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Primitive Substitutive Numbers are Closed Under Rational Multiplication

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Résumé. Soit $M(r)$ l'ensemble des réels α dont le développement en base r contient une queue qui est l'image d'un point fixe d'une substitution primitive par un morphisme de lettres. Nous démontrons que l'ensemble $M(r)$ est stable par multiplication par les rationnels, mais non stable par addition.

Abstract. Let $M(r)$ denote the set of real numbers α whose base-r digit expansion is ultimately primitive substitutive, i.e., contains a tail which is the image (under a letter to letter morphism) of a fixed point of a primitive substitution. We show that the set $M(r)$ is closed under multiplication by rational numbers, but not closed under addition.

1. Introduction

A sequence on a finite alphabet A is called q-automatic if it is the image (under a letter to letter morphism) of a fixed point of a substitution of constant length q. Let $M(q, r)$ denote the set of real numbers whose fractional part has a q-automatic base-r digit expansion. J.H. Loxton and A. van der Poorten stated that each number $\alpha \in M(q, r)$ is either rational or transcendental [LoPo]. Unfortunately a gap has been reported in their proof.

Analogously, a sequence on a finite alphabet A is called *primitive substitutive* if it is the image (under a letter to letter morphism) of a fixed point of a primitive substitution. Let $M(r)$ denote the set of real numbers whose base-r digit expansion is *ultimately primitive substitutive,* i.e., contains a tail which is primitive substitutive. It is believed that a number α in $M(r)$ is either rational or transcendental. This has been verified in a few special cases (see [AlZa], [FeMa], and [RiZa]).

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In [Le] S. Lehr showed that $M(q,r)$ is a $\mathbb{Q}$ -vector space. We will show that $M(r)$ is closed under multiplication by $\mathbb{Q}$ but not closed under addition. Lehr's proof relies in part on a theorem of J.-P. Allouche and M. Mendès France[1]:

Theorem 1 (J.-P. Allouche, M. Mendès France, [AlMe]). *Let $\star$ be an associative binary operation on a finite set $\mathcal{A}$ and let $\omega = \omega_1\omega_2\omega_3\ldots$ be a q-automatic sequence in $\mathcal{A}^{\mathbb{N}}$. Then the induced sequence of partial products*

$$\omega_1, \omega_1 \star \omega_2, \omega_1 \star \omega_2 \star \omega_3, \omega_1 \star \omega_2 \star \omega_3 \star \omega_4, \ldots$$

is q-automatic.

We will use the following analogue of Theorem 1

Theorem 2 (C. Holton, L.Q. Zamboni, [HoZa]). *Let $\star$ be a binary operation on a finite set $\mathcal{A}$ and let $\omega = \omega_1\omega_2\omega_3\ldots$ be an ultimately primitive substitutive sequence in $\mathcal{A}^{\mathbb{N}}$. Then the induced sequence of partial products*

$$\omega_1, \omega_1 \star \omega_2, (\omega_1 \star \omega_2) \star \omega_3, ((\omega_1 \star \omega_2) \star \omega_3) \star \omega_4, \ldots$$

is ultimately primitive substitutive.

2. Main Theorem

Theorem 1. *The set $M(r)$ is closed under multiplication by $\mathbb{Q}$ but not closed under addition.*

Proof. We begin by observing that $\mathbb{Q} \subset M(r)$. In fact, the digit expansion of a rational number is ultimately periodic, and a periodic sequence is primitive substitutive. Let $\xi \in M(r)$. We show that for positive integers n and p, both $n\xi$ and $\frac{\xi}{p}$ are in $M(r)$. In each case we can assume that $0 < \xi < 1$ and that $\xi \notin \mathbb{Q}$. Hence we can write $\xi = \sum_{k=1}^{\infty} \xi_k r^{-k}$ with $\xi_k \in A_r = \{0, 1, \ldots, r-1\}$. The sequence $\{\xi_k\}$ is then ultimately primitive substitutive but not ultimately periodic. We begin by showing that $y = \frac{\xi}{p} \in M(r)$. We write $y = \sum_{k=1}^{\infty} y_k r^{-k}$ with $y_k \in A_r$. Then following [Le] we have

[1]This is a special case of a more general result of F.M. Dekking in [De]

$$\begin{aligned}
y_k &= \left\lfloor \frac{\xi}{p} r^k \right\rfloor \bmod r \\
&= \left\lfloor \frac{r^k}{p} \sum_{i=1}^{\infty} \xi_i r^{-i} \right\rfloor \bmod r \\
&= \left\lfloor \frac{1}{p} \sum_{i=1}^{\infty} \xi_i r^{k-i} \right\rfloor \bmod r \\
&= \left\lfloor \frac{1}{p} \sum_{i=1}^{k} \xi_i r^{k-i} + \frac{1}{p} \sum_{i=k+1}^{\infty} \xi_i r^{k-i} \right\rfloor \bmod r
\end{aligned}$$

