Journal de Théorie des Nombres
de Bordeaux **10** (1998), 49–64

Two problems related to the non-vanishing of $L(1, \chi)$

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Résumé. Dans cet article, nous étudions deux problèmes à priori assez éloignés, l'un se rapportant à la géométrie diophantienne, et l'autre à l'analyse de Fourier. Tous deux induisent en réalité des questions très proches, relatives à l'étude du rang de matrices dont les coefficients sont nuls ou égaux à $((xy/q))$, $(0 \leq x, y < q)$, où $((x)) = x - [x] - 1/2$ désigne la partie fractionnaire "centrée" de x. L'étude de ces rangs est liée au problème d'annulation des fonctions L de Dirichlet au point $s = 1$.

Abstract. We study two rather different problems, one arising from Diophantine geometry and one arising from Fourier analysis, which lead to very similar questions, namely to the study of the ranks of matices with entries either zero or $((xy/q))$, $0 \leq x, y < q$, where $((u)) = u - [u] - 1/2$ denotes the "centered" fractional part of x. These ranks, in turn, are closely connected with the non-vanishing of the Dirichlet L-functions at $s = 1$.

1. Introduction

The following problems, which arise in quite different contexts, have led us to very similar questions.

Problem A. This is a problem in Diophantine geometry and it was suggested by Cellini [2] for the study of the dimension of a commutative descent algebra of the group algebra over $\mathbf{Q}$ of the Weyl groups of type A_n. Let q be a positive integer and for each $a \in \mathbf{Z}$ consider the function $f_{a,q} : \mathbf{Z} \to \mathbf{N}$

$$f_{a,q}(n) = \#\{an < m \leq (a+1)n \mid m \equiv 0 \pmod{q}\}. \tag{1}$$

For fixed q, how many among the functions $f_{a,q}$ are linearly independent?

Manuscrit reçu le 11 mars 1996.

It is clear that $f_{a,q}$ depends only on the congruence class of a modulo q. For a real number α, let $[\alpha]$ denote its integer part, $\{\alpha\}$ denote its fractional part and $((\alpha)) = \{\alpha\} - 1/2$. Then

$$\begin{aligned} f_{a,q}(n) &= [(a+1)n/q] - [an/q] \\ &= n/q - (((a+1)n/q)) + ((an/q)) = n/q + g_{a,q}(n), \end{aligned} \tag{2}$$

whence $f_{a,q}$ is $1/q \cdot$ identity plus a function $g_{a,q}$ which is periodic modulo q. Let also $\phi_{a,q}(n) = ((an/q))$. We shall prove the following

Theorem 1. *The dimension of the vector space generated by the functions $g_{a,q}$ is equal to $[(q-3)/2] + d(q)$ and the dimension of the vector space generated by the functions $f_{a,q}$ is equal to $[(q-1)/2] + d(q)$, where $d(q)$ is the number of divisors of q.*

To prove this we have to study the rank of the matrix

$$M = ((xy/q))_{0 \le x,y < q}, \tag{3}$$

and this will be done in Corollary 2, where the rank will be given in terms of the number of non-vanishing expressions of the form

$$L(\chi) = \sum_{a=0}^{d-1} ((a/d))\chi(d),$$

where χ is a Dirichlet character with modulus a divisor d of q. The method uses a decomposition of the space of functions $\{f \,|\, f : \mathbf{Z}/q\mathbf{Z} \to \mathbf{C}\}$ into an orthogonal sum of subspaces defined in terms of the primitive Dirichlet characters modulo divisors of q. Although this decomposition follows rather easily from the classical theory of Dirichlet characters, we do not have explicit references for this result, so we include its complete proof.

It is rather surprising that Corollary 2 turns out to be equivalent to the non-vanishing of the first Bernoulli numbers relative to odd characters χ, which in turn is equivalent, via the functional equation, to the non-vanishing of the corresponding Dirichlet functions $L(s,\chi)$ at $s = 1$.

Problem B. Let q be a positive integer and let $f : \mathbf{Z} \to \mathbf{C}$ be odd and periodic modulo q. Then the series

$$h(x) = \sum_{n=1}^{\infty} \frac{f(n)}{n} \cos 2\pi n x \tag{4}$$

is convergent everywhere (see Lemma 7 below). Let $\delta_h(d)$ denote the Riemann sum

$$\delta_h(d) = \frac{1}{d}\sum_{a=1}^{d} h(a/d), \tag{5}$$

where $d \geq 1$ is an integer and h is given by (4). Our problem is the following: is it possible to have $\delta_h(d) = 0$ for every $d \geq 1$ and the corresponding $f(n)$ not all zero? The answer is negative (see Theorem 2) and is equivalent to proving that the rank of the matrix $P(ij/q)$ $(1 \leq i, j \leq q)$, where

$$P(x) = \begin{cases} ((x)) & \text{if } x \notin \mathbf{Z} \\ 0 & \text{if } x \in \mathbf{Z}, \end{cases} \tag{6}$$

is $[(q-1)/2]$. In the proof a vital step is provided by the well known identity (see for instance [3] and [4])

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} P(na/q) = -\frac{1}{\pi}\sin(2\pi a/q) \tag{7}$$

which can be deduced from $L(1,\chi) \neq 0$ and the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{\mu(n)\chi(n)}{n}$ for all non-principal Dirichlet characters χ modulo q.

