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On the almost Goldbach problem of Linnik

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Résumé. On démontre que sous GRH et pour $k \geq 200$, tout entier pair assez grand est somme de deux nombres premiers impairs et de k puissances de 2.

Abstract. Under the Generalized Riemann Hypothesis, it is proved that for any $k \geq 200$ there is $N_k > 0$ depending on k only such that every even integer $\geq N_k$ is a sum of two odd primes and k powers of 2.

1. Introduction

In 1951 and 1953, Linnik [L1,L2] investigated the following "almost Goldbach" representation for even integers N:

$$N = p_1 + p_2 + 2^{\nu_1} + 2^{\nu_2} + ... + 2^{\nu_k}, \tag{1.1}$$

where (and throughout) p and ν, with or without subscripts, stand for a prime and a positive integer respectively. He showed that there is a constant $k > 0$ such that every large even integer N can be written as (1.1). This result was generalized by A.I. Vinogradov [Vi] in several directions. In 1975, Gallagher [G] considerably simplified the proofs of Linnik and Vinogradov, and established the following result: For any integer $k \geq 2$ there is a positive constant N_k depending on k only, such that for each even integer $N \geq N_k$,

$$r_k(N) = 2\frac{N \log_2^k N}{\log^2 N}\left\{1 + O\left(\frac{\log^2 k}{k}\right)\right\}, \tag{1.2}$$

where $r_k(N)$ is the number of representations of N in the form of (1.1). Here $\log N$ and $\log_2 N$ in (1.2) correspond to the terms p and 2^ν in (1.1) respectively.

In the above results of Linnik, Vinogradov, and Gallagher, a numerically acceptable value for k still remains unspecified. It is therefore not clear that how many powers of 2 are needed to ensure $r_k(N) > 0$. From (1.2) we see that adding more powers of 2 does not change the constant 2 in

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the main term $N\log_2^k N/\log^2 N$ in (1.2) but gives a better error term only. However, due to the sparsity of the sequence of powers of 2, a desirable error term needs a large number of them. Furthermore, in view of the Hardy-Littlewood theorem [HL] on the exceptional set of the Goldbach conjecture, one may anticipate that even under the Generalized Riemann Hypothesis (GRH) a small k in (1.1) or (1.2) is not sufficient to give the positiveness of $r_k(N)$. Recently, the present authors showed that $k = 770$ is sufficient under the GRH [LLW1], and unconditionally $k = 54000$ is acceptable [LLW2].

The purpose of this paper is to reduce the acceptable value of k to 200 under the GRH. More precisely, we shall prove

Theorem 1. *Assume the GRH. For any integer $k \geq 200$ there exists a positive constant N_k depending on k only, such that if $N \geq N_k$ is an even integer then*

$$r_k(N) \geq \frac{1}{42}\frac{N\log_2^k N}{\log^2 N}. \tag{1.3}$$

In particular, each large even integer is a sum of two primes and 200 powers of 2.

In a letter to Goldbach, Euler asked, and later answered by himself negatively, the problem of representing each sufficiently large odd integer as a sum of a prime and a power of 2. However, Romanoff [R] showed in 1934 that a positive proportion of the odd integers can be written in this way. And Gallagher [G] proved that the density of odd integers n, which can be written as

$$n = p + 2^{\nu_1} + 2^{\nu_2} + \ldots + 2^{\nu_k},$$

tends to 1 as $k \to \infty$, and from this he deduced his result (1.2) for the almost Goldbach problem.

Unlike in [LLW1] where we followed the above approach of Gallagher [G], here we use Linnik's original idea [L1,L2] (see §4 below), a lemma due to Kaczorowski-Perelli-Pintz [KPP] and Languasco-Perelli [LP] (see Lemma 1 below), and a well- known result of Chen [C] obtained by sieve methods (see our Lemma 3).

As usual, $\varphi(n)$ stands for the Euler function, $\mu(n)$ the Möbius function, and $\Lambda(n)$ the von Mangoldt function. Throughout this paper, L always stands for $\log_2 N$. Let χ mod q and χ^0 mod q denote a Dirichlet character and the principal character modulo q respectively. The letter C with subscripts denotes absolute constants, and ε denotes a positive constant which is arbitrarily small.