Since

$$\frac{1}{p} \sum_{i=1}^{k} \xi_i r^{k-i} = \frac{m}{p}$$

for some natural number m, and

$$\frac{1}{p} \sum_{i=k+1}^{\infty} \xi_i r^{k-i} < \frac{1}{p}$$

we obtain

$$y_k = \left\lfloor \frac{1}{p} \sum_{i=1}^{k} \xi_i r^{k-i} \right\rfloor \bmod r = \left\lfloor \frac{\left(\sum_{i=1}^{k} \xi_i r^{k-i} \right) \bmod pr}{p} \right\rfloor \bmod r.$$

$\square$

Consider the sequence $\{(\xi_k, r)\}_{k=1}^{\infty}$ in the alphabet $A_{pr} \times A_{pr}$. Since the sequence $\{\xi_k\}$ is ultimately primitive substitutive, the same is true of the sequence $\{(\xi_k, r)\}$. Let $\star$ denote the associative binary operation on $A_{pr} \times A_{pr}$ given by[2]

$$(a, \alpha) \star (b, \beta) = (a\beta + b \bmod pr, \alpha\beta \bmod pr).$$

For each $k \geq 1$ we set

$$x_k = (\xi_1, r) \star (\xi_2, r) \star \ldots \star (\xi_k, r) = \left(\sum_{i=1}^{k} \xi_i r^{k-i} \bmod pr, r^k \bmod pr \right).$$

[2]There is a typographical error in the definition of the binary operation $*$ given in [Le]. It should be the same as $\star$.

By Theorem 2 the sequence $\{x_k\}$ is ultimately primitive substitutive, and hence so is the sequence $\{y_k\}$ as required.

We next show that $n\xi \in M(r)$.

Lemma 2. *Let $\{\xi_k\}$ be as above. There is a positive integer $M = M(n)$ such that for each $k \geq 0$*

$$\left\lfloor n\sum_{i=1}^{\infty}\frac{\xi_{k+i}}{r^i}\right\rfloor = \left\lfloor n\left(\frac{\xi_{k+1}}{r}+\frac{\xi_{k+2}}{r^2}+\ldots+\frac{\xi_{k+M}}{r^M}\right)\right\rfloor.$$

Proof. Fix a positive integer l so that $r^l \geq n$, and set $(a)_f = a - \lfloor a \rfloor$ for each $a \in \mathbb{R}$. Then

$$\left\lfloor n\sum_{i=1}^{\infty}\frac{\xi_{k+i}}{r^i}\right\rfloor = \left\lfloor n\left(\frac{\xi_{k+1}}{r}+\frac{\xi_{k+2}}{r^2}+\ldots+\frac{\xi_{k+l}}{r^l}\right)\right\rfloor + \left\lfloor S_k + n\sum_{i=l+1}^{\infty}\frac{\xi_{k+i}}{r^i}\right\rfloor$$

where $S_k = \left(n\left(\frac{\xi_{k+1}}{r}+\frac{\xi_{k+2}}{r^2}+\ldots+\frac{\xi_{k+l}}{r^l}\right)\right)_f$. □

Note that for $k \geq 0$ the quantity $\left\lfloor S_k + n\sum_{i=l+1}^{\infty}\frac{\xi_{k+i}}{r^i}\right\rfloor$ is either 0 or 1. Let $\mathcal{S} = \{S_k \mid k \geq 1\}$. Then $\mathrm{Card}(\mathcal{S}) \leq r^l$. For each $s \in \mathcal{S}$ there exist words V_s and U_s (in the alphabet A_r) such that the base-r digit expansion of $\frac{1}{n}(1-s) \in \mathbb{Q}$ is given by $V_sU_s^{\omega}$. Since $\xi \notin \mathbb{Q}$, for each $s \in \mathcal{S}$ there is a positive integer m_s so that the sequence $\{\xi_k\}$ does not contain the subword $U_s^{m_s}$. Set $M_s = |V_s| + m_s|U_s|$ and $M' = \max\{M_s \mid s \in \mathcal{S}\}$. Then for each $k \geq 0$ we have

$$\left\lfloor S_k + n\sum_{i=l+1}^{\infty}\frac{\xi_{k+i}}{r^i}\right\rfloor = 1$$

if and only if

$$\sum_{i=l+1}^{\infty}\frac{\xi_{k+i}}{r^i} > \frac{1}{n}(1-S_k)$$

if and only if

$$\frac{\xi_{k+l+1}}{r^{l+1}}+\frac{\xi_{k+l+2}}{r^{l+2}}+\ldots+\frac{\xi_{k+l+M'}}{r^{l+M'}} > \frac{1}{n}(1-S_k).$$