2. Solution to Problem A

Let $q > 1$ be an integer. We shall denote by ψ a character modulo a divisor of q. If the modulus $m(\psi)$ of ψ is q/d for some divisor d of q, ψ will also be denoted by $\psi^{(d)}$. We shall denote by f_ψ, or by $f_{d,\psi}$ if we want to make explicit the modulus, the function defined on $(\mathbf{Z}/q\mathbf{Z})$ into the complex numbers such that

$$f_\psi(y) = \begin{cases} 0 & \text{if } d \nmid y \\ \psi(y/d) & \text{if } d \mid y. \end{cases}$$

We abbreviate $\mathbf{Z}/q\mathbf{Z}$ by (q) and set $V = \{f \,|\, f : (q) \to \mathbf{C}\}$. The scalar product of two functions $f, g \in V$ is given by

$$(f,g) = \frac{1}{q}\sum_{x\in(q)} f(x)\bar{g}(x). \tag{8}$$

Lemma 1. *There are exactly q functions of the type f_ψ; these functions are pairwise orthogonal and generate V as a vector space. Furthermore*

$$\|f_\psi\|^2 = (f_\psi, f_\psi) = \frac{1}{q}\phi(q/d) = \frac{1}{q}\phi(m(\psi)).$$

Proof. The first assertion follows from the fact that there are $\phi(q/d)$ characters modulo q/d and $\sum_{d \mid q} \phi(q/d) = q$. Let $\psi^{(d)}, \xi^{(e)}$ be characters as above. We have

$$\sum_{x \in (q)} f_\psi(x) \bar{f}_\xi(x) = \sum_{\substack{[d,e] \mid x \\ x \in (q)}} \psi(x/d) \bar{\xi}(x/e).$$

Set $d = hd_1$, $e = he_1$ with $(e_1, d_1) = 1$, hence $[d,e] = hd_1e_1$. Letting $x = y[d,e]$ one sees that the last sum equals

$$\sum_{y \in (q/[d,e])} \psi(ye_1)\bar{\xi}(yd_1) = \psi(e_1)\bar{\xi}(d_1) \sum_{y \in (q/[d,e])} \psi(y)\bar{\xi}(y).$$

Now $hd_1e_1 = de_1$ divides q, whence $e_1 \mid (q/d)$ and, similarly, $d_1 \mid q/e$. Since ψ and ξ have modulus q/d and q/e respectively, we get that $\psi(e_1)\bar{\xi}(d_1) = 0$ unless $e_1 = d_1 = 1$, that is $d = e$. In this case

$$q(f_\psi, f_\xi) = \sum_{y \in (q/d)} \psi(y)\bar{\xi}(y) = \begin{cases} 0 & \text{if } \xi \neq \psi \\ \phi(q/d) & \text{if } \xi = \psi, \end{cases}$$

and the lemma is proved. □

Let χ be a primitive character. If χ induces ψ we write $\chi \mid \psi$. Let V_χ be the vector space generated by the f_ψ such that $\chi \mid \psi$:

$$V_\chi = \langle \{ f_\psi \ : \ \chi \mid \psi \} \rangle.$$

Lemma 2. *V is the direct sum of the subspaces V_χ, where χ runs over all primitive characters whose modulus is a divisor of q.*

Proof. Clear. □

The scalar product on V induces a pairing

$$\{ \cdot , \cdot \} : V \times V \to V$$

as follows:

$$\{f, g\}(y) = \sum_{x \in (q)} f(xy) g(x). \tag{9}$$

To study the properties of this pairing a lemma will be useful. If ψ is a character with modulus q/d and h is a divisor of d we denote by ψ_h the character induced by ψ with modulus $(q/d)h = m(\psi)h$. Hence

$$m(\psi_h) = m(\psi) \cdot h,$$

$$\psi_h = \psi^{(d/h)}.$$

By definition we have

$$\psi_h(x) = \begin{cases} 0 & \text{if } (qh/d, x) > 1 \\ \psi(x) & \text{if } (qh/d, x) = 1. \end{cases}$$

Lemma 3. *The following equality holds:*

$$\psi(x) = \sum_{\substack{\nu \mid h \\ \nu \mid x}} \psi(\nu)\psi_{h/\nu}(x/\nu).$$

