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The integer transfinite diameter of intervals and totally real algebraic integers

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RÉSUMÉ. Dans cet article, nous inspirant de travaux récents d'Amoroso d'une part, de Borwein et Erdélyi d'autre part, nous donnons une majoration et une minoration du diamètre transfini entier de petits intervalles $[\frac{r}{s}, \frac{r}{s} + \delta]$ où $\frac{r}{s}$ est un rationnel fixé et δ tend vers 0. Nous étudions également des fonctions g_-, g, g^+ associées au diamètre transfini d'intervalles de Farey.

Nous introduisons ensuite la notion de polynômes critiques pour un intervalle I. Nous montrons que ces polynômes ont la propriété de diviser tout polynôme à coefficients entiers ayant un maximum suffisamment petit sur I. Aparicio, puis Borwein et Erdélyi ont obtenu des résultats pour le polynôme critique x sur l'intervalle [0,1] ; résultats que nous prolongeons à tout polynôme critique sur un intervalle arbitraire.

Par ailleurs, comme conséquence facile de nos résultats, nous montrons : si α est une entier algébrique totalement réel, de plus petit conjugué α_1 alors, sauf pour un petit nombre d'exceptions explicites, la valeur moyenne de α et de ses conjugués est supérieure à $\alpha_1 + 1.6$.

ABSTRACT. In this paper we build on some recent work of Amoroso, and Borwein and Erdélyi to derive upper and lower estimates for the integer transfinite diameter of small intervals $[\frac{r}{s}, \frac{r}{s} + \delta]$, where $\frac{r}{s}$ is a fixed rational and $\delta \to 0$. We also study functions g_-, g, g^+ associated with transfinite diameters of Farey intervals. Then we consider certain polynomials, which we call *critical* polynomials, associated to a given interval I. We show how to estimate from below the proportion of roots of an integer polynomial which is sufficiently small on I which must also be roots of the critical polynomial. This generalises now classical work of Aparicio, and extends the techniques of Borwein and Erdélyi from the critical polynomial x for $[0, 1]$ to any critical polynomial for an arbitrary interval.

As an easy consequence of our results, we obtain an inequality about algebraic integers of independent interest: if α is totally real, with minimum conjugate α_1, then, with a small number of explicit exceptions, the mean value of α and its conjugates is at least $\alpha_1 + 1.6$.

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1. Introduction. For a set I in the complex plane, its *transfinite diameter* $t(I)$ is given by

$$t(I) = \lim_{n\to\infty} \sup_{z_1,\ldots,z_n \in I} \prod_{i<j} |z_i - z_j|^{1/\binom{n}{2}} \tag{1.1}$$

i.e. the limit, as n tends to infinity, of the supremum of geometric means of the distances between n points in I. This has been computed for many sets I. For a real interval I of length $|I|$ it is $|I|/4$. Fekete [Fek] (see also [Gol]) showed that an equivalent definition of $t(I)$ is

$$t(I) = \inf_P \max_{x\in I} |P(x)|^{1/\partial P}, \tag{1.2}$$

where the infimum is taken over all non-constant monic polynomials P in $\mathbb{C}[x]$. Further, if I is a real set, then this infimum can be restricted to polynomials in $\mathbb{R}[x]$. Clearly "monic" could be replaced by "leading coefficient at least 1" in (1.2).

If I is a real set, and the coefficients of P are restricted to be integers,we can define

$$t_{\mathbb{Z}}(I) = \inf_{\substack{P\in\mathbb{Z}[x]\\ \text{non-constant}}} \max_{x\in I} |P(x)|^{\frac{1}{\partial P}}, \tag{1.3}$$

the *integer transfinite diameter of* I. It is known that

$$t(I) \le t_{\mathbb{Z}}(I) \le \sqrt{t(I)} = \frac{1}{2}\sqrt{|I|}, \tag{1.4}$$

the first inequality being immediate, the second a classical result of Fekete, readily deduced from the discussion on pp.246-248 of [Fek]. While the classical transfinite diameter $t(I)$ is translation-invariant, scales linearly and is therefore additive for abutting intervals, none of these properties holds in general for $t_{\mathbb{Z}}(I)$ (see Corollary (4.3)).

For intervals I of length at least 4, it is known that $t_{\mathbb{Z}}(I) = t(I) = |I|/4$, ([Gol],p.298), so we restrict our attention to studying $t_{\mathbb{Z}}(I)$ for smaller intervals. From (1.4), $t_{\mathbb{Z}}(I) < 1$ for all such intervals. Recently Borwein and Erdélyi [BoEr] pointed out a connection between finding polynomials in $\mathbb{Z}[x]$ whose maximum is small on $[0, 1]$, and finding degree d real algebraic integers of norm N, all of whose conjugates lie in $[1, \infty)$, for which $N^{1/d}$ is small. Their fruitful idea provided the stimulus for this paper. It has also been applied in [Fl2], where it is used to obtain good upper and lower bounds for $t_{\mathbb{Z}}(I)$ for many sub-intervals

$$(1.5) \qquad I = [p/q, r/s] \subset [0,1] \quad \text{with} \quad qr - ps = 1.$$

Essentially, the idea is to use a fractional linear transformation to map certain families of totally positive algebraic integers to families of algebraic numbers with all conjugates in I. For $I = [0,1]$ itself, the result given there is the same lower bound as one due to Aparicio[Ap1], and is indeed equivalent to it ([Fl2]). It was long suspected (see e.g. Chudnovsky[Chu]) that this classical lower bound in fact was the true value of $t_{\mathbb{Z}}(I)$. However, Borwein and Erdélyi [BoEr] show, very surprisingly, that this is not the case. Thus to date no-one has been able to compute $t_{\mathbb{Z}}(I)$ exactly for any interval of length less than 4, and there is now not even a conjectured value for it, for any such interval I!

For an 'integer Chebyshev' polynomial for $[0,1]$, i.e. a polynomial with integer coefficients whose maximum among polynomials of a fixed degree is minimal, Borwein and Erdélyi [BoEr], p. 679 asked whether it must have all its zeroes in $[0,1]$. Recently Habsieger and Salvy[HaSa] showed that it need not, by finding that the degree 70 integer Chebyshev polynomial for $[0,1]$ had a factor with four non-real zeroes.

There have been some applications of estimates for $t_{\mathbb{Z}}(I)$ for intervals. For instance, Schnirelman and Gelfond (see Ferguson[Fer] p143) give a beautiful and short elementary argument proving that

$$\pi(n) \geq \log\left(\frac{1}{t_{\mathbb{Z}}([0,1])}\right)\frac{n}{\log n} \geq 0.865\frac{n}{\log n}$$

for the prime-counting function $\pi(n)$. Also, an upper bound for $t_{\mathbb{Z}}([0,(\sqrt{n}-\sqrt{m})^2])$ (n, m positive integers) gives an irrationality measure for $\log(n/m)$ ([Rh1]).

In Section 2 we introduce a (presumably transcendental) function $g_-(t)$ associated to a family of totally real algebraic integers. This function enables us to give a lower bound for $t_{\mathbb{Z}}(I)$ for *all* intervals I of the type (1.5).

In Section 3 we study a function $g(t)$,closely associated to $t_{\mathbb{Z}}(I)$, show that $g_- \leq g$, and find other bounds for g.

In Section 4 we consider $t_{\mathbb{Z}}(I)$ for very small intervals, i.e. for intervals I where we let the length $|I| = \delta$ tend to 0. Here we generalise and sometimes improve results of Borwein and Erdélyi [BoEr] and Amoroso[Am] on this topic. (See also [La]). (Amoroso's techniques are, however, quite different from ours. He shows that, in (1.3), the norm $\max|\quad|$ can be replaced by the 2-norm $\sqrt{\frac{1}{|I|}\int_I |P|^2}$, and works with this instead.) These results show how very different in relative size $t_{\mathbb{Z}}(I)$ can be, for different intervals of the same length δ , as $\delta \to 0$. We also show, using a nested sequence of Farey intervals, that $t_{\mathbb{Z}}(I)$ can be as big as $0.420726..\sqrt{|I|}$, so that the constant $\frac{1}{2}$ in (1.4) cannot be reduced by much.

In Section 5, we introduce the notion of *critical* polynomials for an interval I. These polynomials have the property that they must divide any integer-coefficient polynomial P which has a sufficiently small maximum on the interval. We prove (extending results of Aparicio[Ap3] and of Borwein and Erdélyi [BoEr]) that not only must critical polynomials Q divide the polynomial P, but that there is a positive constant γ *independent of* P such that $Q^{\gamma\partial P}|P$. As an application, these constants γ are computed in Section 6 for all ten known critical polynomials of $[0,1]$.