Thus $M = l + M'$ satisfies the conclusion of Lemma 2. □

We return to the proof of Theorem 1. Let $z = n\xi$. Then we can write $(z)_f = \sum_{k=1}^{\infty} z_k r^{-k}$. It suffices to show that the sequence $\{z_k\}$ is ultimately primitive substitutive. Let M be as in Lemma 2. Then for $k \geq 0$ we have

$$\begin{aligned} z_k &= \lfloor n\xi r^k \rfloor \bmod r \\ &= \left\lfloor nr^k \sum_{i=1}^{\infty} \xi_i r^{-i} \right\rfloor \bmod r \\ &= \left\lfloor n \sum_{i=1}^{k-1} \xi_i r^{k-i} + n\xi_k + n \sum_{i=k+1}^{\infty} \xi_i r^{k-i} \right\rfloor \bmod r \\ &= \left\lfloor n\xi_k + n \sum_{i=k+1}^{\infty} \xi_i r^{k-i} \right\rfloor \bmod r \\ &= \lfloor n\xi_k \rfloor \bmod r + \left\lfloor n \sum_{i=k+1}^{\infty} \xi_i r^{k-i} \right\rfloor \bmod r \\ &= \lfloor n\xi_k \rfloor \bmod r + \left\lfloor n \left(\frac{\xi_{k+1}}{r} + \frac{\xi_{k+2}}{r^2} + \ldots + \frac{\xi_{k+M}}{r^M} \right) \right\rfloor \bmod r. \end{aligned}$$

Now since $\{\xi_k\}$ is ultimately primitive substitutive, the same is true of the sequence $\{(\xi_k, \xi_{k+1}, \ldots, \xi_{k+M})\}_{k=1}^{\infty}$. In fact if a tail of $\{\xi_k\}$ is the image of a fixed point of a primitive substitution ζ, then the corresponding tail of $\{(\xi_k, \xi_{k+1}, \ldots, \xi_{k+M})\}$ is the image of a fixed point of the primitive morphism ζ_{M+1} defined in [Qu] (see Lemma V.11 and Lemma V.12 in [Qu]). Define $\phi : A_r^{M+1} \to A_r$ by

$$\phi(a_1, a_2, \ldots, a_{M+1}) = \lfloor na_1 \rfloor \bmod r + \left\lfloor n \left(\frac{a_2}{r} + \frac{a_3}{r^2} + \ldots + \frac{a_{M+1}}{r^M} \right) \right\rfloor \bmod r.$$

Then the sequence $\{z_k\} = \{\phi(\xi_k, \xi_{k+1}, \ldots, \xi_{k+M})\}_{k+1}^{\infty}$ is ultimately primitive substitutive as required.

It remains to show that $M(r)$ is not closed under addition. Let τ be the primitive morphism defined by

$$\begin{aligned} 1 &\mapsto 1211 \\ 2 &\mapsto 2112. \end{aligned}$$

Let $a = \{a_i\}$ denote the fixed point of τ beginning in 1 and $b = \{b_i\}$ the fixed point of τ beginning in 2. Let $\alpha = \sum_{i=1}^{\infty} a_i (10)^{-i}$ and $\beta = \sum_{i=1}^{\infty} b_i (10)^{-i}$. Then α and β are each in $M(10)$ but $\alpha + \beta \notin M(10)$. In fact, the digit 3 occurs an infinite number of times in the decimal expansion of $\alpha + \beta$ but not in bounded gap. We note that for each $n \geq 1$ the sequence a begins in $\tau^n(12)\tau^n(1)$ and b begins in $\tau^n(21)\tau^n(1)$. Since $|\tau^n(12)| = |\tau^n(21)|$ it follows that for each $N \geq 1$ we can find $k = k(N)$ so that $a_k a_{k+1} \ldots a_{k+N} = b_k b_{k+1} \ldots b_{k+N}$. If $c = \{c_i\}$ denotes the decimal expansion of $\alpha + \beta$ then the block $c_k c_{k+1} \ldots c_{k+N}$ consists only of the digits 2 and 4. At the same time, for each $n \geq 1$ the sequence a begins in $\tau^n(121)\tau^n(1)$ while b begins in

$\tau^n(211)\tau^n(2)$. Since $|\tau^n(121)| = |\tau^n(211)|$ it follows that $a_j \neq b_j$ (and hence $c_j = 3$) for infinitely many values of j. Thus no tail of the decimal expansion of $\alpha+\beta$ is a minimal sequence. In particular $\{c_i\}$ is not ultimately primitive substitutive. $\square$

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