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An Analogue of Pfister's Local-Global Principle in the Burnside Ring

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Résumé. Soit N/K une extension galoisienne de groupe de Galois $\mathcal{G}$. On étudie l'ensemble $\mathcal{T}(\mathcal{G})$ des combinaisons linéaires sur $\mathbb{Z}$ de caractères de l'anneau de Burnside $\mathcal{B}(\mathcal{G})$, qui induisent des combinaisons $\mathbb{Z}$-linéaires des formes trace de sous-extensions de N/K qui sont triviales dans l'anneau de Witt $\mathrm{W}(K)$ de K. On montre que le sous-groupe de torsion de $\mathcal{B}(\mathcal{G})/\mathcal{T}(\mathcal{G})$ est le noyau de l'homomorphisme signature.

Abstract. Let N/K be a Galois extension with Galois group $\mathcal{G}$. We study the set $\mathcal{T}(\mathcal{G})$ of $\mathbb{Z}$-linear combinations of characters in the Burnside ring $\mathcal{B}(\mathcal{G})$ which give rise to $\mathbb{Z}$-linear combinations of trace forms of subextensions of N/K which are trivial in the Witt ring $\mathrm{W}(K)$ of K. In particular, we prove that the torsion subgroup of $\mathcal{B}(\mathcal{G})/\mathcal{T}(\mathcal{G})$ coincides with the kernel of the total signature homomorphism.

1. Introduction

Let L/K be a finite, separable extension of fields of characteristic $\neq 2$. With it we associate the 'trace form' which is defined by $\mathrm{tr}_{L/K} : L \to K : x \mapsto \mathrm{tr}_{L/K}x^2$. P.E. Conner started to investigate the connection of the trace form of L/K and the trace form of a normal closure N/K of L/K. His work yields some polynomial vanishing theorems for trace forms (see [1]). These identities come from identities in the Burnside ring of the Galois group $\mathcal{G} = G(N/K)$ of N/K. We study the trace ideal $\mathcal{T}(\mathcal{G})$ in $\mathcal{B}(\mathcal{G})$, which is roughly speaking the set of $\mathbb{Z}$-linear combinations of trace forms of subextensions of N/K which are trivial in the Witt ring $\mathrm{W}(K)$ of K.

We first recall the definition of the Burnside ring $\mathcal{B}(\mathcal{G})$ of a finite group $\mathcal{G}$. A theorem of Springer [6] gives rise to a homomorphism $h_{N/K} : \mathcal{B}(\mathcal{G}) \to \mathrm{W}(K)$. The trace ideal $\mathcal{T}(\mathcal{G})$ is a finitely generated subgroup of the free abelian group $\mathcal{B}(\mathcal{G})$. We introduce a signature homomorphism $\mathrm{sign}_\sigma : \mathcal{B}(\mathcal{G}) \to \mathbb{Z}$ for each element $\sigma \in \mathcal{G}$ of order ≤ 2. These signature homomorphisms correspond to signatures of the Witt ring. We conclude that $\mathcal{T}(\mathcal{G})$ is contained in the intersection $L(\mathcal{G})$ of all kernels of signatures. The main theorem states that $\mathcal{T}(\mathcal{G})$ and $L(\mathcal{G})$ are of equal rank. Hence the torsion subgroup of $\mathcal{B}(\mathcal{G})/\mathcal{T}(\mathcal{G})$ is given by the kernel of the total signature homomorphism. In section 7 we reduce our approach to 2-groups. The general case follows by induction via the Frattini subgroup of $\mathcal{G}$.

2. Notation

We first fix our notations. Let K be a field. Then $K^\star$ denotes the multiplicative group of K, $K^{\star 2}$ is the group of squares in $K^\star$. We write K_s for a separable closure of K.

Let N/K be a Galois extension, then $G(N/K)$ denotes the Galois group of N/K. If $\mathcal{H} < G(N/K)$ then $N^{\mathcal{H}}$ is the fixed field of $\mathcal{H}$ in N. Let $\mathrm{Aut}(K)$ be the group of field automorphisms of K.

Now let K be a field of characteristic $\neq 2$. Let ψ, φ be non-degenerate quadratic forms over K. Then $\det_K \psi$ is the determinant of ψ. If $\mathfrak{p}$ is a real place of K then $\mathrm{sign}_{\mathfrak{p}}\psi$ is the signature of ψ with respect to $\mathfrak{p}$. $\psi \otimes \varphi$ is the product of ψ and φ. For $m \in \mathbb{Z}$, $m \times \psi$ is the m-fold sum of ψ. $\psi \simeq \varphi$ indicates the isometry of ψ and φ over K. Let L/K be a field extension. Then ψ_L is the lifting of ψ to a form over L by scalar extension. $\mathrm{W}(K)$ is the Witt ring of K. Let $a_1, \ldots, a_n \in K^\star$. Then $<a_1, \ldots, a_n>$ is the diagonal form $a_1 X_1^2 + \ldots + a_n X_n^2$ over K. $<<a_1, \ldots, a_n>> = \otimes_{i=1,\ldots,n} <1, -a_i>$ is the n-fold Pfister form defined by $a_1, \ldots, a_n$.

Let L/K be a finite and separable field extension. The trace form of L/K is the non-degenerate quadratic form $\mathrm{tr}_{L/K}<1> : L \to K : x \mapsto \mathrm{tr}_{L/K}(x^2)$. We denote the trace form also by $<L/K>$, resp $<L>$ if no confusion can arise.

Let M be a set. Then $\sharp M$ is the cardinality of M. $\mathrm{ord}(\mathcal{G})$, $\mathrm{ord}(\sigma)$ is the order of the finite group $\mathcal{G}$, resp. of the element $\sigma \in \mathcal{G}$.

3. The Burnside ring $\mathcal{B}(\mathcal{G})$

Let $\mathcal{G}$ be a finite group and let $\mathcal{H} < \mathcal{G}$ be a subgroup of $\mathcal{G}$. We denote the transitive action of $\mathcal{G}$ on the set of left cosets $\mathcal{G}/\mathcal{H} = \{a\mathcal{H}, a \in \mathcal{G}\}$ by $(\mathcal{G}, \mathcal{G}/\mathcal{H})$. The transitive and faithful actions of $\mathcal{G}$ on finite sets are in one-to-one correspondence with the set of conjugacy classes of subgroups of $\mathcal{G}$. A subgroup $\mathcal{H}$ of $\mathcal{G}$ induces a transitive action of degree $[\mathcal{G} : \mathcal{H}]$, hence a representation of dimension $[\mathcal{G} : \mathcal{H}]$. Let $\chi_{\mathcal{H}}$ denote the corresponding character. We sometimes write $\chi_{\mathcal{H}}^{\mathcal{G}}$ to indicate that the character is defined on $\mathcal{G}$.

