(1 + r_{nk})(\kappa_n + 1)\left(1 + \frac{\kappa_n\, r_{nk}}{\kappa_n + 1}\right) \ldots (\kappa_n + \alpha - 1)\left(1 + \frac{\kappa_n\, r_{nk}}{\kappa_n + \alpha - 1}\right)}{\alpha!} =$$

$$= d_{\kappa_n}(p^\alpha) \prod_{j=1}^{\alpha} \left(1 + \frac{Br_{nk}}{j}\right) =$$

$$= d_{\kappa_n}(p^\alpha)(1 + Br_{nk} \log \alpha).$$

Hence, for $m \leq n$,

$$d_{\kappa_{n+k}}(m) = \prod_{p^\alpha \| m} d_{\kappa_{n+k}}(p^\alpha) =$$

$$= d_{\kappa_n}(m) = \prod_{p^\alpha \| m} (1 + Br_{nk} \log \alpha) =$$

$$= d_{\kappa_n}(m) \exp\{Br_{nk} \sum_{p^\alpha \| m} \log \alpha\} =$$

$$= d_{\kappa_n}(m) \exp\{Br_{nk} \log\log\, n \sum_{p^\alpha \| m} 1\} =$$

$$= d_{\kappa_n}(m) \exp\{Br_{nk} \log\log\, n\, \log\, n\} =$$

$$= d_{\kappa_n}(m)(1 + Br_{nk} \log\log\, n\, \log\, n). \tag{14}$$

Therefore, from (9), (13) and (14) we have that

$$\int_\Omega |\, W\,|^2\, dm = \frac{Bk^2 \log^2 n \log^2 \log n}{n^2 l_n^2\, \log^2\, l_n} \sum_{l \leq n} \frac{d_{\kappa_n}^2(l)}{l^{2(\sigma_n + u)}}. \tag{15}$$

Repeating the proof of Lemma 3 from [10] and taking into account (10), we see that

$$\sum_{l\le n} \frac{d^2_{\kappa_n}(l)}{l^{2(\sigma_n+u)}} = B.$$

Consequently, this and (15) give the estimate

$$\int\limits_{\Omega} |\ W\ |^2\, dm = \frac{Bk^2 \log^2 n \log^2 \log n}{n^2 l_n^2\ \log^2\ l_n}.$$

From this, (6), (7) and (11) we find that

$$m(A^{\epsilon}_{nk}(K)) = \frac{Bk^2}{\epsilon^2}\left(\frac{1}{n}e^{-c_1\frac{\log n \log^2 l_n}{l_n}} + \frac{\log^2 n \log^2 \log n}{n^2 l_n^2\ \log^2\ l_n}\right)$$

for every $\epsilon > 0$ and $k \in \mathbb{N}$. Thus it follows from the definition of the set $A^{\epsilon}_{nk}(K)$ that

$$m(A^{\epsilon}_{k}(K)) = \lim_{l\to\infty} m(\bigcup_{n>l} A^{\epsilon}_{nk}(K)) =$$

$$= \lim_{l\to\infty} \frac{Bk^2}{\epsilon^2}\sum_{n>l}\left(\frac{1}{n}e^{-c_1\dfrac{\log n \log^2\ l_n}{l_n}} + \frac{\log^2 n \log^2 \log n}{n^2 l_n^2 \log^2\ l_n}\right) = 0.$$

The lemma is proved.

Proof of Theorem. We will deduce the theorem from lemmas 1, 3 and 4. Let

$$(e^{i\tau_1(T)}, e^{i\tau_2(T)}, ...)$$

converges to

$$(e^{i\tau_1}, e^{i\tau_2}, ...)$$

as $T \to \infty$, and let E denote the set $\{(e^{i\tau_1}, e^{i\tau_2}, ...)\}$ of elements of Ω such that

$$h_T(t; e^{i\tau_1(T)}, e^{i\tau_2(T)}, ...)$$

does not converge to some function $h(t; e^{i\tau_1}, e^{i\tau_2}, ...)$ as $T \to \infty$. In order to prove the theorem we must show that $m(E) = 0$.Since Ω is compact, it is separable. Consequently [13], $E \in \mathcal{B}(\Omega)$.