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2. The major arcs of the Goldbach problem

Let N be a large integer, and P, Q parameters satisfying

$$2 \le 2P < Q \le N. \tag{2.1}$$

By Dirichlet's lemma on rational approximation, each $\alpha \in \left[\frac{1}{Q}, 1+\frac{1}{Q}\right]$ may be written in the form

$$\alpha = \frac{a}{q} + \lambda, \quad |\lambda| \le \frac{1}{qQ} \tag{2.2}$$

for some positive integers a, q with $1 \le a \le q, (a,q) = 1$, and $q \le Q$. We denote by $\mathcal{M}(a,q)$ the set of α satisfying (2.2), and put

$$\mathcal{M} = \bigcup_{q \le P} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \mathcal{M}(a,q), \qquad C(\mathcal{M}) = \left[\frac{1}{Q}, 1+\frac{1}{Q}\right] \backslash \mathcal{M}. \tag{2.3}$$

When $q \le P$ we call $\mathcal{M}(a,q)$ a *major arc*. Note that, by (2.1), all major arcs are mutually disjoint. Let $e(\alpha) = \exp(i2\pi\alpha)$ and

$$S(\alpha, N) = \sum_{p \le N} e(p\alpha).$$

The purpose of this section is to establish the following

Theorem 2. *Assume the GRH. Let M be an even integer with $NL^{-2} < M \le N$, and specify the P and Q in (2.1) by putting*

$$P = N^{1/2}L^{-8}, \quad Q = N^{1/2}. \tag{2.4}$$

Then we have

$$\int_{\mathcal{M}} S^2(\alpha, M) e(-M\alpha) d\alpha = \mathfrak{S}(M) \frac{M}{\log^2 M} \left\{1 + O\left(\frac{1}{\log M}\right)\right\}, \tag{2.5}$$

where

$$\mathfrak{S}(M) = 2C_0 \prod_{\substack{p \mid M \\ p>2}} \left(\frac{p-1}{p-2}\right), \qquad C_0 = \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right). \tag{2.6}$$

In the Hardy-Littlewood method, the wider of the major arcs will usually give the better results. Here, we can see the influence of the GRH on the width of the major arcs. Under the GRH we can now widen the length of the major arcs in (2.2) by setting our $Q = N^{1/2}$; otherwise, we have to considerably narrow them, e.g. in [LLW2], (4.1) and (2.1) we set $Q = N^{179/180}L^{-3}$.

To prove Theorem 2, we need the following lemma.

Lemma 1. *Assume the GRH. Let δ_χ be 1 or 0 according as $\chi = \chi^0$ or not. Then*

$$\int_{-1/(qQ)}^{1/(qQ)} \left| \sum_{m \le M} (\Lambda(m)\chi(m) - \delta_\chi) e(m\lambda) \right|^2 d\lambda \ll \frac{M \log^4 M}{qQ}$$

uniformly for any χ mod q and $q \le Q \le M$.

Proof. This is [KPP, Lemma 1] and the first paragraph of [LP,§5].

Now we can give

Proof of Theorem 2. Let

$$T(\alpha, M) = \sum_{m \le M} \Lambda(m) e(m\alpha).$$

It suffices to prove

$$\int_{\mathcal{M}} T^2(\alpha, M) e(-M\alpha) d\alpha = \mathfrak{S}(M) M + O\left(M \log^{-1} M\right), \tag{2.7}$$

since (2.5) follows from (2.7) via partial summation.

Introducing the Dirichlet characters (see [D, §26, (2)]), we have

$$\begin{aligned} T\left(\frac{a}{q} + \lambda, M\right) &= \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \sum_{h=1}^{q} \bar{\chi}(h) e\left(\frac{ah}{q}\right) \sum_{m \le M} \Lambda(m)\chi(m) e(m\lambda) \\ &\quad + O(\log^2(qM)) \\ &= V(\lambda, q, a) + W(\lambda, q, a), \end{aligned}$$

where

$$V(\lambda, q, a) = \frac{\mu(q)}{\varphi(q)} \sum_{m \le M} e(m\lambda),$$

and

$$\begin{aligned} W(\lambda, q, a) &= \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \chi(a) \tau(\bar{\chi}) \sum_{m \le M} (\Lambda(m)\chi(m) - \delta_\chi) e(m\lambda) \\ &\quad + O(\log^2 M), \end{aligned}$$

with $\tau(\chi)=\sum_{h=1}^{q}\chi(h)e(h/q)$ and δ_χ as in Lemma 1. Thus,

$$\begin{aligned}
&\int_{\mathcal{M}} T^2(\alpha,M)\,e(-M\alpha)d\alpha\\
&=\sum_{q\le P}\sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(-\frac{aM}{q}\right)\int_{-1/(qQ)}^{1/(qQ)} V^2(\lambda,q,a)e(-M\lambda)d\lambda\\
&\quad+2\sum_{q\le P}\sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(-\frac{aM}{q}\right)\int_{-1/(qQ)}^{1/(qQ)} V(\lambda,q,a)W(\lambda,q,a)e(-M\lambda)d\lambda\\
&\quad+\sum_{q\le P}\sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(-\frac{aM}{q}\right)\int_{-1/(qQ)}^{1/(qQ)} W^2(\lambda,q,a)e(-M\lambda)d\lambda\\
&=: I_1+I_2+I_3,\qquad \text{say.}
\end{aligned}\tag{2.8}$$