Definition 1. Let $\mathcal{G}$ be a finite group. The *Burnside ring* $\mathcal{B}(\mathcal{G})$ of $\mathcal{G}$ is the free abelian group freely generated by the set
$\{\chi_{\mathcal{H}} \mid \mathcal{H}$ runs over representatives of conjugacy classes of subgroups of $\mathcal{G}\}$

and with multiplication given by

$$\chi_{\mathcal{U}_1} \cdot \chi_{\mathcal{U}_2} = \bigoplus_{\sigma \in \mathcal{U}_1 \backslash \mathcal{G} / \mathcal{U}_2} \chi_{\mathcal{U}_1 \cap \sigma \mathcal{U}_2 \sigma^{-1}},$$

where the sum runs over a set of representatives of the double cosets in $\mathcal{U}_1 \backslash \mathcal{G} / \mathcal{U}_2$.

Remark 2. $\chi_{\mathcal{G}}$ is the multiplicative identity, $\chi_{\{e\}} =: \chi_1$ is the regular character.

Another way of defining the multiplication is as follows. Let $\rho_i : \mathcal{G} \to \mathrm{GL}(V_i), i = 1, 2$ be representations of $\mathcal{G}$. Then $\rho_1 \otimes \rho_2 : \mathcal{G} \times \mathcal{G} \to \mathrm{GL}(V_1 \otimes V_2)$ is a representation of $\mathcal{G} \times \mathcal{G}$ on $V_1 \otimes V_2$. According to the diagonal embedding $\mathcal{G} \to \mathcal{G} \times \mathcal{G}$ the representation $\rho_1 \otimes \rho_2$ restricts to a representation of $\mathcal{G}$ on $V_1 \otimes V_2$. For $\rho_i = (\mathcal{G}, \mathcal{G}/\mathcal{U}_i)$ we get $\rho_1 \otimes \rho_{2|\mathcal{G}} = \oplus_{\sigma \in \mathcal{U}_1 \backslash \mathcal{G} / \mathcal{U}_2} (\mathcal{G}, \mathcal{G}/(\mathcal{U}_1 \cap \sigma \mathcal{U}_2 \sigma^{-1}))$.

4. The homomorphism $h_{N/K} : \mathcal{B}(G(N/K)) \to \mathrm{W}(K)$

Proposition 3 (T.A. Springer)**.** *Let N/K be a finite Galois extension with Galois group $G(N/K) = \mathcal{G}$. Then there is a well-defined ring homomorphism*

$$h_{N/K} : \mathcal{B}(\mathcal{G}) \to \mathrm{W}(K)$$

with

$$h_{N/K}(\chi_{\mathcal{H}}) = <N^{\mathcal{H}}>$$

for all subgroups $\mathcal{H}$ of $\mathcal{G}$.

Proof. Let $\mathcal{H} < \mathcal{G}$ be a subgroup of $\mathcal{G}$. Then $h_{N/K}$ is well-defined as a group homomorphism since $<N^{\sigma \mathcal{H} \sigma^{-1}}> = <\sigma(N^{\mathcal{H}})> = <N^{\mathcal{H}}>$. Now the assertion follows from the next lemma. □

Lemma 4. *Let N/K be a finite Galois extension with Galois group $\mathcal{G} = G(N/K)$. Let $\mathcal{U}_1, \mathcal{U}_2$ be subgroups of $G(N/K)$. Then*

$$<N^{\mathcal{U}_1}> \otimes <N^{\mathcal{U}_2}> = \perp_{\sigma \in \mathcal{U}_1 \backslash \mathcal{G} / \mathcal{U}_2} <N^{\mathcal{U}_1 \cap \sigma \mathcal{U}_2 \sigma^{-1}}>,$$

where the sum runs over a set of representatives of the double cosets $\mathcal{U}_1 \backslash \mathcal{G} / \mathcal{U}_2$.

Proof. (see [2], I.6.2) Let $\alpha \in N$ with $N^{\mathcal{U}_1} = K(\alpha)$ and let $f \in K[X]$ be the minimal polynomial of α over K. Set $L := N^{\mathcal{U}_2}$. From Frobenius reciprocity [5], 2.5.6 we get

$$\begin{aligned} <N^{\mathcal{U}_1}> \otimes <N^{\mathcal{U}_2}> &= <K(\alpha)> \otimes <L> = \mathrm{tr}_{L/K}((\mathrm{tr}_{K(\alpha)/K}<1>)_L) \\ &= \mathrm{tr}_{L/K}<(L[X]/(f))/L> \\ &= \perp_{i=1,\dots,r} \mathrm{tr}_{L/K}<(L[X]/(f_i))/L>, \end{aligned}$$

where $f = f_1 \cdots f_r$ is the decomposition of f into monic irreducible polynomials in $L[X]$. Now consider $\mathrm{tr}_{L/K}<(L[X]/(g))/L>$ for some monic prime divisor $g \in L[X]$ of f. Then g is the minimal polynomial of some conjugate $\sigma(\alpha)$ of α over L. Hence

$$\mathrm{tr}_{L/K}{<}(L[X]/(g))/L{>} = \mathrm{tr}_{L/K}(\mathrm{tr}_{L(\sigma(\alpha))/L}{<}1{>}) = {<}L(\sigma(\alpha)){>} .$$

Now

$$L(\sigma(\alpha)) = L \cdot K(\sigma(\alpha)) = L \cdot \sigma(K(\alpha)) = N^{\mathcal{U}_2} \cdot \sigma(N^{\mathcal{U}_1}) = N^{\mathcal{U}_1 \cap \sigma \mathcal{U}_2 \sigma^{-1}}.$$

The action of $\mathcal{G}$ on the roots of f induces an action of $\mathcal{U}_2$ on the roots of f, which is equivalent to the action of $\mathcal{U}_2$ on $\mathcal{G}/\mathcal{U}_1$. Each orbit of this action corresponds to a monic irreducible factor $g \in L[X]$ of f. □

5. The trace ideal in $\mathcal{B}(\mathcal{G})$

Definition 5. Let $\mathcal{G}$ be a finite group. Set

$$\mathcal{T}(\mathcal{G}) := \cap \ker(h_{N/K}),$$

where the intersection is taken over all Galois extensions N/K over all fields K of characteristic $\neq 2$ with Galois group $G(N/K) \simeq \mathcal{G}$. We call $\mathcal{T}(\mathcal{G})$ the *trace ideal of* $\mathcal{B}(\mathcal{G})$.

6. The Main Results

Theorem 6. *Let $\mathcal{G}$ be a finite group. Then the trace ideal $\mathcal{T}(\mathcal{G})$ of $\mathcal{B}(\mathcal{G})$ is a free abelian group of rank*

$$\mathrm{rank}(\mathcal{T}(\mathcal{G})) = \mathrm{rank}(\mathcal{B}(\mathcal{G})) - \sharp\{\text{conjugacy classes of elements } \sigma \in \mathcal{G} \text{ of order } \leq 2\}.$$

The proof of theorem 6 will be organized as follows. We start by defining in a rather canonical way signatures for elements in the Burnside ring. By lemma 8, the trace ideal is contained in the kernel $L(\mathcal{G})$ of the total signature homomorphism. We compute the rank $L(\mathcal{G})$ in lemma 14. Now the assertion follows from the equality of the ranks of $\mathcal{T}(\mathcal{G})$ and $L(\mathcal{G})$, whose proof will be the subject of sections 7 and 8. In section 7 we reduce the proof of theorem 6 to 2-groups. Section 8 contains the proof of theorem 6 for 2-groups. It runs via induction over the Frattini subgroup of $\mathcal{G}$.