Let E_1 denote the set $\{(e^{i\tau_1}, e^{i\tau_2}, ...)\}$ such that

$$h_T(t; e^{i\tau_1}, e^{i\tau_2}, ...)$$

does not converge to some function $h(t; e^{i\tau_1}, e^{i\tau_2}, ...)$ as $T \to \infty$. We will prove that $m(E_1) = 0$. First we consider the sequence $h_n(t; e^{i\tau_1}, e^{i\tau_2}, ...)$.

Note that there exists a sequence $\{K_j\}$ of compact subsets of $\mathbb{R}$ such that

$$\mathbb{R} = \bigcup_{j=1}^{\infty} K_j,$$

$K_j \subset K_{j+1}$, and if K is as compact of $\mathbb{R}$ then $K \subset K_j$ for some j. Let

$$\rho_j(f,g) = \sup_{t \in K_j} d(f(t), g(t))$$

for $f, g \in C(\mathbb{R})$. Then

$$\rho(f,g) = \sum_{j=1}^{\infty} 2^{-j} \frac{\rho_j(f,g)}{1+\rho_j(f,g)}$$

is a metric in $C(\mathbb{R})$.

Since $C(\mathbb{R})$ is a complete metric space, we have that every fundamental sequence is convergent. Thus it follows from the definition of the fundamental sequence that

$$m((e^{i\tau_1}, e^{i\tau_2}, ...) : h_n(t; e^{i\tau_1}, e^{i\tau_2}, ...) \not\to) =$$

$$= m((e^{i\tau_1}, e^{i\tau_2}, ...) : (e^{i\tau_1}, e^{i\tau_2}, ...) \in \bigcup_{\epsilon>0} \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} A_k^{\epsilon}(K_j)).$$

Thus, by Lemma 4,

$$m((e^{i\tau_1}, e^{i\tau_2}, ...) : h_n(t; e^{i\tau_1}, e^{i\tau_2}, ...) \not\to) = 0. \tag{16}$$

From the definition of the function h_T, using the estimates of types (13) and (14), we find that

$$h_T(t; e^{i\tau_1}, e^{i\tau_2}, ...) = \sum_{k \le [T]} \frac{d_{\kappa_{[T]}}(k)}{k^{\sigma_{[T]}+it} \prod\limits_{p_j^{\alpha_j} \| k} e^{i\alpha_j \tau_j}} + \frac{B}{T^{1/4}}$$

uniformly in $t \in \mathbb{R}$ and in $(e^{i\tau_1}, e^{i\tau_2}, ...) \in \Omega$. Therefore, in view of (16), $m(E_1) = 0$.

We have shown that there exists a function h such that for almost all $(e^{i\tau_1}, e^{i\tau_2}, ...) \in \Omega$

$$\sum_{k\le T} \frac{d_{\kappa_T}(k)}{k^{\sigma_T+it} \prod\limits_{p_j^{\alpha_j}\|k} e^{i\alpha_j\tau_j}} \underset{T\to\infty}{\to} h(t; e^{i\tau_1}, e^{i\tau_2}, ...) \tag{17}$$

uniformly in t on compact subsets of $\mathbb{R}$. Similarly as above in the case of the variable t it can be proved using the Cauchy formula that for almost all $(e^{i\tau_1}, e^{i\tau_2}, \dots) \in \Omega$ the relation (17) is valid uniformly in τ_1 on compact subsets of $\mathbb{R}$, uniformly in τ_2 on compact subsets of $\mathbb{R}, \dots$. Since the family of sets of m–measure one is closed under countable intersection, hence we have that (17) is true for almost all $(e^{i\tau_1}, e^{i\tau_2}, \dots) \in \Omega$ uniformly in t on compact subsets of $\mathbb{R}$, the convergence being uniform in τ_j on compact subsets of $\mathbb{R}$, $j = 1, 2, \dots$.

Since, for every $M > 0$,

$$m\Big(\Big|\sum_{k\le T} \frac{d_{\kappa_T}(k)}{k^{\sigma_T+it} \prod\limits_{p_j^{\alpha_j}\|k} e^{i\alpha_j\tau_j}}\Big| \ge M\Big) \le \frac{1}{M^2}\int\limits_{\Omega} \left|\sum_{k\le T} \frac{d_{\kappa_T}(k)}{k^{\sigma_T+it} \prod\limits_{p_j^{\alpha_j}\|k} e^{i\alpha_j\tau_j}}\right|^2 dm =$$

$$= \frac{1}{M^2}\sum_{k\le T} \frac{d^2_{\kappa_T}(k)}{k^{2\sigma_T}} = \frac{B}{M^2}$$

in view of the estimate

$$\sum_{k\le T} \frac{d^2_{\kappa_T}(k)}{k^{2\sigma_T}} = B,$$

we have that $h(t; e^{i\tau_1}, e^{i\tau_2}, ...) \ne \infty$ for almost all $(e^{i\tau_1}, e^{i\tau_2}, ...) \in \Omega$.