Proof. Let $h = rh_1$, $x = rx_1$, $(h_1, x_1) = 1$. Then the right-hand term equals

$$\sum_{\nu \mid r} \psi(\nu)\psi_{h/\nu}(r/\nu)\psi_{h/\nu}(x_1).$$

The modulus of $\psi_{h/\nu}$ is $qh/d\nu$. Hence if $(h/\nu, r/\nu) > 1$ we have $\psi_{h/\nu}(r/\nu) = 0$. On the other hand $(r/\nu) \mid (h/\nu)$, whence our sum contains at most one non-vanishing term, corresponding to $\nu = r$, namely $\psi(r)\psi_{h_1}(x_1)$. Now $(qh_1/d, x_1) = (q/d, x_1)$, hence $\psi_{h_1}(x_1) = \psi(x_1)$. In conclusion $\psi(r)\psi_{h_1}(x_1) = \psi(r)\psi(x_1) = \psi(x)$. □

Lemma 4. *For a character ψ of modulus q/d the following formula holds:*

$$f_\psi(xy) = \sum_{\nu \mid (d,y)} \psi\left(\frac{y\nu}{(d,y)}\right) f_{\psi_{(d,y)/\nu}}(x).$$

Proof. Let $d = d_1 h$, $y = y_1 h$, $(d_1, y_1) = 1$. Since $f_\psi(xy) = 0$ if $d \nmid xy$, we assume that $d \mid xy$, or, equivalently, $d_1 \mid x$ and $x = d_1 x'$. In this case

$$f_\psi(xy) = \psi(xy/d) = \psi(y_1 x') = \psi(y_1)\psi(x').$$

Using Lemma 3 with $x = x' = x/d_1$ we obtain

$$\psi(x') = \sum_{\substack{\nu \mid h \\ \nu \mid x'}} \psi(\nu)\psi_{h/\nu}(x'/\nu)$$

Let

$$A(x) = \sum_{\nu \mid h} \psi(\nu) f_{\psi_{h/\nu}}(x).$$

We have $m(\psi_{h/\nu}) = (q/d)\cdot(h/\nu) = q/d_1\nu$, whence

$$A(x) = \sum_{\substack{\nu \mid h \\ d_1\nu \mid x}} \psi(\nu)\psi_{h/\nu}(x/d_1\nu) = \sum_{\substack{\nu \mid h \\ \nu \mid x'}} \psi(\nu)\psi_{h/\nu}(x'/\nu),$$

and therefore

$$f_\psi(xy) = \psi(y_1) \sum_{\nu \mid h} \psi(\nu) f_{\psi_{h/\nu}}(x),$$

whenever $d_1 \mid x$. If $d_1 \not| x$ then all terms vanish and the formula holds again. □

Lemma 5. *If $\bar{\xi} \notin \{\psi_\mu : \mu \mid d\}$ one has $\{f_\psi, f_\xi\} = 0$. If $\xi = \bar{\psi}_\mu$ one has $m(\xi) = q\mu/d$ and*

$$\{f_\psi, f_\xi\} = \phi(q\mu/d) \sum_{\nu \mid (d/\mu)} \psi(\nu) f_{\psi_{d/\mu\nu}}. \tag{10}$$

Proof. The first statement follows easily from Lemma 1 and Lemma 4. Suppose now that $\xi = \bar{\psi}_\mu$. If $\mu \not| y$ we have $\{f_\psi, f_\xi\}(y) = 0$ and the same is true for the righthandside of (10). If $\mu \mid y$ one has

$$\begin{aligned} \{f_\psi, f_\xi\}(y) &= q \sum_{\nu \mid (d,y)} \psi(y\nu/(d,y))(f_{\psi_{(d,y)/\nu}}, f_{\psi_\mu}) \\ &= q\|f_{\psi_\mu}\|^2 \psi(y/\mu) = \phi(q\mu/d)\psi(y/\mu). \end{aligned}$$

Applying Lemma 3 with $h = d/\mu$ we have

$$\psi(x) = \sum_{\substack{\nu \mid (d/\mu) \\ \nu \mid x}} \psi(\nu)\psi_{d/\mu\nu}(x/\nu),$$

and, setting $x = y/\mu$,

$$\psi(y/\mu) = \sum_{\nu \mid (d/\mu, y/\mu)} \psi(\nu)\psi_{d/\mu\nu}(y/\mu\nu).$$

□

Corollary 1. *The function $\Gamma_F : V \to V$ given by*

$$\Gamma_F(f) = \{F, f\}$$

maps $V_{\bar{\chi}}$ into V_χ and vice-versa.