If $\mathcal{G}$ is a finite group then $\mathrm{RC}(\mathcal{G})$ denotes a set of representatives of the conjugacy classes of subgroups of $\mathcal{G}$. Further, $\mathrm{RC}_2(\mathcal{G})$ denotes a set of representatives of the conjugacy classes of elements of order 1 or 2 in $\mathcal{G}$. Let $\mathcal{G}_2$ be a 2-Sylow subgroup of $\mathcal{G}$. Then we can choose $\mathrm{RC}_2(\mathcal{G}) \subset \mathcal{G}_2$.

In the sequel we will use the following proposition of Sylvester.

Proposition 7. *Let K be field, $\mathfrak{p}$ be an ordering of K. Then for any separable polynomial $f(X) \in K[X]$ the signature of the trace form of $K[X]/(f(X))$ over K equals the number of real roots of $f(X)$ with respect to the ordering $\mathfrak{p}$.*

For a proof see [7].

Lemma 8. *Let $\mathcal{G}$ be a finite group and let $\sigma \in \mathcal{G}$ be an element of order ≤ 2. Then there is a Galois extension N/K of algebraic number fields and an isomorphism $\iota : \mathcal{G} \xrightarrow{\sim} G(N/K)$ such that*

1. *$K \subset \mathbb{R}$ and $N \subset \mathbb{C}$.*
2. *$\iota(\sigma)$ is induced by the complex conjugation.*

Proof. Set $n := \mathrm{ord}(\mathcal{G})$.

1. $\mathrm{ord}(\sigma) = 2$. If $n = 2$, set $K = \mathbb{Q}, N = \mathbb{Q}(\sqrt{-1})$.
Now let $n = 2m \geq 4$. Consider the quadratic form $\psi = (m-1)\times <1,-1>\perp<1,-2>$ as a form over $\mathbb{Q}$. Then $\det_{\mathbb{Q}}\psi \notin \mathbb{Q}^{\star 2}$ and $\mathrm{sign}_{\mathbb{Q}}\psi = 0$. By theorems 1 and 3 of [4] there is a field extension $L/\mathbb{Q}$ with normal closure $N/\mathbb{Q}$ such that $N \subset \mathbb{C}, G(N/\mathbb{Q}) \simeq \mathfrak{S}_n$ and $L/\mathbb{Q}$ has trace form ψ. Here $\mathfrak{S}_n$ denotes the symmetric group on n elements.

Let $\alpha \in L$ be a primitive element of $L/\mathbb{Q}$. Since $\mathrm{sign}_{\mathbb{Q}} <L>= 0$ no conjugate of α is real (see proposition 7). Let $M := \{\alpha_1, \bar{\alpha}_1, \ldots, \alpha_m, \bar{\alpha}_m\}$ be the set of conjugates of α. $\bar{\alpha}$ is the complex conjugate of $\alpha \in \mathbb{C}$. Let $\varphi : \mathcal{G} \to M$ be a bijection such that for each $a \in \mathcal{G}$ the set $\varphi(\{a, \sigma(a)\})$ consists of a pair of complex conjugate elements of M. Now according to the identification given by φ we get a monomorphism $\iota : \mathcal{G} \hookrightarrow S(\mathcal{G}) \xrightarrow{\sim} S(M) \xrightarrow{\sim} G(N/\mathbb{Q})$. Then $\iota(\sigma)$ is given by the complex conjugation on N. Set $K := N^{\iota(\mathcal{G})}$. Since $\iota(\sigma) \in \iota(\mathcal{G})$ the field K is real.

2. $\sigma = id$. Set $\psi = (n-1)\times <1>\perp<2>$.
Then $\det_{\mathbb{Q}}\psi \notin \mathbb{Q}^{\star 2}$ and $\mathrm{sign}_{\mathbb{Q}}\psi = n$. Now choose L, N and $\alpha \in L$ as above. Since $\mathrm{sign}_{\mathbb{Q}}\psi = \mathrm{sign}_{\mathbb{Q}} <L>= n$ all conjugates of α are real. Hence $L \subset N \subset \mathbb{R}$. Choose any injection $\iota : \mathcal{G} \hookrightarrow G(N/\mathbb{Q})$ and set $K := N^{\iota(\mathcal{G})} \subset \mathbb{R}$. $\square$

Set

$$X = \sum_{\mathcal{H} \in \mathrm{RC}(\mathcal{G})} m_{\mathcal{H}} \cdot \chi_{\mathcal{H}}, \quad m_{\mathcal{H}} \in \mathbb{Z}.$$

Let N/K be a Galois extension with Galois group $G(N/K) = \mathcal{G}$. Let $\mathfrak{p}$ be a real place of K. Then

$$h_{N/K}(X) = \sum_{\mathcal{H} \in \mathrm{RC}(\mathcal{G})} m_{\mathcal{H}} \times <N^{\mathcal{H}}>= 0$$

gives

$$\text{(I)} \qquad \mathrm{sign}_{\mathfrak{p}} h_{N/K}(X) = 0 = \sum_{\mathcal{H}\in \mathrm{RC}(\mathcal{G})} m_{\mathcal{H}} \cdot \mathrm{sign}_{\mathfrak{p}} <N^{\mathcal{H}}>$$

Let $\mathcal{H} < \mathcal{G}$ and $N^{\mathcal{H}} = K(\alpha)$. By proposition 7, $\mathrm{sign}_{\mathfrak{p}} <N^{\mathcal{H}}>$ equals the number of real conjugates of α with respect to the ordering $\mathfrak{p}$. Let $\sigma \in G(N/K)$ be the automorphism which is induced by the complex conjugation. Then $\mathrm{sign}_{\mathfrak{p}} <N^{\mathcal{H}}>$ is the number of fixed points of the action of $<\sigma>$ on the set of conjugates of α, which equals the number of fixed points of the action of $<\sigma>$ on $\mathcal{G}/\mathcal{H}$. Therefore the equation (I) is already determined by $\mathcal{G}$ and the conjugacy class of the complex conjugation in $\mathcal{G}$. This leads to the following definition.

Definition 9. Let $\sigma \in \mathcal{G}$ be an element of order ≤ 2. Let $\mathcal{H}$ be a subgroup of $\mathcal{G}$ and let $\chi_{\mathcal{H}} \in \mathcal{B}(\mathcal{G})$ be the corresponding character. Set

$$\mathrm{sign}_{\sigma}\chi_{\mathcal{H}} = \sharp\{\text{fixed points of } (<\sigma>, \mathcal{G}/\mathcal{H})\}.$$

Of course, $\mathrm{sign}_{\sigma}\chi_{\mathcal{H}} = \chi_{\mathcal{H}}(\sigma)$. Since our approach is motivated by quadratic form considerations we feel it is more convenient to talk about signatures.