Finally, in Section 7, we prove the following result of independent interest, which follows easily from a result (Proposition 7.1) which we need for the results of Section 4.

THEOREM 1.1. *Let α be a totally real algebraic integer of degree $\partial\alpha$ with least conjugate α_1. Then*

$$\frac{\mathit{Trace}(\alpha)}{\partial\alpha} > 1.6 + \alpha_1 \tag{1.6}$$

unless, for some rational integer k, $\alpha + k$ is a zero of one of the polynomials given in Table 1.

We also list (Table 6) all polynomials of degree up to 6 with Trace/degree $-\alpha_1$ less than 1.7.

2. The function g_-, and a lower bound for $t_{\mathbb{Z}}([p/q, r/s])$. Firstly, we define two families $\{U_k\}$ and $\{V_k\}$ of polynomials, all of whose zeroes lie on the imaginary axis. These are the polynomials such that $\frac{U_k(z)}{V_k(z)}$ is the kth iterate of the function $G(z) := z + 1/z$. They are closely related to the *Gorškov* polynomials[Gor]. (See the Appendix for the precise connection, for a summary of properties of Gorškov polynomials, and for related references.) The U_k and V_k are defined inductively by $U_0 = z$, $V_0 = 1$ and

$$U_{k+1} = U_k^2 + V_k^2, \tag{2.1}$$

$$V_{k+1} = U_k V_k \tag{2.2}$$

Polynomial	$\dfrac{Tr(\alpha)}{\partial\alpha} - \alpha_1$
x	0.0000
x^2-3x+1	1.1180
x^2-4x+2	1.4142
x^3-5x^2+6x-1	1.4686
$x^4-7x^3+13x^2-7x+1$	1.5222
x^3-6x^2+9x-3	1.5321
$x^4-7x^3+14x^2-8x+1$	1.5771
$x^3-7x^2+14x-7$	1.5803.

Table 1. List of all monic irreducible polynomials with all roots real, least root in $[0,1)$, and (Trace/Degree) $-\alpha_1$ at most 1.6.

for $k \geq 0$. Note that $\partial U_k = 2^k$, $\partial V_k = 2^k - 1$. If we put $x_k := U_k/V_k$ then

$$x_{k+1} = x_k + x_k^{-1} \quad \text{and} \quad U_k^2/U_{k+1} = x_k/x_{k+1} \qquad (k \geq 0).$$

Also, it is known that U_k and V_k, or U_k and $U_{k'}$ with $k < k'$ have no common zero, and that U_k is irreducible (See Appendix). The Julia set of the map $G(z)$ is the imaginary axis J, and the (Lyubich) invariant measure μ is defined as the weak limit, as $k \to \infty$ of the atomic probability measure having equal weights at the zeroes of U_k. (See [St], p164, and the Appendix). [Here invariance means that $\mu(G^{-1}(E)) = \mu(E)$ for every Borel set $E \subset J$.] This measure gives rise to the logarithmic potential

$$p_\mu(z) := -\int_J \log(|z-x|)d\mu(x) \tag{2.3}$$

which is a harmonic function on $\mathbb{C}\backslash J$ (see [St], p17). In fact, as $\Re(z-x) > 0$ for $\Re z > 0$, $x \in J$, the 'complex potential'

$$p_\mu^{\mathbb{C}}(z) := -\int_J \log(z-x)d\mu(x) = -\lim_{k\to\infty} \frac{1}{2^{k-1}} \log U_k(z)$$

(taking the principal value of log) is readily shown to be analytic in the right half-plane $\Re z > 0$. We now define a function $g_-(z)$ for z in this half-plane by

$$g_-(z) := z \exp(p_\mu^{\mathbb{C}}(z)) = z / \lim_{k\to\infty} U_k(z)^{1/2^{k-1}}.$$

Then we claim

LEMMA 2.1. *On the half-plane $\Re z > 0$ the function g_- is analytic, is given by*

$$g_-(z) = \prod_{k=1}^{\infty} x_k^{-\frac{1}{2^k}} \tag{2.4}$$

and satisfies the functional equations

$$g_-(z) = g_-(\frac{1}{z}) \tag{2.5}$$

and

$$g_-(z + \frac{1}{z}) = (z + \frac{1}{z}) g_-(z)^2. \tag{2.6}$$

Furthermore, for real, positive t we have the bounds

$$te^{-2t^2} < g_-(t) < \frac{t}{1+t^2} \qquad (t > 0) \tag{2.7}$$

and hence certainly

$$t - 2t^3 < g_-(t) < t \qquad (t > 0). \tag{2.8}$$

Proof. We have, for $\Re z > 0$,

$$\begin{aligned}
g_-(z) : &= z / \lim_{k\to\infty} U_k^{1/2^{k-1}} \\
&= \lim_{k\to\infty} \frac{1}{z} (\frac{U_0^2}{U_1})^{\frac{1}{2^0}} (\frac{U_1^2}{U_2})^{\frac{1}{2^1}} \ldots (\frac{{U_{k-1}}^2}{U_k})^{\frac{1}{2^{k-1}}} \\
&= \lim_{k\to\infty} \frac{1}{z} (\frac{x_0}{x_1})^{\frac{1}{2^0}} (\frac{x_1}{x_2})^{\frac{1}{2^1}} \ldots (\frac{x_{k-1}}{x_k})^{\frac{1}{2^{k-1}}} \\
&= \lim_{k\to\infty} {x_1}^{-\frac{1}{2^1}} {x_2}^{-\frac{1}{2^2}} \ldots x_{k-1}^{-\frac{1}{2^{k-1}}} . {x_k}^{-\frac{1}{2^{k-1}}} \\
&= \prod_{k=1}^{\infty} x_k^{-\frac{1}{2^k}},
\end{aligned}$$

as $x_0 = z$. Convergence of this product can be verified directly from the fact that $\Re x_{k+1} > \Re x_k$ and $\Im x_{k+1} < \Im x_k$. Equation (2.5) now follows straight from the fact that $x_1(1/z) = x_1(z)$. To prove (2.6), note that, from (2.4),

$$g_-(z)^2 = x_1^{-1} g_-(x_1),$$

which gives the result.

Now take $z = t > 0$. To prove (2.7), first note that, from (2.4),

$$\log\left(\frac{t}{g_-(t)}\right) = \sum_{k=1}^{\infty} \frac{\log(tx_k)}{2^k} \tag{2.9}$$

We now claim that

$$tx_k \le 1 + kt^2. \tag{2.10}$$

This is readily verified by induction, using $x_{k+1} = x_k + x_k^{-1}$. Hence

$$\log\left(\frac{t}{g_-(t)}\right) \le \sum_k 2^{-k}\log(1+kt^2) \le \sum_k 2^{-k} kt^2 = 2t^2,$$

which gives the left-hand inequality. For the right-hand inequality, use the fact that $tx_k \ge tx_1 = 1 + t^2$.

Another functional equation

$$g_-(t)^2 = (\frac{t}{1+t^2}) g_-(\frac{t}{1+t^2})$$

follows straight from the lemma, from which it is easy to produce the asymptotic series

$$g_-(t) = t - 2t^3 + 7t^5 - 38t^7 + 295t^9 - 3074t^{11} + 40804t^{13} - \ldots$$
$$\frac{1}{g_-(t)} = \frac{1}{t} + 2t - 3t^3 + 18t^5 - 162t^7 + 1920t^9 - 28113t^{11} + \ldots$$
$$g_-(t+\frac{1}{t}) = t - 3t^3 + 14t^5 - 86t^7 + 687t^9 - 7069t^{11} + \ldots.$$

The graph of g_- is shown in Fig. 1. Its maximum value is $g_-(1) = 0.420726377$.

With the aid of the function g_- we can reformulate a result in [Fl2] (Theorem 1.2. See also [Fl3]):

Proposition 2.2. *For a sub-interval* $I = [\frac{p}{q}, \frac{r}{s}]$ *of* $[0,1]$ *with* $qr - ps = 1$

$$t_{\mathbb{Z}}(I) \ge \frac{1}{\sqrt{qs}} g_-\left(\sqrt{\frac{q}{s}}\right).$$

This also follows from Lemma 3.1 and Proposition 3.3.

Fig. 1. The function g_- and some computed values of g^+.

We now use this result to show

Corollary 2.3. *Given a positive irrational number v, there are arbitrarily small intervals I containing v which have*

$$t_{\mathbb{Z}}(I) \geq g_-(\limsup\big(\sqrt{\frac{q_n}{q_{n+1}}}\big))\sqrt{|I|}.$$

Here $\{\frac{p_n}{q_n}\}$ are the convergents in the continued fraction expansion of v.

The proof is immediate, from the consideration of the Farey intervals with endpoints $\frac{p_n}{q_n}$ and $\frac{p_{n+1}}{q_{n+1}}$.