Proof. Clear. □

Let $F \in V$. Put

$$F = \sum_{\chi \text{ primitive}} F_\chi,$$

where χ runs over primitive characters with modulus dividing q and $F_\chi \in V_\chi$, and set

$$F_\chi = \sum_{\chi \mid \psi} \alpha_\psi f_\psi.$$

Let ξ be a character induced by $\bar{\chi}$. From Lemma 5, putting $m(\xi) = q/r$, we get

$$\begin{aligned}\{F, f_\xi\} &= \phi(m(\xi)) \sum_{\bar{\psi} \mid \xi} \alpha_\psi \sum_{\nu \mid r} \psi(\nu) f_{\psi_{r/\nu}} \\ &= \phi(q/r) \sum_{\chi \mid \zeta = \zeta^{(s)}} f_\zeta \sum_{\substack{\nu \mid r \\ \psi_{r/\nu} = \zeta \\ \bar{\psi} \mid \xi}} \psi(\nu) \alpha_\psi.\end{aligned}$$

Let $\chi = \chi^{(a)}$, that is $q/a = m(\chi)$. Then $r \mid a$. If $\psi = \psi^{(d)}$, then $d \mid a$. Moreover $\bar{\psi} \mid \xi$ implies $r \mid d$ whence $r \mid d \mid a$. Since also $s \mid a$, $\psi_{r/\nu} = \zeta$ is equivalent to $(q/d)(r/\nu) = q/s$, that is $rs = d\nu$.

If χ is a non-real (resp. real) character, the matrix associated to the restriction of the function Γ_F to $V_\chi \oplus V_{\bar{\chi}}$ (resp. V_χ) is of type

$$\begin{pmatrix} 0 & A \\ \bar{A} & 0 \end{pmatrix}$$

(resp. of type (A)) where A is the matrix of a linear function $\tilde{\Gamma}$ on a vector space with basis $\{Y_s\}_{s \mid a}$ such that

$$\tilde{\Gamma}(Y_r) = \phi(q/r) \sum_{s \mid a} Y_s \sum_{\substack{rs = d\nu \\ r \mid d \mid a \\ \nu \mid r}} \psi(\nu) \beta_d, \tag{11}$$

where ψ is the unique character of modulus q/d induced by χ and $\beta_d = \alpha_\psi$. The conditions on d and ν can also be written as $\nu \mid (r, s)$ and $d = (rs/\nu) \mid a$. Moreover

$$\psi(\nu) = \begin{cases} 0 & \text{if } (\nu, q/d) = (\nu, q\nu/rs) > 1 \\ \chi(\nu) & \text{otherwise.} \end{cases}$$

Hence (11) becomes

$$\tilde{\Gamma}(Y_r) = \phi(q/r) \sum_{s \mid a} Y_s \sum_{\substack{\nu \mid (r,s) \\ rs \mid a\nu \\ (\nu, q/d) = 1}} \chi(\nu) \beta_{rs/\nu}.$$

We note that

$$1 = (\nu, q/d) = (\nu, q\nu/rs) \Leftrightarrow \nu(rs, q) = rs \Leftrightarrow \nu = rs/(rs, q). \tag{12}$$

Further, if ν satisfies (12) then $\nu \mid (r, s)$ since $[r, s] \mid q$; so the inner sum contains at most one term. Hence we have proved

Lemma 6.

$$\tilde{\Gamma}(Y_r) = \phi(q/r) \sum_{\substack{s \mid a \\ (rs,q) \mid a}} Y_s\, \chi(rs/(rs,q))\, \beta_{(rs,q)}.$$

We consider now the special case $F(x) = ((x/q))$. Observe that in this case

$$\Gamma_F(f)(y) = \sum_{x \in (q)} ((xy/q)) f(x),$$

hence Γ_F can be represented by the matrix M defined in (3).

Proposition 1. *The rank of Γ_F is maximal on $V_\chi + V_{\bar{\chi}}$ when $L(\bar{\chi}) \neq 0$ and is zero when $L(\bar{\chi}) = 0$.*

Let ψ be induced by $\chi = \chi^{(a)}$ and $m(\psi) = q/d$. We have

$$\begin{aligned}
\alpha_\psi &= \frac{1}{\phi(q/d)} \sum_{y=1}^{q/d} ((dy/q)) \bar{\psi}(y) \\
&= \frac{1}{\phi(q/d)} \sum_{y=1}^{q/d} ((dy/q)) \bar{\chi}(y) \sum_{\substack{c \mid y \\ c \mid (q/d)}} \mu(c) \\
&= \frac{1}{\phi(q/d)} \sum_{c \mid (q/d)} \mu(c) \sum_{z=1}^{q/dc} ((dcz/q)) \bar{\chi}(c) \bar{\chi}(z) \\
&= \frac{1}{\phi(q/d)} \sum_{c \mid (q/d)} \mu(c) \bar{\chi}(c) \sum_{z=1}^{q/dc} ((dcz/q)) \bar{\chi}(z).
\end{aligned}$$