As usual $C_{\mathcal{G}}(\sigma)$ denotes the centralizer of σ in $\mathcal{G}$. Let $\mathcal{G}\sigma = \{\rho^{-1}\sigma\rho \mid \rho \in \mathcal{G}\}$ be the set of conjugates of σ in $\mathcal{G}$.

Proposition 10. *Let $\mathcal{G}$ be a finite group, $\mathcal{H} < \mathcal{G}$ a subgroup of $\mathcal{G}$. Let $\sigma \in \mathcal{G}$ be an element of order ≤ 2. Then*

$$\mathrm{sign}_{\sigma}\chi_{\mathcal{H}} = \frac{\mathrm{ord}(C_{\mathcal{G}}(\sigma))\sharp(\mathcal{G}\sigma \cap \mathcal{H})}{\mathrm{ord}(\mathcal{H})} = \frac{[\mathcal{G}:\mathcal{H}]\sharp(\mathcal{G}\sigma \cap \mathcal{H})}{\sharp\mathcal{G}\sigma}$$

Proof. Consider the action of $<\sigma>$ on $\mathcal{G}/\mathcal{H}$. Let $\rho \in \mathcal{G}$. Then $\rho\mathcal{H}$ is a fixed point if and only if $\rho^{-1}\sigma\rho \in \mathcal{H}$. Hence we can assume that

$$\mathcal{G}\sigma \cap \mathcal{H} = \{\sigma_1, \dots, \sigma_r\}$$

is a set of $r > 0$ elements. Let

$$M = \{(\rho, \sigma_i) \mid \rho^{-1}\sigma\rho = \sigma_i\} \subset \mathcal{G} \times \{\sigma_1, \dots, \sigma_r\}.$$

Obviously the cardinality of M is the product of $\mathrm{ord}(\mathcal{H})$ and the number of fixed points. Further, for $i = 1, \dots, r$ we get

$$\sharp\{\rho \in \mathcal{G} \mid (\rho, \sigma_i) \in M\} = \mathrm{ord}(C_{\mathcal{G}}(\sigma)).$$

Hence $\sharp M = \mathrm{ord}(C_{\mathcal{G}}(\sigma)) \cdot \sharp\mathcal{G}\sigma \cap \mathcal{H}$. □

We abbreviate $\chi_{<\tau>}$ to χ_{τ}.

Corollary 11. *In the situation of proposition 10 we get*

1. $\mathrm{sign}_{\sigma}\chi_{\mathcal{H}} \equiv [\mathcal{G}:\mathcal{H}] \bmod 2$.
2. $\mathrm{sign}_{id}\chi_{\mathcal{H}} = [\mathcal{G}:\mathcal{H}]$.

3. $\mathrm{sign}_\sigma \chi_{\mathcal{H}} \neq 0$ *if and only if* $\mathcal{H}$ *contains some conjugate of* σ.
4. *Let* $\tau \in \mathcal{G}$ *be an element of order* ≤ 2. *Then* $\mathrm{sign}_\sigma \chi_\tau \neq 0$ *if and only if* σ *and* τ *are conjugate or* $\sigma = \mathrm{id}$.
5. *Let* τ *and* σ *be two conjugate involutions. Then*

$$2 \cdot \sharp \mathcal{G}\sigma \cdot \mathrm{sign}_\sigma \chi_\tau = \mathrm{ord}(\mathcal{G}).$$

6. *If* $\mathcal{H}$ *is a normal subgroup of* $\mathcal{G}$, *then* $\mathrm{sign}_\sigma \chi_{\mathcal{H}} = 0$ *or* $= [\mathcal{G} : \mathcal{H}]$.

sign_σ extends to a homomorphism on $\mathcal{B}(\mathcal{G})$.

Proposition 12. *Let* $\mathcal{G}$ *be a finite group and let* $\sigma \in \mathcal{G}$ *be an element of order* ≤ 2. *Then there is a unique homomorphism*

$$\mathrm{sign}_\sigma : \mathcal{B}(\mathcal{G}) \to \mathbb{Z}$$

with $\mathrm{sign}_\sigma \chi_{\mathcal{U}} = \sharp\{\text{fixed points of } (<\sigma>, \mathcal{G}/\mathcal{U})\}$ *for all subgroups* $\mathcal{U}$ *of* $\mathcal{G}$.

Proof. We consider the representations and characters over fields of characteristic 0. Let $\rho : \mathcal{G} \to GL(V)$ be the underlying representation of $\chi_{\mathcal{U}}$. Hence we get $\mathrm{sign}_\sigma \chi_{\mathcal{U}} = trace(\rho(\sigma)) = \chi_{\mathcal{U}}(\sigma)$. Since $trace(A \otimes B) = trace(A) \cdot trace(B)$, sign_σ is a ring homomorphism. □

We conclude that $\mathcal{T}(\mathcal{G})$ is contained in the intersection of all kernels of signature homomorphisms.

Definition 13. Let $\mathcal{G}$ be a finite group. Set

$$L(\mathcal{G}) := \Big\{ \sum_{\mathcal{H} \in \mathrm{RC}(\mathcal{G})} m_{\mathcal{H}} \chi_{\mathcal{H}} \mid \sum_{\mathcal{H} \in \mathrm{RC}(\mathcal{G})} m_{\mathcal{H}} \cdot \mathrm{sign}_\sigma \chi_{\mathcal{H}} = 0 \quad \text{for all } \sigma \in \mathrm{RC}_2(\mathcal{G}) \Big\} \subset \mathcal{B}(\mathcal{G}).$$

Lemma 14. *Let* $\mathcal{G}$ *be a finite group of order* n. *The system of linear equations given by*

$$\sum_{\mathcal{H} \in \mathrm{RC}(\mathcal{G})} \mathrm{sign}_\sigma \chi_{\mathcal{H}} \cdot x_{\mathcal{H}} = 0, \ \ \sigma \in \mathrm{RC}_2(\mathcal{G})$$

has rank $\sharp \mathrm{RC}_2(\mathcal{G})$.

Proof. Let $\sigma_1 = \mathrm{i}d, \sigma_2 \dots, \sigma_r$ be the r distinct elements of $\mathrm{RC}_2(\mathcal{G})$. Consider the coefficients $\mathrm{sign}_{\sigma_j} \chi_{<\sigma_i>}$ for $i, j = 1, \dots, r$. We get $\mathrm{sign}_{\mathrm{id}} \chi_{<\sigma_i>} = \mathrm{ord}(\mathcal{G})/\mathrm{ord}(\sigma_i) \in \{n, n/2\}$ for $i = 1, \dots, r$. For $j = 2, \dots, r$ we have $\mathrm{sign}_{\sigma_j} \chi_{<\sigma_i>} \neq 0$ if and only if $i = j$. □

Remark 15. By lemma 14, $L(\mathcal{G})$ is a free abelian group of rank
$$\mathrm{rank}(\mathcal{B}(\mathcal{G})) - \sharp\mathrm{RC}_2(\mathcal{G}).$$
Further, $\mathcal{T}(\mathcal{G}) \subset L(\mathcal{G})$ by lemma 8 and the remarks following it. We get $\mathrm{rank}(\mathcal{T}(\mathcal{G})) = \mathrm{rank}(L(\mathcal{G}))$ if and only if there exists a positive integer $a \in \mathbb{Z}$ with $a \cdot L(\mathcal{G}) \subset \mathcal{T}(\mathcal{G})$.