Corollary 2.4. *There are arbitrarily short intervals I for which the constant $\frac{1}{2}$ in the upper bound (1.4) for $t_{\mathbb{Z}}(I)$cannot be reduced below $g_-(1) =$*

0.420726377.... *Further, in any inequality* $t_{\mathbb{Z}}(I) \le c|I|^\alpha \quad (c, \alpha \ge 0)$ *valid for sufficiently short intervals* I, *we have* $\alpha \le \frac{1}{2}$.

Proof. Let v be given by the continued fraction $[1,2,1,3,1,4,1,5,\ldots]$.Then, from the recurrence for the $\{q_n\}$, it follows that $\limsup(\frac{q_n}{q_{n+1}}) = 1$. The second part follows immediately.

3. The function g, and upper bounds for $t_{\mathbb{Z}}$. It is convenient to work here with *degree d polynomial-powers*, which we define to be expressions of the type

$$X(x) = P(x)^{d/\partial P}$$

for some polynomial $P(x) \in \mathbb{Z}[x]$, of degree ∂P, and $d =: \partial X$ a non-negative real number. (While the phase of X is not well-defined, $|X|$ is, which is all we need.) Then by definition

$$t_{\mathbb{Z}}(I) = \inf_{\substack{X \\ \partial X = 1}} \max_{x \in I} |X(x)|.$$

To define the g-function, we first define a function $g_{\mathbf{P}}$ as follows: fix a finite set $\mathbf{P} = \{P_j\}_{j \in J}$ of polynomials, and $\lambda = \{\lambda_j\}_{j \in J}$ a corresponding set of non-negative real numbers, and put

$$X_\lambda(x) := \prod_{j \in J} P_j(x)^{\lambda_j/\partial P_j} \tag{3.1}$$

a polynomial-power of degree $d = \sum \lambda_j$. Then $g_{\mathbf{P}}$ is defined for $t > 0$ by

$$g_{\mathbf{P}}(t) := \min_\lambda \sup_{x>0} \frac{|X_\lambda(x)|}{t + x/t}, \tag{3.2}$$

of degree d, where $0 \le d \le 1$. (The search for an optimal λ is a problem in "semi-infinite LP", and can be carried out by, for instance, the Remez algorithm [Che]). Thus $g_{\mathbf{P}}(t)$ is the least number such that, for some $X_\lambda(x)$,

$$\log(t^2 + x) - \log|X_\lambda(x)| \ge \log\left(\frac{t}{g_{\mathbf{P}}(t)}\right) \qquad \text{for} \qquad x > 0. \tag{3.3}$$

Next, put

$$g(t) := \inf_{\mathbf{P}} g_{\mathbf{P}}(t). \tag{3.4}$$

Note immediately that

Lemma 3.1. *For $qr - ps = 1$ we have*

$$t_{\mathbb{Z}}([\frac{p}{q}, \frac{r}{s}]) = \frac{1}{\sqrt{qs}} g\left(\sqrt{\frac{q}{s}}\right) = \inf_{\mathbf{P},\lambda} \sup_{x>0} \frac{|X_\lambda(x)|}{q + xs}. \tag{3.5}$$

Proof. We have, for X_λ as defined above,

$$\frac{1}{\sqrt{qs}} g\left(\sqrt{\frac{q}{s}}\right) = \inf_{\mathbf{P},\lambda} \sup_{x>0} \frac{|X_\lambda(x)|}{q + xs} = \inf_{Y_\lambda} \sup_{t\in[\frac{p}{q}, \frac{r}{s}]} |Y_\lambda(t)(r - ts)^{1-d_\lambda}| \tag{3.6}$$

on putting

$$x := (tq - p)/(r - ts) \tag{3.7}$$

and

$$Y_\lambda(t) := (r - ts)^d X_\lambda\left(\frac{tq - p}{r - ts}\right), \tag{3.8}$$

a degree d polynomial-power in t. Since $Y_\lambda(t)(r-ts)^{1-d}$ is a general degree 1 polynomial-power, we have the result.

Corollary 3.2. *We have*

$$t_{\mathbb{Z}}([\frac{p}{q}, \frac{r}{s}]) \le \frac{1}{\sqrt{qs}} g_{\mathbf{P}}\left(\sqrt{\frac{q}{s}}\right) \tag{3.9}$$

for any set $\mathbf{P}$ *of polynomials.*

Informally, we can define a function $g^+(t)$ by choosing, for a given t, a set $\mathbf{P}_t$ of polynomials for which $g_{\mathbf{P}_t}(t)$ is small, and putting $g^+(t) := g_{\mathbf{P}_t}(t)$. Then

$$t_{\mathbb{Z}}([\frac{p}{q}, \frac{r}{s}]) \le \frac{1}{\sqrt{qs}} g^+\left(\sqrt{\frac{q}{s}}\right). \tag{3.10}$$

Some values of such a g^+ are shown in Fig. 1, using computations from [Fl2].

We now observe that

Proposition 3.3. *For all $t \geq 0$,*

$$g(t) \geq g_-(t).$$

Proof. The proof is in essence the same as that of the classical lower bound for $t_{\mathbb{Z}}([0,1])$, which is equivalent to the inequality $g(1) \geq g_-(1)$. We note first that for $\alpha_1, \ldots, \alpha_d$ a complete set of conjugate positive algebraic integers

$$\prod_{i=1}^{d} |X_\lambda(\alpha_i)| \geq 1 \qquad \text{or} \qquad = 0,$$

so that

$$\begin{aligned} g_{\mathbf{P}}(t) &\geq \min_\lambda \left(\prod_{i=1}^{d} \frac{|X_\lambda(\alpha_i)|}{t + \alpha_i/t} \right)^{\frac{1}{d}} \\ &\geq \frac{1}{\prod_{i=1}^{d}(t+\alpha_i/t)^{1/d}} \\ &= \frac{t}{P_k(t)^{1/2^{k-1}}} \end{aligned}$$

for $\pm\sqrt{-\alpha_i}$ the zeroes of P_k, and k large enough so that no power of P_k divides X_λ. The result follows on letting $k \to \infty$.

From [BoEr], Theorem 3.4, we now know, surprisingly, that $g(1) > g_-(1)$, so that $g_- \neq g$. Presumably Borwein and Erdélyi's method will extend to show that $g(t) > g_-(t)$ for $t > 0$.

Next, we derive a functional equation and a functional inequality for g:

Proposition 3.4. *The function g satisfies for all $t > 0$*

$$g\left(\frac{1}{t}\right) = g(t) \tag{3.11}$$

and, for $0 < t \leq 1$

$$t^2\left(t + \frac{1}{t}\right) g(t)^2 \leq g\left(t + \frac{1}{t}\right) \leq \left(t + \frac{1}{t}\right) g(t)^2. \tag{3.12}$$

[Compare Lemma 2.1]

Proof. With X_λ as in (3.1), of degree $d \leq 1$, we have

$$g_{\mathbf{P}}\left(\frac{1}{t}\right) = \min_\lambda \sup_{\frac{1}{x}>0} \frac{|X_\lambda\left(\frac{1}{x}\right)|}{\frac{1}{t} + \frac{t}{x}} = \min_\lambda \sup_{x>0} \frac{|x^{1-d}.x^d X_\lambda\left(\frac{1}{x}\right)|}{t + x/t} = g_{\mathbf{P}^*}(t)$$

where $\mathbf{P}^* = \{x\} \cup \{x^{\partial P_j} P_j\left(\frac{1}{x}\right)\}_{j\in J}$. Since every set of polynomials belongs to a set with $\mathbf{P} = \mathbf{P}^*$, we have

$$g(t) = \min_{\mathbf{P}} g_{\mathbf{P}}(t) = \min_{\mathbf{P}:\mathbf{P}=\mathbf{P}^*} g_{\mathbf{P}}(t)$$

and (3.11) follows.