But $d \mid a$ and $\bar{\chi}(c) = 0$ if $(c, q/a) > 1$. Hence, since $c \mid (q/d) = (q/a) \cdot (a/d)$, one may assume $c \mid (a/d)$ and

$$\alpha_\psi = \frac{1}{\phi(q/d)} \sum_{c \mid (a/d)} \mu(c) \bar{\chi}(c) \sum_{z=0}^{(q/a)-1} \bar{\chi}(z) \sum_{b=0}^{(a/dc)-1} ((dc(z + bq/a)/q)).$$

The inner sum is equal to $((a/qz))$, whence

$$\alpha_\psi = \frac{L(\bar{\chi})}{\phi(q/d)} \sum_{c \mid (a/d)} \mu(c) \bar{\chi}(c),$$

where

$$L(\bar{\chi}) = \sum_{z \in (q/a)} ((a/qz)) \bar{\chi}(z).$$

It follows that if $L(\bar\chi) = 0$ then the linear function $\tilde\Gamma$ is identically zero. Assume $L(\bar\chi) \neq 0$. The action of $\tilde\Gamma$ is given by

$$\tilde\Gamma(Y_r) = \phi(q/r)L(\bar\chi) \sum_{\substack{s \mid a \\ (rs,q) \mid a}} Y_s \frac{\chi(rs/(rs,q))\lambda(a/(rs,q))}{\phi(q/(rs,q))},$$

where $\lambda(a/d) = \sum_{c \mid (a/d)} \mu(c)\bar\chi(c)$ is multiplicative.

Let $\delta_{r,s}$ be the coefficient of Y_s in the last expression. We have to prove that $\det(\delta_{r,s}) \neq 0$. Let p be a prime divisor of q. For an integer m we set $m = p^\rho m^*$ where $\rho \geq 0$ and $(p, m^*) = 1$. Let $r = p^\rho r^*$, $s = p^\sigma s^*$, $q = p^\kappa q^*$. Then

$$(rs,q)^* = (r^*s^*, q^*)$$

Let $\epsilon_{\rho,\sigma} = \epsilon_{\rho,\sigma}(p) = \delta_{p^\rho,p^\sigma}$. By multiplicativity of the functions involved one gets

$$\delta_{r,s} = \epsilon_{\rho,\sigma} \cdot \delta_{r^*,s^*}.$$

If $a = p^\alpha a^*$, whence $0 \leq \rho \leq \alpha$, $0 \leq \sigma \leq \alpha$, the matrix $\delta_{r,s}$ is of type

$$\begin{pmatrix} \epsilon_{0,0}A^* & \cdots & \epsilon_{0,\alpha}A^* \\ \cdots & \cdots & \cdots \\ \epsilon_{\alpha,0}A^* & \cdots & \epsilon_{\rho,\sigma}A^* \end{pmatrix}$$

and

$$\det(\delta_{r,s}) = \det(\epsilon_{\rho,\sigma})^{\dim A^*}(\det \ A^*)^{\alpha+1}$$

(see for intance [1, Prop. 2.14, p. 202] for the last formula). By induction on the number of prime factors of a we obtain

$$\det(\delta_{r,s}) = \prod_{p \mid a} \det(\epsilon_{\rho,\sigma}(p))^{e_p},$$

for suitable integer exponents e_p. Hence it suffices to prove that

$$\det(\epsilon_{\rho,\sigma}) = \det(\epsilon_{\rho,\sigma}(p)) \neq 0$$

for any p. Consider $\epsilon_{\rho,\sigma}$. We have $\epsilon_{\rho,\sigma} = 0$ if $\min(\rho+\sigma,\kappa) > \alpha$. Otherwise

$$\epsilon_{\rho,\sigma} = \frac{\chi(p^{\rho+\sigma-\min(\rho+\sigma,\kappa)})\lambda(p^{\alpha-\min(\rho+\sigma,\kappa)})}{\phi(p^{\kappa-\min(\rho+\sigma,\kappa)})}.$$

Observe that $\lambda(1) = 1$, $\lambda(p) = \lambda(p^2) = \cdots = 1 - \bar\chi(p)$. Moreover $\chi(p) = 0 \Leftrightarrow p \mid (q/a) \Leftrightarrow \alpha < \kappa$. Assume first $\alpha < \kappa$. Then $\lambda(p^\mu) = 1$, $\forall \mu$. We have

$$\epsilon_{\rho,\sigma} = \begin{cases} 0 & \text{if } \rho+\sigma > \alpha \\ 1/\phi(p^{\kappa-\rho-\sigma}) & \text{otherwise.} \end{cases}$$