By Pfisters local-global principle, $L(\mathcal{G})$ is the set of all $X \in \mathcal{B}(\mathcal{G})$ such that $h_{N/K}(X)$ is a torsion form for any Galois extension N/K with $G(N/K) \simeq \mathcal{G}$. Hence the rank formula of theorem 6 is equivalent to the existence of an integer $l \in \mathbb{Z}, l \geq 0$ depending only on $\mathcal{G}$ such that 2^l annihilates $h_{N/K}(L(\mathcal{G}))$ for any Galois extension N/K with Galois group $\mathcal{G}$.

Since $\mathcal{T}(\mathcal{G}) \subset L(\mathcal{G})$ each signature homomorphism sign_σ induces a unique signature homomorphism $\mathrm{sign} : \mathcal{B}(\mathcal{G})/\mathcal{T}(\mathcal{G}) \to \mathbb{Z}$. Hence we easily get from Theorem 6:

Theorem 16 (**Local-Global Principle**). *An element $X \in \mathcal{B}(\mathcal{G})$ is a torsion element in $\mathcal{B}(\mathcal{G})/\mathcal{T}(\mathcal{G})$ if and only if* $\mathrm{sign}_\sigma(X) = 0$ *for every $\sigma \in \mathcal{G}$ of order ≤ 2. Every torsion element of $\mathcal{B}(\mathcal{G})/\mathcal{T}(\mathcal{G})$ has 2-power order.*

7. Reduction to 2-groups

Proposition 17. *Let $\mathcal{G}$ be a group of odd order. Then*
$$\mathcal{T}(\mathcal{G}) = L(\mathcal{G}).$$
Hence $\mathrm{rank}(\mathcal{T}(\mathcal{G})) = \mathrm{rank}(\mathcal{B}(\mathcal{G})) - 1$.

Proof. Let N/K be a Galois extension with Galois group $G(N/K) \simeq \mathcal{G}$. Let L be an intermediate field of N/K. Then $<L> = [L:K] \times <1>$ (see [2], cor. I.6.5). Let $X = \sum_{\mathcal{H}\in\mathrm{RC}(\mathcal{G})} m_\mathcal{H} \cdot \chi_\mathcal{H}$. Then $h_{N/K}(X) = \sum_{\mathcal{H}\in\mathrm{RC}(\mathcal{G})} m_\mathcal{H} \cdot [\mathcal{G} : \mathcal{H}] \times <1>$. Since $\mathrm{ord}(\mathcal{G})$ is odd, $L(\mathcal{G})$ is defined by the equation $\sum_{\mathcal{H}\in\mathrm{RC}(\mathcal{G})} m_\mathcal{H} \cdot [\mathcal{G} : \mathcal{H}] = 0$ (see corollary 11). Now the statement about the ranks follows from remark 15. □

Let $\mathcal{H}$,$\mathcal{U}$ be subgroups of $\mathcal{G}$. Then the representation defined by the action of $\mathcal{G}$ on $\mathcal{G}/\mathcal{U}$ restricts to a representation of $\mathcal{H}$ on $\mathcal{G}/\mathcal{U}$. This defines a ring homomorphism
$$\mathrm{res}^\mathcal{G}_\mathcal{H} : \mathcal{B}(\mathcal{G}) \to \mathcal{B}(\mathcal{H}),$$
the 'restriction map'. We get
$$\mathrm{res}^\mathcal{G}_\mathcal{H}\chi^\mathcal{G}_\mathcal{U} = \oplus_{\sigma\in\mathcal{H}\backslash\mathcal{G}/\mathcal{U}}\ \chi^\mathcal{H}_{\mathcal{H}\cap\sigma\mathcal{U}\sigma^{-1}} \in \mathcal{B}(\mathcal{H}),$$
where $\chi^\mathcal{H}_{\mathcal{H}\cap\sigma\mathcal{U}\sigma^{-1}} \in \mathcal{B}(\mathcal{H})$ is a character of $\mathcal{H}$.

Proposition 18. *Let $\mathcal{G}$ be a finite group and let $\mathcal{H} < \mathcal{G}$. Let $\sigma \in \mathcal{H}$ be an element of order ≤ 2. Then*

$$\begin{tikzcd} \mathcal{B}(\mathcal{G}) \arrow[r, "\mathrm{sign}_\sigma"] \arrow[d, "\mathrm{res}^{\mathcal{G}}_{\mathcal{H}}"'] & \mathbb{Z} \\ \mathcal{B}(\mathcal{H}) \arrow[ur, "\mathrm{sign}_\sigma"'] & \end{tikzcd}$$

commutes.

Proof. Let $\mathcal{U} < \mathcal{G}$. We compute the signature of $\mathrm{res}^{\mathcal{G}}_{\mathcal{H}}\chi^{\mathcal{G}}_{\mathcal{U}}$ as follows: Restrict the action of $\mathcal{G}$ on $\mathcal{G}/\mathcal{U}$ to $\mathcal{H}$. Then count the number of fixed points of $<\sigma>$ according to this action. Of course, this number equals $\mathrm{sign}_\sigma\chi^{\mathcal{G}}_{\mathcal{U}}$. $\square$

There is an additive but not multiplicative *corestriction* map $\mathrm{cor}^{\mathcal{G}}_{\mathcal{H}} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{G})$ defined by $\mathrm{cor}^{\mathcal{G}}_{\mathcal{H}}\chi^{\mathcal{H}}_{\mathcal{U}} = \chi^{\mathcal{G}}_{\mathcal{U}}$.

Proposition 19. *Let $\mathcal{G}$ be a finite group, $\mathcal{H} < \mathcal{G}$. Let N/K be a Galois extension with $G(N/K) = \mathcal{G}$. Let $s^* : \mathrm{W}(K) \to \mathrm{W}(N^{\mathcal{H}})$ be the lifting homomorphism. Then*

$$\begin{tikzcd} \mathcal{B}(\mathcal{G}) \arrow[r, "h_{N/K}"] \arrow[d, "\mathrm{res}^{\mathcal{G}}_{\mathcal{H}}"'] & \mathrm{W}(K) \arrow[d, "s^*"] \\ \mathcal{B}(\mathcal{H}) \arrow[r, "h_{N/N^{\mathcal{H}}}"] & \mathrm{W}(N^{\mathcal{H}}) \end{tikzcd}$$

and

$$\begin{tikzcd} \mathcal{B}(\mathcal{H}) \arrow[r, "h_{N/N^{\mathcal{H}}}"] \arrow[d, "\mathrm{cor}^{\mathcal{G}}_{\mathcal{H}}"'] & \mathrm{W}(N^{\mathcal{H}}) \arrow[d, "tr_{N^{\mathcal{H}}/K}"] \\ \mathcal{B}(\mathcal{G}) \arrow[r, "h_{N/K}"] & \mathrm{W}(K) \end{tikzcd}$$

commute.