Now we proceed to estimate I_1, I_2 and I_3. Clearly,

$$|I_3|\le\sum_{q\le P}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{-1/(qQ)}^{1/(qQ)}|W(\lambda,q,a)|^2d\lambda=:I_3^*,\qquad\text{say.}$$

Using the orthogonality of characters, the bound $|\tau(\chi)|^2\le q$, and then Lemma 1, one gets

$$\begin{aligned}
I_3^*\le 2\sum_{q\le P}\frac{1}{\varphi^2(q)}&\sum_{\chi_1\bmod q}\ \sum_{\chi_2\bmod q}\tau(\bar\chi_1)\overline{\tau(\bar\chi_2)}\left\{\sum_{\substack{a=1\\(a,q)=1}}^{q}\chi_1(a)\bar\chi_2(a)\right\}\\
&\times\int_{-1/(qQ)}^{1/(qQ)}\left\{\sum_{m\le M}(\Lambda(m)\chi_1(m)-\delta_{\chi_1})e(m\lambda)\right\}\\
&\times\left\{\sum_{m\le M}(\Lambda(m)\bar\chi_2(m)-\delta_{\bar\chi_2})e(-m\lambda)\right\}d\lambda\\
&+O\left\{\sum_{q\le P}\sum_{a=1}^{q}\int_{-1/(qQ)}^{1/(qQ)}\log^4 M d\lambda\right\}
\end{aligned}\tag{2.9}$$

$$
\begin{aligned}
&= 2\sum_{q\le P}\frac{1}{\varphi(q)}\sum_{\chi \bmod q}|\tau(\chi)|^2\int_{-1/(qQ)}^{1/(qQ)}\left|\sum_{m\le M}(\Lambda(m)\chi(m)-\delta_\chi)e(m\lambda)\right|^2 d\lambda\\
&\qquad + O\left(\frac{P\log^4 M}{Q}\right)\\
&\ll \frac{PM\log^4 M}{Q}.
\end{aligned}
$$

Applying Cauchy's inequality and then using the elementary estimate

$$
\sum_{m\le M} e(m\lambda) \ll \min(M,|\lambda|^{-1}), \tag{2.10}
$$

one has

$$
\begin{aligned}
|I_2|^2 &\ll I_3^*\sum_{q\le P}\frac{1}{\varphi^2(q)}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{-1/(qQ)}^{1/(qQ)}\left|\sum_{m\le M}e(m\lambda)\right|^2 d\lambda\\
&\ll I_3^*\sum_{q\le P}\frac{1}{\varphi(q)}\int_{-1/(qQ)}^{1/(qQ)}\min(M^2,|\lambda|^{-2})d\lambda\\
&\ll I_3^*\sum_{q\le P}\frac{1}{\varphi(q)}\left\{\int_0^{1/M}M^2 d\lambda+\int_{1/M}^{\infty}\lambda^{-2}d\lambda\right\}\ll I_3^* M\log M.
\end{aligned}
$$

It therefore follows from (2.9) that

$$
I_2 \ll \frac{P^{1/2}M\log^3 M}{Q^{1/2}}. \tag{2.11}
$$

Now we treat the main term I_1. Let

$$
c_q(m) = \sum_{\substack{h=1\\(h,q)=1}}^{q} e\left(\frac{mh}{q}\right).
$$

Applying the elementary estimate (2.10) again, one obtains

$$\begin{aligned} I_1 &= \sum_{q\le P}\frac{\mu^2(q)}{\varphi^2(q)}c_q(-M)\int_{-1/(qQ)}^{1/(qQ)}\left(\sum_{m\le M}e(m\lambda)\right)^2 e(-M\lambda)d\lambda \\ &= \sum_{q\le P}\frac{\mu^2(q)}{\varphi^2(q)}c_q(-M) \\ &\quad\times\left\{\int_{-1/2}^{1/2}\left(\sum_{m\le M}e(m\lambda)\right)^2 e(-M\lambda)d\lambda + O\left(\int_{1/(qQ)}^{\infty}|\lambda|^{-2}d\lambda\right)\right\} \\ &= \sum_{q\le P}\frac{\mu^2(q)}{\varphi^2(q)}c_q(-M)\left\{M+O(PQ)\right\}. \end{aligned} \tag{2.12}$$