and the matrix $(\epsilon_{\rho,\sigma})$ becomes

$$\begin{pmatrix} \frac{1}{\phi(p^\kappa)} & \dots & \dots & \frac{1}{\phi(p^{\kappa-\alpha})} \\ \frac{1}{\phi(p^{\kappa-1})} & \dots & \frac{1}{\phi(p^{\kappa-\alpha})} & 0 \\ \dots & \dots & \dots & \dots \\ \frac{1}{\phi(p^{\kappa-\alpha})} & 0 & \dots & 0 \end{pmatrix}$$

and has non-zero determinant. Suppose now $\alpha = \kappa$. In this case we have $\chi(p) \neq 0$, $\lambda(1) = 1$, $\lambda(p^\mu) = 1 - \bar{\chi}(p)$ for $\mu \geq 1$. We have, for $0 \leq \rho, \sigma \leq \kappa$,

$$\begin{aligned} \epsilon_{\rho,\sigma} &= \chi(p^{\rho+\sigma-\min(\rho+\sigma,\kappa)})\frac{\lambda}{\phi}(p^{\kappa-\min(\rho+\sigma,\kappa)}) \\ &= \chi(p^{\rho+\sigma})\bar{\chi}(p^\kappa)\frac{\chi\lambda}{\phi}(p^{\kappa-\min(\rho+\sigma,\kappa)}). \end{aligned}$$

Regarding $\det(\epsilon_{\rho,\sigma})$, note that $\bar{\chi}(p^\kappa))$ is a non-vanishing factor of all terms, hence it may be neglected. Also we may neglect $\chi(p^{\rho+\sigma}) = \chi(p^\rho)\chi(p^\sigma)$ by first dividing the ρ-th row by $\chi(p^\rho)$ and after dividing the σ-th column by $\chi(p^\sigma)$. Setting $\eta = \chi\lambda/\phi$ we must then compute $\det(\eta(p^{\kappa-\min(\rho+\sigma,\kappa)}))$, that is

$$R_p(\kappa) = \det \begin{pmatrix} \eta(p^\kappa) & \eta(p^{\kappa-1}) & \dots & \eta(p) & \eta(1) \\ \eta(p^{\kappa-1}) & \dots & \eta(p) & \eta(1) & \eta(1) \\ \dots & \dots & \eta(1) & \eta(1) & \eta(1) \\ & & & & \\ \eta(1) & \dots & \dots & \dots & \eta(1) \end{pmatrix}.$$

Subtracting the second last column from the last one it becomes clear that

$$R_p(\kappa) = \pm(\eta(1) - \eta(p))R_p(\kappa - 1)$$

whence

$$R_p(\kappa) = \pm(\eta(1) - \eta(p))^\kappa.$$

On the other hand

$$\eta(1) - \eta(p) = 1 - \frac{\chi(p)(1 - \bar{\chi}(p))}{\phi(p)} = 1 - \frac{\chi(p) - 1}{p - 1} = \frac{p - \chi(p)}{p - 1} \neq 0.$$

Corollary 2. *The rank of* Γ_F *is* $[(q-1)/2] + d(q)$, *where* $d(n)$ *is the number of divisors of* n.

Proof. $L(\chi) = B_{1,\chi}$ is in fact the first Bernoulli number associated with χ. It is well known that $B_{1,\chi} = 0$ for all non-principal even characters, whereas $B_{1,\chi} \neq 0$ if χ is odd or χ is the principal character (see for instance [5, p. 31] for details). By Proposition 1 the rank of Γ_F is the sum of dimension of V_χ for χ either odd or principal. The principal character $\chi = 1$ induces exactly one character for each modulus dividing q, hence $\dim V_1 = d(q)$. Moreover, a character ψ is induced by an odd primitive character χ (or is primitive odd itself) if and only if ψ is odd. So the dimensions of V_χ for χ

odd equals the number of odd characters modulo a divisor of q. It is well known that, for $d \neq 1, 2$, there are exactly $\phi(d)/2$ odd characters, while for $d = 1, 2$ there is only one character (the principal one). Hence we have

$$\sum_{\chi \text{ odd primitive}} \dim V_\chi = \sum_{\substack{d \mid q \\ d>2}} \phi(d)/2 = \begin{cases} (q-1)/2 & \text{if } q \text{ is odd} \\ (q-2)/2 & \text{if } q \text{ is even,} \end{cases}$$

and the corollary follows. □

We finally go back to the proof of Theorem 1.

Proof of Theorem 1. Let $\mu : V \to V$ be the linear function defined by

$$\mu(f)(x) = f(x) - f(x+1)$$

The $q \times q$ matrix associated to μ is

$$B = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & \dots & 0 & 0 \\ & & & & & \\ & . & . & . & . & \\ & & & & & \\ 0 & 0 & 0 & \dots & 1 & -1 \\ -1 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Then the dimension of the vector space spanned by the functions $g_{a,q}$ is equal to the rank of the matrix BM, where M has rank $[(q-1)/2] + d(q)$ by Corollary 2. Sincle clearly B has rank $q-1$, the rank of BM is equal either to $\text{rank}(M) - 1$ or to $\text{rank}(M)$ depending on whether the image of Γ_F intersects or not the kernel of the linear function μ. Taking for f the function which is 1 at the zero class and zero elsewhere, we get that

$$\Gamma_F(f)(y) = ((0/q)) = -1/2. \qquad \forall y$$

whence Γ_F is a constant and is contained in the kernel of μ.