Proof. We use the notation of lemma 4 and its proof. Set $L := N^{\mathcal{H}}$. Then

$$\begin{aligned} h_{N/L}(\mathrm{res}^{\mathcal{G}}_{\mathcal{H}}(\chi^{\mathcal{G}}_{\mathcal{U}})) &= \perp_{\sigma\in\mathcal{U}\backslash\mathcal{G}/\mathcal{H}}\ h_{N/L}(\chi^{\mathcal{H}}_{\mathcal{H}\cap\sigma\mathcal{U}\sigma^{-1}}) \\ &= \perp_{\sigma\in\mathcal{U}\backslash\mathcal{G}/\mathcal{H}}<N^{\mathcal{H}\cap\sigma\mathcal{U}\sigma^{-1}}/L>=\perp_{i=1,\dots,r}<(L[X]/(f_i))/L> \\ &= <(L[X]/(f_1\cdots f_r))/L>=<(K[X]/(f))\otimes L> \\ &= s^*(<N^{\mathcal{U}}/K>) = s^*\circ h_{N/K}(\chi^{\mathcal{G}}_{\mathcal{U}}). \end{aligned}$$

Let $\mathcal{U} < \mathcal{H} < \mathcal{G}$. Then

$$h_{N/K} \circ \mathrm{cor}^{\mathcal{G}}_{\mathcal{H}}(\chi^{\mathcal{H}}_{\mathcal{U}}) = h_{N/K}(\chi^{\mathcal{G}}_{\mathcal{U}}) =< N^{\mathcal{U}} >= \\ tr_{N^{\mathcal{H}}/K} < N^{\mathcal{U}}/N^{\mathcal{H}} >= tr_{N^{\mathcal{H}}/K}(h_{N/N^{\mathcal{H}}}(\chi^{\mathcal{H}}_{\mathcal{U}})).$$

□

Lemma 20. *Let $\mathcal{H} < \mathcal{G}$ be finite groups.*

1. *Then* $\mathrm{res}^{\mathcal{G}}_{\mathcal{H}}(L(\mathcal{G})) \subset L(\mathcal{H})$.
2. *Let $[\mathcal{G} : \mathcal{H}]$ be odd.*
   (a) *Then* $\mathrm{res}^{\mathcal{G}}_{\mathcal{H}}(X) \in L(\mathcal{H})$ *if and only if* $X \in L(\mathcal{G})$.
   (b) $\mathrm{res}^{\mathcal{G}}_{\mathcal{H}}(X) \in \mathcal{T}(\mathcal{H})$ *implies* $X \in \mathcal{T}(\mathcal{G})$.

Proof. 1. follows from proposition 18.
2. Choose $\mathrm{RC}_2(\mathcal{G}) \subset \mathrm{RC}_2(\mathcal{H})$ and apply proposition 18.
(b) Let N/K be a Galois extension with $G(N/K) = \mathcal{G}$ and let $X \in \mathcal{B}(\mathcal{G})$ with $\mathrm{res}^{\mathcal{G}}_{\mathcal{H}}(X) \in \mathcal{T}(\mathcal{H})$. Now $h_{N/N^{\mathcal{H}}} \circ \mathrm{res}^{\mathcal{G}}_{\mathcal{H}}(X) = 0 = s^* \circ h_{N/K}(X)$ by proposition 19. By a theorem of Springer s^* is injective (see [5], 2.5.3). Thus $h_{N/K}(X) = 0$. □

From $X \in \mathcal{T}(\mathcal{G})$ we get $X \in \ker(h_{N/K})$, hence $\mathrm{res}^{\mathcal{G}}_{\mathcal{H}}(X) \in \ker(h_{N/N^{\mathcal{H}}})$. But we do not get $\mathrm{res}^{\mathcal{G}}_{\mathcal{H}}(X) \in \mathcal{T}(\mathcal{H})$. We only get $\mathrm{res}^{\mathcal{G}}_{\mathcal{H}}(X) \in \bigcap \ker(h_{N/K})$, where the intersection runs over all Galois extensions N/K with Galois group $\mathcal{G}$ and such that $\mathcal{G} < \mathrm{Aut}(N)$.

Let $\exp(\mathcal{G})$ denote the exponent of $\mathcal{G}$.

Proposition 21. *Let $\mathcal{G}$ be a finite group and let $\mathcal{G}_2$ be a 2-Sylow subgroup of $\mathcal{G}$.*

1. *Then the rank formula of theorem 6 holds for $\mathcal{G}$ if it holds for any 2-Sylow subgroup of $\mathcal{G}$, in which case* $\exp(L(\mathcal{G})/\mathcal{T}(\mathcal{G}))$ *divides the exponent of $L(\mathcal{G}_2)/\mathcal{T}(\mathcal{G}_2)$.*
2. *Suppose there is a set $\mathcal{X}$ of fields such that $\mathcal{G} < \mathrm{Aut}(N)$ for any $N \in \mathcal{X}$ and such that*
$$\mathcal{T}(\mathcal{G}_2) = \bigcap_{N \in \mathcal{X}} \bigcap_{\mathcal{U} < \mathrm{Aut}(N), \mathcal{U} \simeq \mathcal{G}_2} \ker(h_{N/N^{\mathcal{U}}}).$$
   Then $X \in \mathcal{T}(\mathcal{G})$ if and only if $\mathrm{res}^{\mathcal{G}}_{\mathcal{G}_2}(X) \in \mathcal{T}(\mathcal{G}_2)$. Hence $L(\mathcal{G})/\mathcal{T}(\mathcal{G})$ is isomorphic to a subgroup of $L(\mathcal{G}_2)/\mathcal{T}(\mathcal{G}_2)$.