The relation (17) and the uniform convergence imply that for almost all $(e^{i\tau_1}, e^{i\tau_2}, ...) \in \Omega$

$$\sum_{k\le T} \frac{d_{\kappa_T}(k)}{k^{\sigma_T+it} \prod\limits_{p_j^{\alpha_j}\|k} e^{i\alpha_j\tau_j(T)}} \underset{T\to\infty}{\to} h(t; e^{i\tau_1}, e^{i\tau_2}, ...)$$

uniformly in t on compact subsets of $\mathbb{R}$. This yields $m(E) = 0$.The latter equality together with Lemmas 1 and 3 proves the theorem.

Now let $n_T = T^{\frac{\kappa_T}{2}}$.

COROLLARY. *There exists a probability measure P on $(C(\mathbb{R}), \mathcal{B}\ (C(\mathbb{R})))$ such that $P_{T,S_{n_T}}$ converges weakly to P as $n \to \infty$.*

Proof. Let K be a compact subset of $\mathbb{R}$. Denote by $Z_T(it+i\tau)$ the diference

$$S_T(\sigma_T + it + i\tau) - S_{n_T}(\sigma_T + it + i\tau) = \sum_{n_T < k \leq T} \frac{d_{\kappa_T}(k)}{k^{\sigma_T + it + i\tau}}.$$

Let $\epsilon_T = (\log l_T)^{-1}$.Then

$$\nu_T^\tau(\sup_{t \in K} \mid Z_T(it + i\tau) \geq \epsilon_T) \leq \frac{1}{\epsilon_T^2 T} \int_0^T \sup_{t \in K} \mid Z_T(it + i\tau) \mid^2 d\tau. \tag{18}$$

In view of the Cauchy formula

$$\sup_{t \in K} \mid Z_T(it + i\tau) \mid = Bl_T \int_L \mid Z_T(z + i\tau) \mid^2 \ \mid dz \mid$$

where L is the contour similar to that in the proof of Lemma 4. Hence we find by the Montgomery–Vaughan theorem for trigonometrical polynomials [15], [12] that

$$\int_0^T \sup_{t \in K} \mid Z_T(it + i\tau) \mid^2 d\tau = Bl_T \sup_{z \in L} \int_0^T \mid Z_T(z + i\tau) \mid^2 d\tau =$$

$$= Bl_T T \sup_{z \in L} \sum_{n_T < k \leq T} \frac{d_{\kappa_T}^2(k)}{k^{2\sigma_T + 2u}} = Bl_T T^{1 - \kappa_T \frac{c_2 \log^2 l_T}{l_T}} \sum_{n_T < k \leq T} \frac{d_{\kappa_T}^2(k)}{k} =$$

$$= BT^{1 - \kappa_n \frac{c_2 \log^2 \ l_T}{l_T}} \log\ T \sum_{k \leq T} \frac{1}{k} = BT^{1 - c_3 \frac{\log^{\frac{3}{2}}\ l_T}{l_T}} \log^2\ T.$$

From this and from (18) we deduce that

$$\nu_T^\tau(\sup_{t \in K} \mid Z_T(it + i\tau) \mid \geq \epsilon_T) = o(1) \tag{19}$$

as $T \to \infty$. Clearly, from the definition of the metric ρ, for $\epsilon > 0$,

$$\nu_T^\tau(\rho(S_T(\sigma_T + it + i\tau), S_{n_T}(\sigma_T + it + i\tau)) \geq \epsilon) \leq$$

$$\leq \frac{1}{\epsilon T}\int_0^T \rho(S_T(\sigma_T+it+i\tau), S_{n_T}(\sigma_T+it+i\tau))\,d\tau \leq$$

$$\leq \frac{1}{\epsilon}\sum_{j=1}^{\infty} 2^{-j}\frac{1}{T}\int_0^T \frac{2\sup\limits_{t\in K_j} \mid Z_T(it+i\tau)\mid}{1+2\sup\limits_{t\in K_j}\mid Z_T(it+i\tau)\mid}\,d\tau =$$