As to the second part, note that the equality

$$\sum_{a \in (q)} f_{a,q} = \sum_{a \in (q)} (\frac{1}{q}\text{identity} + g_{a_q}) = \text{identity}$$

implies that the identity belongs to the vector space generated by the $f_{a,q}$, which therefore can be spanned by the identity together with the $g_{a,q}$. Now the identity is clearly linearly independent from all the functions $g_{a,q}$, since all these functions are periodic, and the result follows. □

3. Solution to Problem B

We begin by proving the following

Lemma 7. *Let $f : \mathbf{Z} \to \mathbf{C}$ be odd and periodic modulo q. Then the series*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \cos 2\pi nx$$

converges everywhere and we have the identity

$$h(x) = \sum_{n=1}^{\infty} \frac{f(n)}{n} \cos 2\pi nx = -i\pi \sum_{m=1}^{q-1} g(m) P(x + m/q) \tag{13}$$

where

$$g(m) = \sum_{n=0}^{q-1} f(n) e^{-2\pi i nm/q}$$

and $P(x)$ is given by (6).

Proof. Let us consider the equivalent relations

$$f(n) = \sum_{m=0}^{q-1} g(m) e^{2\pi i \frac{mn}{q}} \iff g(m) = \frac{1}{q} \sum_{n=0}^{q-1} f(n) e^{-2\pi i nm/q}. \tag{14}$$

It is easy to see that f is odd if and only if such is g. First suppose that f is odd: then we have

$$\begin{aligned} g(-m) &= \frac{1}{q} \sum_{n=0}^{q-1} f(n) e^{2\pi i nm/q} = \frac{1}{q} \sum_{n=0}^{q-1} f(q-1-n) e^{2\pi i (q-1-n)m/q} \\ &= -\frac{1}{q} \sum_{n=0}^{q-1} f(n+1) e^{-2\pi i (n+1)m/q} = -g(m). \end{aligned}$$

The implication in the other way is proved similarly.

Since f is odd and periodic modulo q so is g and this implies that

$$g(q-n) = -g(n) \quad \forall\ n \in \mathbf{Z} \tag{15}$$

$$g(0) = 0 \tag{16}$$

and

$$g(q/2) = 0$$

if q is even. Now consider the sum

$$\sum_{n=1}^{N} \frac{f(n)}{n} \cos 2\pi nx\ .$$

If we substitute (14) for $f(n)$ and exchange the sums we get

$$(17)\quad \sum_{n=1}^{N}\frac{f(n)}{n}\cos 2\pi nx = \\ = \frac{1}{2}\sum_{m=0}^{q-1} g(m)\Big(\sum_{n=1}^{N}\frac{e^{2\pi in(m/q+x)}}{n}\Big) + \frac{1}{2}\sum_{\ell=0}^{q-1} g(\ell)\Big(\sum_{n=1}^{N}\frac{e^{2\pi in(\ell/q-x)}}{n}\Big).$$

Taking into account (15) and (16) we can write (17) in the form

$$(18)\quad \begin{aligned} &\sum_{n=1}^{N}\frac{f(n)}{n}\cos 2\pi nx = \\ = \ &\frac{1}{2}\sum_{m=1}^{q-1} g(m)\sum_{n=1}^{N}\frac{\exp(2\pi in(m/q+x))}{n} \\ &\quad +\frac{1}{2}\sum_{m=1}^{q-1} g(q-m)\sum_{n=1}^{N}\frac{\exp(2\pi in((q-m)/q-x))}{n} \\ = \ &\frac{1}{2}\sum_{m=1}^{q-1} g(m)\sum_{n=1}^{N}\frac{\exp(2\pi in(m/q+x))-\exp(-2\pi in(m/q+x))}{n} \\ = \ &\frac{1}{2}\sum_{m=1}^{q-1} g(m)\Big(2i\sum_{n=1}^{N}\frac{\sin 2\pi n(m/q+x)}{n}\Big). \end{aligned}$$

Since $P(y) = -\pi^{-1}\sum_{n=1}^{\infty}\frac{\sin 2\pi ny}{n}$, $\forall\ y\in\mathbf{R}$, we see that (13) follows from (18) when $N\to\infty$. $\square$

We now prove that the coefficients $f(n)/n$ of the function $h(x)$ can be recovered from the corresponding Riemann sums $\delta_h(d)$ given by (5) of the introduction.