Proof. 1. If the rank formula holds for $\mathcal{G}_2$, then by remark 15 there is a positive integer a with $a \cdot L(\mathcal{G}_2) \subset \mathcal{T}(\mathcal{G}_2)$. Let $X \in L(\mathcal{G})$. Then $\mathrm{res}^{\mathcal{G}}_{\mathcal{G}_2}(X) \in L(\mathcal{G}_2)$ and $\mathrm{res}^{\mathcal{G}}_{\mathcal{G}_2}(aX) = a \cdot \mathrm{res}^{\mathcal{G}}_{\mathcal{G}_2}(X) \in a \cdot L(\mathcal{G}_2) \subset \mathcal{T}(\mathcal{G}_2)$. Hence $aX \in \mathcal{T}(\mathcal{G})$ by lemma 20(2)(b). The proof of (2) is left to the reader. □

8. Proof of theorem 6

Let $\mathcal{J}_2(\mathcal{G})$ be the set of involutions of the 2-group $\mathcal{G}$. For a subgroup $\mathcal{H}$ of $\mathcal{G}$ define

$$X_{\mathcal{H}}^{\mathcal{G}} := X_{\mathcal{H}} := \operatorname{ord}(\mathcal{H}) \cdot \chi_{\mathcal{H}}^{\mathcal{G}} - \chi_1^{\mathcal{G}} + \sum_{\tau \in \mathrm{RC}_2(\mathcal{G}), \tau \neq 1} \sharp(\mathcal{G}\tau \cap \mathcal{H}) \cdot (\chi_1^{\mathcal{G}} - 2 \cdot \chi_{\tau}^{\mathcal{G}})$$

and let

$$M_{\mathcal{G}} := \{X_{\mathcal{H}} \mid \mathcal{H} \in \mathrm{RC}(\mathcal{G}) - \mathrm{RC}_2(\mathcal{G})\}.$$

By proposition 10 and corollary 11, M is a free subset of $L(\mathcal{G})$ which consists of $\operatorname{rank}(L(\mathcal{G}))$ elements. We will prove by induction that $M_{\mathcal{G}}$ is contained in $\mathcal{T}(\mathcal{G})$.

Lemma 22. *Let $\mathcal{G}$ be a 2-group. Then $M_{\mathcal{G}}$ is a free subset of $\mathcal{T}(\mathcal{G})$ consisting of* $\operatorname{rank}(L(\mathcal{G}))$ *elements.*

Proof. Observe that $\mathcal{T}(\mathbb{Z}_2) = 0$. Let $\mathcal{G}$ be a group of order $2^l \geq 4$ and let N/K be a Galois extension with Galois group $\mathcal{G}$. Now we proceed by induction.

1. Let $\mathcal{H}$ be a subgroup with $\mathcal{H} \neq \mathcal{G}$. Let $\tau, \tau' \in \mathcal{G}$ be involutions. Then $\chi_\tau = \chi_{\tau'}$ if and only if $\tau' \in \mathcal{G}\tau$. Since $\mathcal{J}_2(\mathcal{G})$ is the disjoint union of the conjugacy classes of the involutions of $\mathcal{G}$ we get

$$\mathcal{J}_2(\mathcal{H}) = \mathcal{J}_2(\mathcal{G}) \cap \mathcal{H} = \bigcup_{\tau \in \mathrm{RC}_2(\mathcal{G}), \tau \neq 1} \mathcal{G}\tau \cap \mathcal{H}.$$

Let $\mathcal{U} < \mathcal{G}$ be a maximal subgroup of $\mathcal{G}$ which contains $\mathcal{H}$. Then

$$\begin{aligned} X_{\mathcal{H}}^{\mathcal{U}} &= \operatorname{ord}(\mathcal{H}) \cdot \chi_{\mathcal{H}}^{\mathcal{U}} - \chi_1^{\mathcal{U}} + \sum_{\tau \in \mathrm{RC}_2(\mathcal{U}), \tau \neq 1} \sharp(\mathcal{U}\tau \cap \mathcal{H})(\chi_1^{\mathcal{U}} - 2 \cdot \chi_{\tau}^{\mathcal{U}}) \\ &= \operatorname{ord}(\mathcal{H}) \cdot \chi_{\mathcal{H}}^{\mathcal{U}} - \chi_1^{\mathcal{U}} + \sum_{\tau \in \mathcal{J}_2(\mathcal{H})} (\chi_1^{\mathcal{U}} - 2 \cdot \chi_{\tau}^{\mathcal{U}}). \end{aligned}$$

Now $X_{\mathcal{H}}^{\mathcal{U}} \in \mathcal{T}(\mathcal{U})$ by induction hypothesis. Hence $h_{N/N^{\mathcal{U}}}(X_{\mathcal{H}}^{\mathcal{U}}) = 0$, which gives $h_{N/K}(X_{\mathcal{H}}^{\mathcal{G}}) = \operatorname{tr}_{N^{\mathcal{U}}/K}(h_{N/N^{\mathcal{U}}}(X_{\mathcal{H}}^{\mathcal{U}})) = 0$ (see proposition 19). Hence $X_{\mathcal{H}}^{\mathcal{G}} \in \mathcal{T}(\mathcal{G})$ if $\mathcal{H} \neq \mathcal{G}$.

2. Next we have to prove $X_{\mathcal{G}}^{\mathcal{G}} \in \mathcal{T}(\mathcal{G})$. First we consider an elementary abelian group. Then

$$X_{\mathcal{G}}^{\mathcal{G}} = 2^l \cdot \chi_{\mathcal{G}}^{\mathcal{G}} + (2^l - 2) \cdot \chi_1^{\mathcal{G}} - 2 \cdot \sum_{\tau \in \mathcal{G}, \tau \neq 1} \chi_{\tau}^{\mathcal{G}}.$$

Let $N = K(\sqrt{a_1}, \ldots, \sqrt{a_l})$. We know that $<N>=<2^l>\otimes<<-a_1, \ldots, -a_l>>$ (see [3], prop. 1).

Now expand the Pfister form $<<-a_1, \ldots, -a_l>>=<1, b_2, \ldots, b_{2^l}>$. Then the entries $b_2, \ldots, b_{2^l}$ are in one-to-one correspondence with the quadratic subextensions of N/K. There are exactly $2^{l-1}-1$ elements $\tau \in \mathcal{G}, \tau \neq \mathrm{id}$ such that $K(\sqrt{b_i}) \subset N^\tau$. Hence

$$\begin{aligned} h_{N/K}(X_{\mathcal{G}}^{\mathcal{G}}) &= 2^l \times <1> \perp (2^l-2) \times <N> -2 \sum_{\tau \in \mathcal{G}, \tau \neq \mathrm{id}} <N^\tau> \\ &= 0. \end{aligned}$$

Now we can assume that $\mathcal{G}$ is not an elementary abelian group. Let $\mathcal{U}_1, \ldots, \mathcal{U}_m$ be the maximal subgroups of $\mathcal{G}$. Since $\mathcal{G}$ is not a group of order 2, we get $\mathcal{J}_2(\mathcal{G}) \subset \cup_{i=1}^m \mathcal{U}_i$. This gives

$$\sum_{\tau \in \mathcal{J}_2(\mathcal{G})} (\chi_1 - 2 \cdot \chi_\tau) = \sum_{\mathcal{U}=\mathcal{U}_{i_1} \cap \ldots \cap \mathcal{U}_{i_r}} (-1)^{r+1} \sum_{\tau \in \mathcal{J}_2(\mathcal{U})} (\chi_1 - 2 \cdot \chi_\tau),$$

where the sum runs over the set of all non-empty subsets of $\{1, \ldots, m\}$. Let $\Phi(\mathcal{G})$ denote the Frattini subgroup of $\mathcal{G}$. Let 2^k be its order and set $\mathcal{V} = \mathcal{G}/\Phi(\mathcal{G})$. Let F be the fixed field of $\Phi(\mathcal{G})$. Then F/K is an elementary abelian extension. Let $\{i_1, \ldots, i_r\} \subset \{1, \ldots, m\}$ be a set of r different indices. Set $\mathcal{H} = \mathcal{U}_{i_1} \cap \ldots \cap \mathcal{U}_{i_r}$. Then $X_{\mathcal{H}}^{\mathcal{H}} \in \mathcal{T}(\mathcal{H})$ by induction hypothesis. We get $h_{N/N^{\mathcal{H}}}(X_{\mathcal{H}}^{\mathcal{H}}) = 0$, which implies

$$\sum_{\tau \in \mathcal{J}_2(\mathcal{H})} (<N/N^{\mathcal{H}}> -2\times <N^\tau/N^{\mathcal{H}}>) = <N/N^{\mathcal{H}}> -\mathrm{ord}(\mathcal{H}) \times <1> .$$