To prove (3.12) note first that, for any set $\mathbf{P}$

$$\begin{aligned} g_{\mathbf{P}}\left(t+\frac{1}{t}\right) &= \min_{\lambda} \sup_{x>0} \frac{|X_\lambda(x)|}{\left(t+\frac{1}{t}\right) + x/\left(t+\frac{1}{t}\right)} \\ &= \left(t+\frac{1}{t}\right) \min_{\lambda} \sup_{x>0} \frac{|X_\lambda(x+\frac{1}{x}-2)|}{\left(t+\frac{1}{t}\right)^2 + x + \frac{1}{x} - 2}. \end{aligned} \tag{3.13}$$

Now, for any polynomial P,

$$P\left(x+\frac{1}{x}-2\right) = R(x)x^{-\partial P}$$

for some reciprocal polynomial $R(x) = x^{2\partial P} R\left(\frac{1}{x}\right)$, so that

$$P\left(x+\frac{1}{x}-2\right)^2 = R(x)R\left(\frac{1}{x}\right).$$

Thus, on defining

$$\mathbf{R} := \{R_j : R_j(x)R_j\left(\frac{1}{x}\right) = P_j\left(x+\frac{1}{x}-2\right)^2, P_j \in \mathbf{P}\} \cup \{x\},$$

we get

$$\begin{aligned} X_\lambda\left(x+\frac{1}{x}-2\right) := &\prod_j P_j\left(x+\frac{1}{x}-2\right)^{\lambda_j/\partial P_j} \\ &= \prod R_j(x)^{\lambda_j/2\partial P_j} \prod R_j\left(\frac{1}{x}\right)^{\lambda_j/2\partial P_j} \\ &= X'_\lambda(x)X'_\lambda\left(\frac{1}{x}\right) \qquad \text{say,} \end{aligned}$$

as $\partial R_j = 2\partial P_j$, for X'_λ a polynomial-power, again of degree $d(=\sum \lambda_j)$. So from (3.13)

$$\begin{aligned}
g_{\mathbf{P}}\left(t+\frac{1}{t}\right) &= \left(t+\frac{1}{t}\right)\min_{\lambda}\sup_{x>0}\frac{|X'_\lambda(x)X'_\lambda\left(\frac{1}{x}\right)|}{\left(t+\frac{x}{t}\right)\left(t+\frac{1}{xt}\right)} \\
&= \left(t+\frac{1}{t}\right)\min_{\lambda}\sup_{x>0}\frac{|x^{(1-d)/2}X'_\lambda(x).x^{(1-d)/2}.x^d X'_\lambda\left(\frac{1}{x}\right)|}{\left(t+\frac{x}{t}\right)^2}.\frac{\left(t+\frac{x}{t}\right)}{\left(xt+\frac{1}{t}\right)} \\
&\geq t^2\left(t+\frac{1}{t}\right)\min_{\lambda}\sup_{x>0}\left(\frac{x^{(1-d)/2}X'_\lambda(x)}{t+\frac{x}{t}}\right)^2 \\
&= t^2\left(t+\frac{1}{t}\right)g_{\mathbf{R}}(t)^2 \geq t^2\left(t+\frac{1}{t}\right)g(t)^2 \qquad (t\leq 1)
\end{aligned}$$

as $(t+\frac{x}{t})/(xt+\frac{1}{t}) \geq t^2$ for $t \leq 1$. Then the lower bound follows on using (3.4). For the upper bound, note that

$$\begin{aligned}
g_{\mathbf{P}}(t)^2 &= \min_{\lambda}\sup_{x>0}\frac{|X_\lambda(x)|}{t+\frac{x}{t}}.\min_{\lambda}\sup_{x>0}\frac{|X_\lambda\left(\frac{1}{x}\right)|}{t+\frac{1}{xt}} \\
&= \min_{\lambda}\left(\sup_{x>0}\frac{|X_\lambda(x)|}{t+\frac{x}{t}}.\sup_{x>0}\frac{|X_\lambda\left(\frac{1}{x}\right)|}{t+\frac{1}{xt}}\right) \\
&\geq \min_{\lambda}\sup_{x>0}\frac{|X_\lambda(x)X_\lambda\left(\frac{1}{x}\right)|}{(t+\frac{x}{t})(t+\frac{1}{xt})}
\end{aligned}$$

as both minima are attained at the same λ.

Now for any P_j

$$P_j(x)P_j\left(\frac{1}{x}\right) = Q_j(x+\frac{1}{x}-2)$$

for some Q_j, so, for $Y_\lambda = \prod_j Q_j^{\lambda_j/\partial Q_j}$

$$\begin{aligned}
g_{\mathbf{P}}(t)^2 &\geq \min_{\lambda}\sup_{x>0}\frac{|Y_\lambda(x+\frac{1}{x}-2)|}{\left(t+\frac{1}{t}\right)^2+x+\frac{1}{x}-2} \\
&= \left(t+\frac{1}{t}\right)^{-1}\min_{\lambda}\sup_{y>0}\frac{|Y_\lambda(y)|}{\left(t+\frac{1}{t}\right)+\frac{y}{\left(t+\frac{1}{t}\right)}} \\
&= \left(t+\frac{1}{t}\right)^{-1} g_{\mathbf{Q}}\left(t+\frac{1}{t}\right),
\end{aligned}$$

where $\mathbf{Q} = \{Q_j\}$. Finally, from the definition of g and the fact that $g_{\mathbf{Q}} \geq g$, we obtain the upper bound for $g\left(t + \frac{1}{t}\right)$ in (3.12).

4. Small intervals with one rational endpoint. In this section we bound $t_{\mathbb{Z}}(I)$ for small intervals of length δ with one fixed rational endpoint $\frac{r}{s}$. Our main result is the following:

THEOREM 4.1. *There is a numerically determined constant c and a function $m(b)$ $(0 \leq b \leq 1)$ for which the following estimates hold. Let $\epsilon > 0$ be arbitrary, r/s be a rational number in $(0, 1]$, and let $\delta < \min(1/c, 2\epsilon/c^2)/s^2$. Let p/q be the Farey fraction of largest denominator with*

$$\frac{p}{q} = \frac{r}{s} - \frac{1}{qs} \leq \frac{r}{s} - \delta,$$

and put

$$b := \left\{ \frac{1}{\delta s^2} - \frac{q}{s} \right\},$$

where $\{\ \}$ denotes fractional part. Then the interval $[\frac{r}{s} - \delta, \frac{r}{s}]$ has integer transfinite diameter in the range

$$\delta s - (3 - b)\delta^2 s^3 \leq t_{\mathbb{Z}}([\frac{r}{s} - \delta, \frac{r}{s}]) \leq \delta s - \delta^2 s^3 (m(b) - b - \epsilon). \tag{4.1}$$

Here

$$1.6 \leq m(b) - b \leq 1.7719,$$

the upper bound being attained at $b = 0$ and 1. further, if $b = 0$, then the left-hand side of (4.1) can be replaced by $\delta s - 2\delta^2 s^3$.

The constant c is that in Proposition 7.1. The theorem generalises results of [Am] and [BoEr] for $s = 1$. However, for the upper bound, Amoroso had the better constant 1.648 instead of our $1.6 - \epsilon$ above (in the worst case). We expect that we should be able to improve the constant 1.6 in Proposition 7.1 to greater than 1.65, with a corresponding improvement here. (See the remarks after the proof of Prop. 7.1). For the lower bound, Amoroso already had the above bound (i.e. $\delta - 2\delta^2$) in the case $s = 1, b = 0$, while Borwein and Erdélyi showed that the 2 could be replaced by a number less than 2.

COROLLARY 4.2. *The same bounds apply to* $t_{\mathbb{Z}}([\frac{r}{s}, \frac{r}{s}+\delta])$ *for* $\frac{r}{s} \in [0,1)$, $0 < \delta < 1 - \frac{r}{s}$.

Proof of 4.2. This follows from the fact that $t_{\mathbb{Z}}([a,b]) = t_{\mathbb{Z}}([1-b, 1-a])$ for $0 < a < b \leq 1$, which in turn comes from replacing x by $1-x$ in a "good" polynomial $P(x)$ on $[a,b]$.

The following result is well-known:

COROLLARY 4.3. *None of the putative properties* $t_{\mathbb{Z}}(cI) = |c| t_{\mathbb{Z}}(I)$,

$$t_{\mathbb{Z}}([a,b]) + t_{\mathbb{Z}}([b,c]) = t_{\mathbb{Z}}([a,c]) \tag{4.2}$$

or $t_{\mathbb{Z}}(I+c) = t_{\mathbb{Z}}(I)$ *holds in general.*

Proof of 4.3. For counterexamples to the first two, note that

$$t_{\mathbb{Z}}([0,1]) = t_{\mathbb{Z}}([1,2]) \geq g_{-}(1) > 0.42$$

using Theorem 2.2, while $t_{\mathbb{Z}}([0,2]) \leq \frac{1}{\sqrt{2}}$ by (1.4). For the third, note that, from Theorem 4.1,

$$t_{\mathbb{Z}}([0,\delta]) \leq \delta - \delta^2(m(b) - \epsilon) < 2\delta - 8(3-b)\delta^2 \leq t_{\mathbb{Z}}([\frac{1}{2}, \frac{1}{2} + \delta]) \tag{4.3}$$

for δ sufficiently small.

See also Rhin [Rh2] for another example of this kind.