Theorem 2. *Let* $h(x) = \sum_{n=1}^{\infty}\frac{f(n)}{n}\cos 2\pi nx$ *as in Lemma 7. Then we have*

$$(19)\qquad \sum_{k=1}^{\infty}\mu(k)\delta_h(kd) = f(d)/d$$

for every integer $d\geq 1$.

Proof. By (13) of Lemma 7 we can write

$$d\delta_h(d) = \sum_{a=1}^{d} h(a/d) = -i\pi \sum_{m=1}^{q-1} g(m) \sum_{a=1}^{d} P(a/d + m/q). \tag{20}$$

Since $P(x) = -\pi^{-1} \sum_{n=1}^{\infty} \frac{\sin 2\pi n x}{n}$ for every $x \in \mathbf{R}$ we have

$$-\pi \sum_{a=1}^{d} P(a/d + m/q) = \sum_{n=1}^{\infty} \frac{1}{n} \Big(\sum_{a=1}^{d} \sin 2\pi n(a/d + m/q) \Big). \tag{21}$$

It is easy to see that

$$\sum_{a=1}^{d} \sin 2\pi n(a/d + m/q) = \begin{cases} d \sin 2\pi n m/q & \text{if } n \equiv 0 \pmod{d} \\ 0 & \text{otherwise.} \end{cases} \tag{22}$$

From (21) and (22) follows the identity

$$-\pi \sum_{a=1}^{d} P(a/d + m/q) = \sum_{\substack{n=1 \\ n \equiv 0 \pmod{d}}}^{\infty} \frac{d}{n} \sin 2\pi n m/q = -\pi P(md/q),$$

so that (20) becomes

$$d\delta_h(d) = -i\pi \sum_{m=1}^{q-1} g(m) P(md/q). \tag{23}$$

From (23) we obtain

$$\sum_{k=1}^{N} \mu(k)\delta_h(kd) = -i\pi \sum_{m=1}^{q-1} g(m) \Big(\sum_{k=1}^{N} \frac{\mu(k)}{k} P(mkd/q) \Big) \frac{1}{d}. \tag{24}$$

Now recall that $\sum_{k=1}^{\infty} \frac{\mu(k)}{k} P(mkd/q) = -\frac{1}{\pi} \sin 2\pi md/q$: if we let $N \to \infty$ in (24) by the first one of the two reciprocal relations (14) we get

$$\begin{aligned} \sum_{k=1}^{\infty} \mu(k)\delta_h(kd) &= -i\pi \sum_{m=1}^{q-1} g(m) \left(-\frac{1}{\pi} \sin 2\pi md/q \right) \frac{1}{d} \\ &= \frac{1}{d} \sum_{m=1}^{q-1} g(m) e^{2\pi i m d/q} = f(d)/d, \end{aligned}$$

since $\sum_{m=1}^{q-1} g(m) \cos 2\pi md/q = 0$ because g is odd. $\square$

Theorem 2 implies the following

Corollary 3. *For every integer $q \geq 3$ we have*

$$\det\left|P(md/q)\right| \neq 0 \quad , \quad 1 \leq d, m \leq [(q-1)/2]. \tag{25}$$

Proof. We shall show that if we suppose that (25) is false for an integer $q \geq 3$ we reach a contradiction.

If (25) is false then there exist values $g(m)$ not all zero such that

$$\sum_{m=1}^{[(q-1)/2]} g(m)P(md/q) = 0 \qquad \text{if} \quad 1 \leq d \leq [(q-1)/2]. \tag{26}$$

Let us now consider the odd continuation with period q of the $g(m)$ which are solutions of the system (26), i.e. we require that $g : \mathbf{Z} \to \mathbf{C}$ is periodic modulo q and $g(q-m) = -g(m) \ \ \forall\ m \in \mathbf{Z}$.

Let $f(n)$, $n \in \mathbf{Z}$, be defined by the first relation in (14), that is $f(n) = \sum_{m=0}^{q-1} g(m)e^{2\pi imn/q}$.

As already noted $f(n)$ is also odd and periodic modulo q: moreover since the $g(m)$ are not all zero, the same is true for the $f(n)$. Let us now put $h(x) = \sum_{n=1}^{\infty} \frac{f(n)}{n} \cos 2\pi nx$ as in lemma 1. Equality (23) can be written in the form

$$d\delta_h(d) = -2\pi i \sum_{m=1}^{[(q-1)/2]} g(m)P(md/q) \tag{27}$$

because we have

$$g(q-m)P((q-m)d/q) = g(m)P(md/q)$$

since both g and P are odd.

From (26) and (27) it follows $\delta_h(d) = 0 \ \ \forall\ d \geq 1$ which implies, by (19) of Theorem 2, $f(d) = 0 \ \ \forall\ d \geq 1$. But this is a contradiction, since f is not identically zero. This proves the corollary. □

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