Set $\mathcal{V}' = \mathcal{H}/\Phi(\mathcal{G})$ and suppose $\mathcal{H} \neq \Phi(\mathcal{G})$. By (1) we know that $X_{\mathcal{V}'}^{\mathcal{V}} \in \mathcal{T}(\mathcal{V})$ for all subgroups $\mathcal{V}'$ of $\mathcal{V}$ with $\mathcal{V}' \neq 1$. This gives

$$\mathrm{ord}(\mathcal{H}/\Phi(\mathcal{G})) \times <1> = <F/N^{\mathcal{H}}> - \sum_{\tau \in \mathcal{J}_2(\mathcal{V}')} (<F/N^{\mathcal{H}}> -2\cdot <F^\tau/N^{\mathcal{H}}>).$$

We further get

$$\begin{aligned}
h_{N/K}\Big(\sum_{\tau\in\mathcal{J}_2(\mathcal{H})}(\chi_1^{\mathcal{G}}-2\cdot\chi_\tau^{\mathcal{G}})\Big) &= \sum_{\tau\in\mathcal{J}_2(\mathcal{H})}(<N> -2\times <N^\tau>)\\
&= \mathrm{tr}_{N^{\mathcal{H}}/K}\Big[\sum_{\tau\in\mathcal{J}_2(\mathcal{H})}(<N/N^{\mathcal{H}}> -2\times <N^\tau/N^{\mathcal{H}}>)\Big]\\
&= \mathrm{tr}_{N^{\mathcal{H}}/K}(<N/N^{\mathcal{H}}> -\mathrm{ord}(\mathcal{H})\times <1>)\\
&= (<N> -\mathrm{ord}(\mathcal{H})\times <N^{\mathcal{H}}>)\\
&= <N> -2^k\times \mathrm{tr}_{N^{\mathcal{H}}/K}(\mathrm{ord}(\mathcal{H}/\Phi(\mathcal{G}))\times <1>)\\
&= <N> -2^k\times \mathrm{tr}_{N^{\mathcal{H}}/K}(<F/N^{\mathcal{H}}>\\
&\quad -\sum_{\tau\in\mathcal{J}_2(\mathcal{H}/\Phi(\mathcal{G}))}(<F/N^{\mathcal{H}}> -2\times <F^\tau/N^{\mathcal{H}}>))\\
&= <N> -2^k\times <F>\\
&\quad +2^k\times\sum_{\tau\in\mathcal{J}_2(\mathcal{H}/\Phi(\mathcal{G}))}(<F> -2\times <F^\tau>)
\end{aligned}$$

If $\mathcal{H}=\Phi(\mathcal{G})$, then $\mathcal{J}_2(\mathcal{H}/\Phi(\mathcal{G}))$ is empty and $N^{\mathcal{H}}=F$. Hence the formula also holds in this situation.
Now $\sum_{r=0}^n(-1)^r\binom{n}{r}=0$ implies

$$\begin{aligned}
h_{N/K}(X_{\mathcal{G}}^{\mathcal{G}}) &= 2^l\times <1> - <N> +\sum_{\mathcal{H}}(-1)^{r+1}\sum_{\tau\in\mathcal{J}_2(\mathcal{H})}(<N> -2\times <N^\tau>)\\
&= 2^l\times <1> -2^k\times <F>\\
&\quad +2^k\times\sum_{\mathcal{H}}(-1)^{r+1}\sum_{\tau\in\mathcal{J}_2(\mathcal{H}/\Phi(\mathcal{G}))}(<F> -2\times <F^\tau>)\\
&= 2^k\times[\mathrm{ord}(\mathcal{V})\times <1> - <F> +h_{F/K}\Big(\sum_{\tau\in\mathcal{J}_2(\mathcal{V})}(\chi_1^{\mathcal{V}}-2\cdot\chi_\tau^{\mathcal{V}})\Big]\\
&= 2^k h_{F/K}(X_{\mathcal{V}}^{\mathcal{V}})=0
\end{aligned}$$

by the above. $\square$

9. Open Questions

We conclude with some open questions. How does the exponent of $\mathcal{B}(\mathcal{G})/\mathcal{T}(\mathcal{G})$ depend on $\mathcal{G}$?

Proposition 23. *Let $\mathcal{G}$ be a finite group. If a 2-Sylow subgroup $\mathcal{G}_2$ of $\mathcal{G}$ is a normal subgroup of $\mathcal{G}$, then the restriction homomorphism induces an epimorphism*

$$\mathrm{res}: L(\mathcal{G})\to L(\mathcal{G}_2)/\mathcal{T}(\mathcal{G}_2)$$

Proof. Let cor : $\mathcal{B}(\mathcal{G}_2) \to \mathcal{B}(\mathcal{G})$ be the corestriction. This is an additive homomorphism. Since $\mathcal{G}_2$ is normal in $\mathcal{G}$ we get $\mathrm{res} \circ \mathrm{cor} = [\mathcal{G} : \mathcal{G}_2] \cdot \mathrm{i}d$. By Theorem 6 there is an integer $l \in \mathbb{N}$ such that $2^l \cdot L(\mathcal{G}_2) \subset \mathcal{T}(\mathcal{G}_2)$. Let $k, t \in \mathbb{Z}, k > 0$ with $k \cdot [\mathcal{G} : \mathcal{G}_2] = 1 + t \cdot 2^l$. Then $\mathrm{res} \circ \mathrm{cor}(kX) = X + t \cdot 2^l X \equiv X \bmod \mathcal{T}(\mathcal{G}_2)$. □

This leads to the following question: Does the restriction homomorphism induces an isomorphism

$$\mathrm{res} : L(\mathcal{G})/\mathcal{T}(\mathcal{G}) \to L(\mathcal{G}_2)/\mathcal{T}(\mathcal{G}_2)?$$

We know that the answer is affirmative if $\mathcal{G}$ is an abelian group whose 2-Sylow subgroup is cyclic or elementary abelian. In these cases $L(\mathcal{G})/\mathcal{T}(\mathcal{G})$ has exponent 2. If $\mathcal{G}$ is the dihedral group of order 8, then the exponent is 2. In the case of the quaternion group Q_8 of order 8 we get $\exp(L(Q_8)/\mathcal{T}(Q_8)) = 4$.

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