For the proof of Theorem 4.1, we need the following

LEMMA 4.4. *For* $0 < \lambda < \frac{1}{6}$, $b \in [0,1)$ *and* $x > b$ *we have*

$$f_{\lambda}(x) := \frac{1}{\lambda} \log(1 + \lambda(x-b)) - 3\log(x/3) \geq 3 - b - 9\lambda.$$

Proof. It is readily checked that $f_{\lambda}(x)$ has minimum value v in $[b, \infty)$ of

$$v = \frac{1-3\lambda}{\lambda} \log\left(1 + \frac{(3-b)\lambda}{1-3\lambda}\right)$$

at $x = \frac{3(1-b\lambda)}{1-3\lambda}$. Then the trivial bound

$$\log(1+t) \geq t - t^2/2 \qquad (t \geq 0) \tag{4.4}$$

shows that
$$v \geq 3 - b - \frac{(3-b)^2\lambda}{2(1-3\lambda)} \geq 3 - b - 9\lambda.$$

Proof of Theorem 4.1. The basis of the proof is the use of two Farey intervals $[\frac{p'}{q'}, \frac{r}{s}]$ and $[\frac{p''}{q''}, \frac{r}{s}]$ with
$$[\frac{p''}{q''}, \frac{r}{s}] \subset [\frac{r}{s} - \delta, \frac{r}{s}] \subset [\frac{p'}{q'}, \frac{r}{s}].$$

For our given rational $\frac{r}{s}$, we choose the rational $\frac{p}{q} < \frac{r}{s}$ to be the adjacent rational in the Farey series of largest denominator s. Thus $qr - ps = 1$. Then also
$$\frac{r}{s} - \frac{p+kr}{q+ks} = \frac{1}{s(q+ks)} \quad (k = 0, 1, 2 \ldots).$$

Choose k so that
$$\frac{1}{s(q+(k+1)s)} < \delta \leq \frac{1}{s(q+ks)}$$

i.e. $q' := q + ks \leq \frac{1}{s\delta} < q' + s =: q''$, or
$$k = \lfloor \frac{1}{s^2\delta} - \frac{q}{s} \rfloor.$$

Then for $p' := p + kr$ (and, for later use, $p'' := p + (k+1)r$) and $a := r/s - \delta$
$$b := \frac{aq' - p'}{r - as} = \frac{1 - q's\delta}{s^2\delta} = \{\frac{1}{s^2\delta} - \frac{q}{s}\},$$

so that $b \in [0, 1)$. Note also that

$$\frac{1}{q'} = \frac{s\delta}{1 - bs^2\delta} \quad , \quad \frac{1}{q''} = \frac{s\delta}{1 + (1-b)s^2\delta}. \tag{4.5}$$

Now suppose that we have found a positive function $m_1(b)$ $(0 \leq b < 1)$ such that, for each b there is a polynomial-power U_b of degree d^* with

$$\log(q' + sx) - \log|U_b(x)| \geq m_1(b) \quad \text{for} \quad x \in [b, \infty) \tag{4.6}$$

Note that $d^* \leq 1$, as otherwise the left-hand side of (4.6) would be negative for large x. Then, in a similar way to (4.1-2),

$$|(r - ts)^{1-d^*} V_b(t)| \leq e^{-m_1(b)} \quad \text{for} \quad t \in [a, \frac{r}{s}] \tag{4.7}$$

where

$$V_b(t) = (r - ts)^{d^*} U_b(\frac{tq' - p'}{r - ts})$$

is of degree d^* in t. Thus the left-hand side of (4.7) is a degree 1 polynomial-power in t, showing that

$$T := t_{\mathbb{Z}}([\frac{r}{s} - \delta, \frac{r}{s}]) \le e^{-m_1(b)}.$$

To find $m_1(b)$ for δ small, we note that, from the definition of b above,

$$\log(q' + sx) = \log(\frac{1}{\delta s}) + \log(1 + s^2\delta(x - b)). \tag{4.8}$$

Next, we use the fact that, by Proposition 7.1, we can find a constant c and, for each $b \in [0, 1)$ a constant $m(b)$ with $m(b) - b > 1.6$ such that there is a polynomial-power W_b with

$$x - \log|W_b(x)| \ge m(b) \qquad (b \le x \le c)$$

and

$$|W_b(x)| \le (x/3)^3 \qquad (x > c).$$

Then, for each $\epsilon > 0$, and $\lambda := s^2\delta$, using (4.4) and then Lemma 4.4 we have

$$\log(1 + \lambda(x - b)) - \lambda \log|W_b(x)| \ge$$

$$\begin{aligned}
&\ge \begin{cases} \lambda(m(b) - b) - (\lambda(x - b))^2/2 & (b \le x \le c) \\ \lambda f_\lambda(x) & (x > c) \end{cases} \\
&\ge \begin{cases} \lambda(m(b) - b) - \lambda^2 c^2/2 & (b \le x \le c) \\ \lambda(3 - b - 9\lambda) & (x > c) \end{cases} \\
&\ge \quad \lambda(m(b) - b - \epsilon) \qquad\qquad (4.9)
\end{aligned}$$

for $\lambda \le \min(1/c, 2\epsilon/c^2, \epsilon/9) =: \lambda_0$ say, i.e. $\delta \le \lambda_0/s^2$. Hence from (4.8) and (4.9)

$$\log(q' + sx) - \lambda \log|W_b(x)| \ge \log(\frac{1}{\delta s}) + \lambda(m(b) - b - \epsilon)$$

for $\lambda \le \lambda_0$ and $x > b$. Then, putting $U_b := (W_b)^\lambda$ we can take

$$m_1(b) := \log(\frac{1}{\delta s}) + \delta s^2(m(b) - b - \epsilon)$$

so that

$$T \le e^{-m_1(b)} \le \delta s - \delta^2 s^3(m(b) - b - \epsilon),$$

giving the upper bound of (4.1).

For the lower bound, choose p'' and q'' as at the start of the proof. Next, note that, from (2.8), $t''g(t'') > t^2 - 2t^4$ if $\frac{1}{2} > t'' > t$. Now, with the help of (4.4) choose t' and t'' to be

$$t''^2 := \frac{s}{q''} > s^2\delta - (1-b)s^4\delta^2 =: t^2.$$

Then, by Theorem 2.2,

$$\begin{aligned} T &\ge t_{\mathbb{Z}}([p''/q'', r/s]) \\ &\ge \frac{1}{s}\sqrt{\frac{s}{q''}}\, g_-\left(\sqrt{\frac{s}{q''}}\right) \\ &\ge \frac{1}{s}(t^2 - 2t^4) \ge \frac{1}{s}(t^2 - 2(s^2\delta)^2) \\ &\ge s\delta - (3-b)s^3\delta^2 \end{aligned}$$

as $t^2 \le s^2\delta$. This completes the proof of Theorem 4.1.

5. Critical polynomials and lower bounds for exponents. Consider an irreducible polynomial $P(x) = a_d x^d + \cdots \in Z[x], a_d > 0$, all of whose zeros α_i lie in an interval I, and for which its *critical value* $c_P := a_d^{-1/d}$ is greater than $t_{\mathbb{Z}}(I)$. We call such a polynomial a *critical polynomial (for I)*. In practice, of course, $t_{\mathbb{Z}}(I)$ is not known exactly, so that in order to identify a polynomial P as being critical for I we need to find another polynomial Q in $Z[x]$ whose maximum

$$m_I(Q) := \max_{x \in I} |Q(x)|^{1/\partial Q}$$

is less than c_P, so that

$$c_P > m_I(Q) \ge t_{\mathbb{Z}}(I).$$

Now, by a classical argument (see [Chu], [BoEr]) if P and Q are relatively prime, and P has all its zeros in I, then $m_I(Q) \ge c_P$. Thus if P is critical for I then P and Q must have a non-trivial common factor, so that in fact P divides Q, since P is irreducible. Note that then, writing $Q = P^k R$ we have that

$$\prod_i |R(\alpha_i)|^{1/(\partial P \partial R)} \geq c_P. \tag{5.1}$$

We now show something stronger than $P|Q$, namely that

THEOREM 5.1. *Suppose that the polynomial P is critical for I, with critical value c_P, and that $Q \in Z[x]$ has $m := m_I(Q) < c_P$. Then P^k divides Q, where $k \geq \gamma \partial Q$, where $\gamma > 0$ depends only on P and m.*

The proof of Theorem 5.1 follows straight from Proposition 5.3 below, on taking P to be critical for I, and $M := c_P$. Specific lower estimates for γ are also given. As examples, the lower bounds γ are then computed in Section 6 for all known critical polynomials of $[0,1]$.