$$= \frac{1}{\epsilon}\sum_{j=1}^{\infty} 2^{-j}\frac{1}{T}\left(\int\limits_{\substack{0\\ \sup\limits_{t\in K_j}\mid Z_T(it+i\tau)\mid\leq \epsilon_T}}^{T} + \int\limits_{\substack{0\\ \sup\limits_{t\in K_j}\mid Z_T(it+i\tau)\mid\geq \epsilon_T}}^{T}\right)\times$$

$$\times\frac{2\sup\limits_{t\in K_j} \mid Z_T(it+i\tau)\mid}{1+2\sup\limits_{t\in K_j}\mid Z_T(it+i\tau)\mid}\,d\tau. \tag{20}$$

By (19) the second integral in the latter formula is $o(T)$ as $T\to\infty$, and the first integral trivially is $B\epsilon_T T$. Hence and from (20)

$$\nu_T^\tau = (\rho(S_T(\sigma_T+it+i\tau), S_{n_T}(\sigma_T+it+i\tau))\geq\epsilon) = o(1)$$

as $T\to\infty$ for every $\epsilon>0$. Thus, the corollary follows from Theorem and Theorem 4.1 from [13]: Let (S,ρ) be a separable space and X_n and Y_n be S-valued random elements. If $X_n \underset{n\to\infty}{\xrightarrow{\mathcal{D}}} X$ and $\rho(X_n,Y_n)\underset{n\to\infty}{\xrightarrow{P}} 0$, then $Y_n \underset{n\to\infty}{\xrightarrow{\mathcal{D}}} X$.

Acknowledgement. For the warm hospitality during our stay at the University Bordeaux I we express our deep gratitude to Prof. M. MENDÈS FRANCE, Prof. M. BALAZARD and to their colleagues. We also would like to thank Prof. J.-P. ALLOUCHE and the referee for several helpful remarks.

References

[1] H. Bohr and B. Jessen, *Über die Wertverteilung der Riemannschen Zeta funktion*, Ernste Mitteilung, Acta Math. **51** (1930), 1–35.

[2] H. Bohr and B. Jessen, *Über die Wertverteilung der Riemannschen Zeta funktion*, Zweite Mitteilung, Acta Math. **58** (1932), 1–55.

[3] B. Jessen and A. Wintner, *Distribution functions and the Riemann zeta-function*, Trans.Amer.Math.Soc. **38** (1935), 48–88.

[4] V. Borchsenius and B. Jessen, *Mean motions and values of the Riemann zeta-function*, Acta Math. **80** (1948), 97–166.

[5] A. Laurinčikas, *Limit theorems for the Riemann zeta-function on the complex space*, Prob. Theory and Math. Stat., 2, Proceedings of the Fifth Vilnius Conference, VSP/Mokslas (1990), 59–69.

[6] B. Bagchi, *The Statistical behaviour and universality properties of the Riemann zeta-function and other allied Dirichlet series,*, Ph. D. Thesis, Indian Statist. Inst., Calcutta, (1981).

[7] A.P.Laurinčikas, *Distribution of values of complex-valued functions*, Litovsk. Math. Sb. **15 Nr.2** (1975), 25–39, (in Russian); English transl. in Lithuanian Math. J., 15, 1975.

[8] D. Joyner, *Distribution Theorems for L-functions*, John Wiley (986).

[9] A.P. Laurinčikas, *A limit theorem for the Riemann zeta-function close to the critical line. II*, Mat. Sb., 180, **6** (1989), 733-+749, (in Russian); English transl. in Math. USSR Sbornik, 67, 1990.

[10] A. Laurinčikas, *A limit theorem for the Riemann zeta-function in the complex space*, Acta Arith. **53** (1990), 421–432.

[11] D. R. Heath-Brown, *Fractional moments of the Riemann zeta-function*, J.London Math. Soc. **24(2)** (1981), 65–78.

[12] A.Ivič, *The Riemann zeta-function*, John Wiley, 1985.

[13] P. Billingsley, Convergence of Probability Measures, John Wiley, 1968.

[14] H.Heyer, *Probability measures on locally compact groups*, Springer-Verlag, Berlin-Heidelberg-New York (1977).

[15] H. L. Montgomery and R. C. Vaughan, *Hilbert's inequality*, J. London Math. Soc. **8(2)** (1974), 73–82.

Antanas Laurinčikas
Department of Mathematics
Vilnius University
Naugarduko 24
2006 Vilnius, Lithuania