This result is essentially a generalisation of a result of Borwein and Erdélyi [BoEr], where they prove this result in the special case of the critical polynomial x for $I = [0,1]$. The basic idea is as follows: for any zero α in I of a critical polynomial, re-parametrize by $y \in [0,1]$ a sub-interval of I having α as one endpoint. Then apply their argument, making use of the y-parametrization.

However, we first need a slight generalisation of a result of theirs ([BoEr], Theorem 3.1):

LEMMA 5.2. *Let c, m, M be positive constants with $m < M$. Suppose that there is a real polynomial*

$$Q(x) := a_k x^k + a_{k+1} x^{k+1} + \cdots + a_n x^n,$$

where $0 \leq k \leq n$, with $|a_k| \geq c^k M^n$ and $m_{[0,1]}(Q) \leq m$. Then $k \geq \gamma n$, where γ is the least positive root of

$$\frac{(1+x)^{1+x}}{(1-x)^{1-x}(2x)^{2x}c^x} = \frac{M}{m}. \tag{5.2}$$

Proof. This is essentially the proof given in [BoEr] for $c = 1$. However, it has been modified so that it is valid for all n, instead of only for n sufficiently large.

We apply the Gram-Schmidt process to the inner-product space of real polynomials with ordered basis elements $x^n, x^{n-1}, \ldots, x^k$, and inner-product $\langle p, q\rangle := \int_0^1 pq dx$. This gives [BoEr] the Muntz-Legendre polynomials

$$L_i(x) := \sum_{j=i}^{n} (-1)^{n-j} \binom{n+1+j}{n-i} \binom{n-i}{n-j} x^j \quad (i = n, n-1, \ldots, k)$$

with $\langle L_i, L_j\rangle = \delta_{ij}/(2i+1)$. Now, writing $Q(x) = \sum_{i=k}^{n} \lambda_i L_i$ we have $\int_0^1 Q^2 dx \geq \lambda_k^2/(2k+1)$, and, since a_k is the coefficient of x^k in $\lambda_k L_k$,

$$m^n \geq \max_{x\in[0,1]} |Q(x)| \geq \sqrt{\int_0^1 Q^2 dx} \geq \frac{|\lambda_k|}{\sqrt{2k+1}} = \frac{|a_k|}{\sqrt{2k+1}\binom{n+1+k}{n-k}}$$
$$\geq \frac{c^k M^n}{\sqrt{2k+1}\binom{n+1+k}{n-k}}.$$

Next, from the simple inequality

$$((n-k)+2k)^{n+k} \geq \binom{n+k}{n-k} (n-k)^{n-k} (2k)^{2k}$$

we obtain, for $\alpha = k/n$ and

$$f(\alpha) := \frac{(1+\alpha)^{1+\alpha}}{(1-\alpha)^{1-\alpha}(2\alpha)^{2\alpha}}$$

that $\binom{n+k}{n-k}^{1/n} \leq f(\alpha)$ and hence that

$$\frac{f(\alpha)}{c^\alpha} \left(\frac{n+k+1}{\sqrt{2k+1}}\right)^{1/n} \geq \frac{M}{m}.$$

Finally, since any power Q^N of Q also satisfies the conditions of the the lemma, we can replace n by nN and k by kN and let $N \to \infty$, giving simply

$$\frac{f(\alpha)}{c^\alpha} \geq \frac{M}{m},$$

from which the result follows.

We now apply the lemma to an arbitrary finite interval I.

Proposition 5.3. *Let P be a real polynomial with all its zeroes α_i distinct and lying in I, and R another real polynomial such that*

$$(5.3)\qquad (i)\qquad \prod_i |R(\alpha_i)|^{1/(\partial P \partial R)} \geq M$$

$$(ii)\qquad m_I(P^k R) \leq m,$$

for some positive integer k, where $m < M \leq 1$. Then there is a positive constant γ, independent of R (but in general depending on I, m, M, and P) such that $k \geq \gamma n$. Explicitly, γ is given by Lemma 5.2 with the constant c defined as follows:

let $I = [t_-, t_+]$, $\alpha'_1, \ldots, \alpha'_{\partial P-1}$ be the zeros of P', and $x_0 := t_-$, $x_{\partial P} := t_+$, while for $i = 2, \ldots, \partial P - 1$ let x_i be specified by $\{\alpha_i, x_i\}$ being the two roots of $P(x)/(x - \alpha_i) = P'(\alpha_i)$ in $(\alpha_{i-1}, \alpha_{i+1})$. Then c can be taken to be

$$\max(|P(t_-)|, |P(t_+)|)$$

if $\partial P = 1$, while if $\partial P > 1$

$$c := \min_{i=1,\ldots,\partial P} \rho_i$$

where

$$\rho_1 := \max(|P(t_-)|, |P(\alpha'_1)|)$$

$$\rho_{\partial P} := \max(|P(\alpha'_{\partial P-1})|, |P(t_+)|)$$

and for $i = 2, \ldots, \partial P - 1$

$$\rho_i := \begin{cases} \max(|P(x_i)|, |P(\alpha'_i)|) & \text{if } x_i < \alpha'_{i-1} \\ \max(|P(\alpha'_{i-1})|, |P(\alpha'_i)|) & \text{if } x_i \in [\alpha'_{i-1}, \alpha'_i] \\ \max(|P(\alpha'_{i-1})|, |P(x_i)|) & \text{if } x_i > \alpha'_i. \end{cases}$$

Proof. From (i), there is some α_i, say α, such that $|R(\alpha)|^{1/\partial R} \geq M$. Let u be a point in I such that there are no zeros of P in $[u, \alpha)$ (or $(\alpha, u]$ if $u > \alpha$). Put $d := u - \alpha$ (possibly negative), $P_0(x) := P(x)/(x - \alpha)$ and $y := (x - \alpha)/d$. Then for $0 \leq y \leq 1$, $x \in I$ and $P_0(x) \neq 0$. Hence (ii) implies that

$$(5.4)\qquad \max_{y \in [0,1]} |(yd)^k P_0(yd + \alpha)^k R(yd + \alpha)| \leq m^n$$

where $n := \partial(P^k R)$. Thus if

$$r := \min_{y\in[0,1]} |P_0(yd+\alpha)| > 0, \tag{5.5}$$

then

$$\max_{y\in[0,1]} |y^k(rd)^k R(yd+\alpha)| \le m^n \le m^{k+\partial R}$$

as $m < 1$. Hence we can apply Lemma 5.2 to the polynomial $y^k(rd)^k R(yd+\alpha)$, whose coefficient of y^k is at least $(\frac{r|d|}{M})^k M^{k+\partial R}$ in modulus. Thus we can take $c := r|d|/M$ in the lemma.

In order to get good estimates for γ for particular critical polynomials of a specific interval I, we need to choose u so that the value $c = r|d|$ is as large as possible. Working *for* us is the fact that we can choose d to be of either sign, allowing us to choose whichever sign maximises c. Working *against* us, however, is the fact that we don't know which α_i is α, so we must minimise over all i.

To obtain a good value of c, we first assemble some elementary facts about P, P', the P_i and their roots:

(5.6) (i) $t_- \le \alpha_1 < \alpha'_1 < \alpha_2 \cdots < \alpha_{\partial P-1} < \alpha'_{\partial P-1} < \alpha_{\partial P} \le t_+$

(ii) $P_i(x) := P(x)/(x-\alpha_i)$ has $P_i(\alpha_i) = P'(\alpha_i) \quad (i = 1,\dots,\partial P)$

(iii) For $\partial P = 1$ and $i = 2,\dots,\partial P-1$, $|P_i(x)|$ has a unique maximum in $[\alpha_{i-1},\alpha_{i+1}]$, and the equation $P_i(x) = P_i(\alpha_i)$ has exactly two roots α_i and (say) x_i in this interval.

(iv) For each i, $P_i(x)$ is monotonic in $[t_-,\alpha_1]$ and in $[\alpha_{\partial P}, t_+]$.

We now apply these results to the proof. The case $\partial P = 1$ is trivial, so we can assume that $\partial P > 1$. For each zero α_i of P we choose, successively for each I, the number u above to be one of α'_{i-1}, α'_i or x_i, as follows:

For $d = u - \alpha_i$ we want to choose d so that

$$w_i(d) := |d| \min_{y\in[0,1]} |P_i(\alpha_i + yd)| \tag{5.7}$$

is as large as possible. Clearly we need $u \in (\alpha_{i-1},\alpha_{i+1})$ so that $w_i(d) \ne 0$. Now, by (iii) above, $|P_i(\alpha_i+yd)|$ has no local minimum for $y \in [0,1]$, and hence

$$\begin{aligned} w_i(d) &= \min(|d||P_i(\alpha_i)|, |d||P_i(\alpha_i+d)|)\\ &= \min(|d||P'(\alpha_i)|, |P(\alpha_i+d)|). \end{aligned}$$

Now $P'(\alpha_i)d$ and $P(\alpha_i + d)$ have the same sign in the range of d under discussion, so the maximum of $w_i(d)$ occurs when $P'(\alpha_i)d = P(\alpha_i + d)$ or when $P'(\alpha_i + d) = 0$ or when $\alpha_i + d$ is an endpoint of I (i.e. $d = \alpha'_{i-1} - \alpha_i$ or $\alpha'_i - \alpha_i$ in either of these last two cases.) If α'_i (respectively α'_{i-1}) is between α_i and x_i then the maximum of $w_i(d)$ on $[\alpha_i, \alpha_{i+1}]$ (respectively $[\alpha_{i-1}, \alpha_i]$) occurs at x_i (i.e. for $d = x_i - \alpha_i$). Otherwise it occurs at α'_i (respectively α'_{i-1}). The final value of c is obtained by minimising over all i. This completes the proof.

6. Application to $[0,1]$. In this section we apply our results to the interval $[0,1]$. To do this, we make use of some computations from [Fl1]. All the polynomials

$$
\begin{aligned}
&P_1(x) = x\\
&P_2(x) = x - 1\\
&P_3(x) = x^2 - 3x + 1\\
&P_4(x) = x^4 - 7x^3 + 13x^2 - 7x + 1\\
&P_5(x) = x^3 - 5x^2 + 6x - 1\\
&P_6(x) = x^3 - 6x^2 + 5x - 1\\
&P_7(x) = x^8 - 15x^7 + 83x^6 - 220x^5 + 303x^4 - 220x^3 + 83x^2 - 15x + 1\\
&P_8(x) = x^4 - 7x^3 + 14x^2 - 8x + 1\\
&P_9(x) = x^4 - 8x^3 + 14x^2 - 7x + 1
\end{aligned}
$$

have all their zeroes in $[0, \infty)$, and among such polynomials, have small absolute Mahler measure (see [Sm1]). Following [BoEr], [Fl2] define polynomials Q_i by $Q_0(t) = t - 1$ and

$$Q_i(t) = P_i\left(\frac{t}{1-t}\right)(1-t)^{\partial P_i} \qquad (i = 1, \dots, 9)$$

and exponents $e_i, (i = 0, \dots, 10)$ as

$$
\begin{aligned}
e_0 = e_1 &= 0.31784899\\
e_2 &= 0.11621266\\
e_3 &= 0.03824029\\
e_4 &= 0.01501115\\
e_5 = e_6 &= 0.00624421\\
e_7 &= 0.00575228\\
e_8 = e_9 &= 0.00321130\\
e_{10} &= 0.00119514
\end{aligned}
$$

Then, from [Fl1], pp. 67-68 the polynomial-power $Q(t) = \prod_{i=0}^{9} Q_i(t)^{e_i}(6t^2 - 6t + 1)^{e_{10}}$ has $m_{[0,1]}(Q) = 0.42353115$, so that any polynomial $at^d + \dots$ with all zeroes in $[0,1]$ and $a^{-1/d} > 0.42353115$ is critical. In particular, $Q_0, Q_1, \dots, Q_9$ are all critical. Now apply Theorem 5.1 with the precise lower bound for γ given by Lemma 5.4, for $M := c_P$ and $m := m_{[0,1]}(Q)$. We see that if a polynomial P with integer coefficients satisfies

$$m_{[0,1]}(P) < 0.42353115,$$

then $Q_i^{\gamma_i \partial P}$ divides P. The polynomials Q_i, the corresponding exponents γ_i and critical values c_{Q_i} are as follows:

Polynomials Q_i	γ_i	c_{Q_i}
$Q_0(t) = t - 1$	0.264151	1
$Q_1(t) = t$	0.264151	1
$Q_2(t) = 2t - 1$	0.021963	0.5
$Q_3(t) = 5t^2 - 5t + 1$	0.005285	0.4472136
$Q_4(t) = 29t^4 - 58t^3 + 40t^2 - 11t + 1$	0.001065	0.4309238
$Q_5(t) = 13t^3 - 20t^2 + 9t - 1$	0.000232	0.4252904
$Q_6(t) = 13t^3 - 19t^2 + 8t - 1$	0.000232	0.4252904
$Q_7(t) = 941t^8 - 3764t^7 + 6349t^6 - 5873t^5 +$		
$+3243t^4 - 1089t^3 + 216t^2 - 23t + 1$	0.000136	0.4249143
$Q_8(t) = 31t^4 - 63t^3 + 44t^2 - 12t + 1$	0.000026	0.4237987
$Q_9(t) = 31t^4 - 61t^3 + 41t^2 - 11t + 1$	0.000026	0.4237987

Table 2. Ten critical polynomials for the interval $[0,1]$.

Earlier Aparicio [Ap3] had obtained the values $\gamma_0 = \gamma_1 = 0.1456, \gamma_2 = 0.016$ and $\gamma_3 = 0.0037$. Borwein and Erdélyi [BoEr] improved γ_0, γ_1 substantially to 0.26. Further, they showed (Corollary 2.3) that these polynomials Q_i must all be factors of any "integer Chebyshev polynomial" of $[0,1]$, which has sufficiently high degree, i.e. of any polynomial P for which $m_{[0,1]}(P)$ is minimal among all integer polynomials of that degree. Here we quantify their result by providing lower bounds for the power of the factor Q_i dividing P, relative to the degree of P.

7. The trace of totally real algebraic integers. The main result of this section is the following, which was needed in Section 4:

Proposition 7.1. *There is a constant c such that, for every b in $[0,1)$ there is a polynomial-power X_b such that for $x \geq b$*

$$x - \log|X_b(x)| \geq 1.6 + b \tag{7.1}$$

and for $x \geq c$

$$|X_b(x)| \leq (x/3)^3. \tag{7.2}$$

Proof. For $b = 0$, one can check that, for

$$\begin{aligned}
X_0(x) &= x^{0.59316418}(x-1)^{0.54297203}(x-2)^{0.08906340}(x^2-3x+1)^{0.19812041}\times\\
&\times(x^2-4x+2)^{0.00951759}(x^3-5x^2+6x-1)^{0.10300271}(x^3-6x^2+9x-1)^{0.01373640}\times\\
&\times(x^4-7x^3+13x^2-7x+1)^{0.05116436}(x^4-7x^3+14x^2-8x+1)^{0.03662990}\times\\
&\times(x^5-9x^4+28x^3-35x^2+15x-1)^{0.01268943}(x^2-4x+1)^{0.00980493}\times\\
&\times(x^2-5x+5)^{0.00074611}(x^3-6x^2+9x-3)^{0.00635328}\times\\
&\times(x^5-9x^4+27x^3-31x^2+12x-1)^{0.01050115}\times\\
&\times(x^6-11x^5+43x^4-72x^3+51x^2-14x+1)^{0.00134609}
\end{aligned}$$

and all $x > 0$ we have

$$x - \log|X_0(x)| \geq 1.7719.$$

This $X_0(x)$ was the polynomial-power used in [Sm2](see also [Sm3]) to prove that $\mathrm{Trace}(\alpha)/\partial\alpha \geq 1.7719$ for all totally positive algebraic integers not a zero of X_0. (X_0 did not appear in [Sm2] as the table containing it was removed to save space). We refer to this computation as *Run 0* in Table 3. We have now found, by a Remez-type optimisation algorithm, 10 further polynomial-powers $X_{b_j}(j = 1, \dots, 10)$ where, for $b = b_j$

$$x - \log|X_b(x)| \geq m(b) \qquad \text{for all} \qquad x \geq b. \tag{7.3}$$

The b_j, $m(b_j)$ and $X_{b_j} = \prod_{x=1}^{N_j} P_{n_{ij}}(x)^{e_{ij}}$ are given by Tables 3,4 and 5. The polynomials used in the optimisation, and those in Table 6, were found by the same search procedure as used in [Sm3]. However, the polynomials searched for were specified to have their zeroes in the intervals $[0.05\,k, \infty), (k = 0, 1 \dots, 19)$ instead of only $[0, \infty)$ as in [Sm3].

Note that

$$m(b_j) - b_{j+1} > 1.6 \qquad (j = 0, \dots, 10), \tag{7.4}$$

(this being of course the basis by which the b_j have been chosen) so that certainly (7.1) holds for $b = b_j$. But also, any inequality (7.3), valid for b is also trivially valid (with $X_{b'} := X_b$) for $x \geq b'$ with $b' > b$. Thus, if we put $X_b := X_{b_j}$ and $m(b) := m(b_j)$ for $b \in [b_j, b_{j+1})$ then, because of (7.4),

Run j	b_j	$m(b_j) - b_j$	$m(b_j) - b_{j+1}$
0	0.00	1.7719	1.6018
1	0.17	1.6592	1.6092
2	0.22	1.6844	1.6044
3	0.30	1.6926	1.6026
4	0.39	1.7143	1.6043
5	0.50	1.6501	1.6001
6	0.55	1.6930	1.6030
7	0.64	1.6654	1.6054
8	0.70	1.6620	1.6020
9	0.76	1.7568	1.6068
10	0.91	1.7515	1.6615
	(1.00	1.7719)	

Table 3. The values of b used in the proof of Proposition 7.1, and required functions of $m(b)$.

(7.1) holds for all $b \in [0,1)$. Finally, (7.2) is a consequence of the fact that X_0 above, and each of the X_b's in Table 5 is $O(x^{2.83})$.

As mentioned earlier, we expect that, with the use of substantially more than ten values b_j of b, we should be able to improve the constant 1.6 to at least 1.65, in Theorem 4.1, Proposition 7.1 and Theorem 1.1. This is because all the values in Column 3 of Table 3 are at least this value. Of course further improvements may also be possible perhaps using extra polynomials. One polynomial which may give such an improvement is the factor of Habsieger and Salvy's polynomial mentioned in the introduction.

The proof of Theorem 1.1 now follows easily. Let α and α_1 be as in the statement of the theorem. By replacing α by $\alpha - \lfloor \alpha_1 \rfloor$ we can assume that $\alpha_1 \in [0,1)$. Then we take $b = \alpha_1$ in Proposition 7.1, and so by (7.1)

$$\frac{\mathrm{Trace}(\alpha)}{\partial\alpha} \geq 1.6 + \alpha_1 + \log |\prod_i X_{\alpha_1}(\alpha_i)|^{1/\partial\alpha} \geq 1.6 + \alpha_1$$

unless $X_{\alpha_1}(\alpha_i) = 0$, as $\prod_i X_{\alpha_1}(\alpha_i)$ is a product of positive powers of the resultants of the minimal polynomial of α and the polynomials making up X_{α_1}. Hence (1.6) holds unless α_1 is a root of one of the polynomials of Table 5, and only those listed in the statement of the theorem actually have $\mathrm{Trace}(\alpha)/\partial\alpha < 1.6 + \alpha_1$.

TABLE 4. The polynomial factors of the X_b in Proposition 7.1

Table 5. $X_{b_i} = \prod_j (\text{Poly } \#_j)^{e_{ij}}$

degree	trace/degree $-r_1$	r_1	discriminant	coefficients						
1	0.0	0.0	1	1	0					
2	1.1180	0.3820	5	1	-3	1				
2	1.4142	0.5858	8	1	-4	2				
3	1.4686	0.1981	49	1	-5	6	-1			
4	1.5222	0.2278	725	1	-7	13	-7	1		
3	1.5321	0.4679	81	1	-6	9	-3			
4	1.5771	0.1729	1125	1	-7	14	-8	1		
3	1.5803	0.7530	49	1	-7	14	-7			
4	1.6057	0.6443	725	1	-9	27	-31	11		
5	1.6688	0.1312	38569	1	-9	26	-29	11	-1	
6	1.6703	0.1630	1134389	1	-11	42	-68	46	-12	1
6	1.6704	0.4963	485125	1	-13	64	-151	177	-96	19
3	1.6751	0.3249	148	1	-6	8	-2			
6	1.6805	0.1528	966125	1	-11	42	-67	45	-12	1
5	1.6866	0.5134	36497	1	-11	41	-64	41	-9	
5	1.6866	0.1134	36497	1	-9	27	-31	12	-1	
6	1.6866	0.4801	434581	1	-13	62	-138	149	-73	13
4	1.6913	0.5587	1957	1	-9	26	-28	9		
3	1.6920	0.3080	49	1	-6	5	-1			
4	1.6935	0.8065	725	1	-10	34	-45	19		
5	1.6965	0.3035	24217	1	-10	35	-51	29	-5	

Table 6. List of irreducible polynomials up to degree 6 with all zeroes real, minimum zero r_1 in [0,1], and trace/degree $-r_1$ less than 1.7000.

Appendix: The Gorškov polynomials. These polynomials G_k were defined originally by Gorškov[Gor], and later independently by Wirsing and Montgomery [Mo], p.183 and by Smyth[Sm1]. See also [Ap2], p.6, and [BoEr], p.667. They are monic, with integer coefficients. One could say that the Gorškov polynomials bear a comparable relationship to the positive half line, where all their zeroes lie, as the cyclotomic polynomials do to the unit circle.

Let $Hz := z - 1/z$, and let $H^k z$ be its kth iterate. Put $G_0(y) := y - 1$, $D_0(y) = 1$, so that $Hz = \frac{G_0(z^2)}{zD_0(z^2)}$. The kth Gorškov polynomial G_k of degree 2^k is then defined by

$$H^k z = \frac{G_{k-1}(z^2)}{zD_{k-1}(z^2)}, \tag{A.1}$$

so that, for $k = 1, 2, \ldots$

$$H^{k+1} z = \frac{G_k(z^2)}{zD_k(z^2)} = H^k z - \frac{1}{H^k z} = \frac{G_{k-1}(z^2)^2 - z^2 D_{k-1}(z^2)^2}{zG_{k-1}(z^2)D_{k-1}(z^2)}. \tag{A.2}$$

From this we obtain, for $k = 1, 2, \ldots$

$$G_k(y) = G_{k-1}(y)^2 - yD_{k-1}(y)^2 \tag{A.3}$$

$$D_k(y) = G_{k-1}(y)D_{k-1}(y) = \prod_{j=0}^{k-1} G_j(y). \tag{A.4}$$

The first four Gorškov polynomials are the polynomials P_2, P_3, P_4, P_7 at the start of Section 6. The G_k are known to be irreducible, as was proved by Smyth[Sm1], Lemma 4, and by Wirsing(see [Mo], p.187). Wirsing's elegant proof is self-contained and elementary. Note, however, that $G_k(z^2)$ is reducible, as it is the difference of two squares in (A.2). Thus, for any zero α of $G_k(y)$, $\sqrt{\alpha}$ is one of $\pm G_{k-1}(\alpha)/D_{k-1}(\alpha)$. It follows straight from the definitions that each $G_k(y)$ is monic with integral coefficients, and that all its zeroes are real and positive. Also, from $H^{k+1}z = H^k H z$ we have for $k = 1, 2, \ldots$ that

$$G_k(y) = y^{2^{k-1}} G_{k-1}(y + \frac{1}{y} - 2).$$

Further, as observed by Wirsing and Montgomery[Mo], p.184, we have, for $k = 1, 2, \ldots$ the recurrence

$$G_{k+1}(y) = G_k(y)^2 + G_k(y)G_{k-1}(y)^2 - G_{k-1}(y)^4. \tag{A.5}$$

To prove this, note that from (A.3) and (A.4)

$$G_{k+1}(y) = G_k(y) - yD_{k-1}(y)G_{k-1}(y). \tag{A.6}$$

Now eliminate $yD_k(y)$ using (A.3). (In fact Wirsing and Montgomery worked with $f_k(z) := z^{2^k} G_k(\frac{1}{z} - 1)$, which has all its zeroes in $[0, 1]$, instead of with G_k.)

To see the connection between the polynomials of Section 2 and Gorškov polynomials, define the map I by $Iz := iz$, and take $Gz := z + 1/z$, as in Section 2. Then $Gz = I^{-1}HIz$, so that the kth iterate $G^k z$ is given by

$$G^k z = I^{-1}H^k I z = \frac{G_{k-1}(-z^2)}{-zD_{k-1}(-z^2)}.$$

Hence the polynomials U_k, V_k of Section 2 are, for $k \geq 2$, given by $U_k(z) = G_{k-1}(-z^2), V_{k-1}(z) = -zD_k(-z^2)$. Note that $U_k(z)$ is irreducible, because any root β of U_k is imaginary, so that

$$[\mathbb{Q}(\beta) : \mathbb{Q}] = [\mathbb{Q}(\beta) : \mathbb{Q}(\beta^2)][\mathbb{Q}(\beta^2) : \mathbb{Q}] = 2\deg(G_{k-1}) = \deg(U_k),$$

since $G_{k-1}(z)$ is irreducible.

It is clear that G_k and D_k, and G_k and $G_{k'}$ with $k' < k$ have no common zeroes([Mo], p.186). Thus the same applies to U_k, V_k and to $U_k, U_{k'}$. The distribution, as $k \to \infty$ of the zeroes of G_k, is highly irregular. In fact their limiting probability density has Hausdorff dimension 0.800611138269168784. For details see [DaSm], in which also the density function of the 32768 zeroes of G_{15} is illustrated.

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