The contribution of the last O-term to I_1 is

$$\ll PQ\sum_{q\le P}\frac{\mu^2(q)}{\varphi^2(q)}|c_q(-M)| \ll PQ\sum_{q\le P}\frac{1}{\varphi(q)} \ll PQ\log M. \tag{2.13}$$

Since

$$\begin{aligned} \sum_{q>P}\frac{\mu^2(q)}{\varphi^2(q)}c_q(-M) &= \sum_{q>P}\frac{\mu^2(q)}{\varphi^2(q)}\frac{\mu(q/(q,M))\varphi(q)}{\varphi(q/(q,M))} \\ &\ll \frac{1}{P}\sum_{m|M}\frac{\mu^2(m)m}{\varphi(m)} \ll \frac{d(M)\log M}{P}, \end{aligned}$$

we have

$$\sum_{q\le P}\frac{\mu^2(q)}{\varphi^2(q)}c_q(-M) = \mathfrak{S}(M) + O\left(\frac{d(M)\log M}{P}\right). \tag{2.14}$$

Inserting (2.13) and (2.14) into (2.12), we obtain

$$I_1 = \mathfrak{S}(M)M + O\left(\frac{d(M)M\log M}{P} + PQ\log M\right). \tag{2.15}$$

Therefore we conclude from (2.8), (2.9), (2.11), and (2.15) that

$$\begin{aligned} &\int_{\mathcal{M}} T^2(\alpha, M)e(-M\alpha)d\alpha = \mathfrak{S}(M)M \\ &\quad + O\left(\frac{d(M)M\log M}{P} + PQ\log M + \frac{PM\log^4 M}{Q} + \frac{P^{1/2}M\log^3 M}{Q^{1/2}}\right). \end{aligned}$$

By (2.4) and $NL^{-2} < M \le N$, the last O-term is $\ll M\log^{-1} M$. This proves (2.7) and hence Theorem 2.

3. An exponential sum over powers of 2

The results of this section do not depend on the GRH.

Let

$$G(\alpha) = \sum_{\nu \le L} e(2^\nu \alpha).$$

Lemma 2. *Let $\eta < 1/(7e)$. Then the set $\mathcal{E}$ of $\alpha \in (0,1]$ for which $|G(\alpha)| \ge (1-\eta)L$ has measure $\le L^{5/2} N^{\Theta - 1}$, where*

$$\begin{aligned} \Theta = \Theta(\eta) &= \frac{1}{\log 2} \eta \csc^2(\pi/8) \log \frac{1}{\eta \csc^2(\pi/8)} \\ &+ \frac{1}{\log 2}(1 - \eta \csc^2(\pi/8)) \log \frac{1}{1 - \eta \csc^2(\pi/8)}. \end{aligned}$$

Proof. This is [LLW1, Lemma 3].

Lemma 3. *Let h be any even integer, and N sufficiently large. Then the number of solutions of the equation $h = p_1 - p_2$ with $p_j \le N$ is*

$$< 7.8342 C_0 \prod_{\substack{p \mid h \\ p > 2}} \left(\frac{p-1}{p-2} \right) \frac{N}{\log^2 N}$$

uniformly for all even integers $h \ne 0$. Here C_0 is as in (2.6).

Proof. This is [C, Theorem 3].

Lemma 4. *We have*

$$\int_0^1 |S(\alpha, N) G(\alpha)|^2 d\alpha \le \frac{2}{\log^2 2} C_0 C_1 N \tag{3.1}$$

where $C_1 < 24.4189$.

The above integral is trivially $\gg N$, so the upper bound in (3.1) is of correct order of magnitude.

The inequality (3.1) with an unspecified constant in the upper bound was obtained by Romanoff [R] (see also [P], Satz 8.1 on p.173). Therefore Romanoff completed the qualitative estimate for the integral in (3.1) which is, in fact, the sum of squares of the representations of those integers of the form $p + 2^\nu$, with p and ν in suitable ranges. From this he deduced, by Cauchy's inequality, that a positive proportion of the odd integers can be written as $p + 2^\nu$. What we do in our Lemma 4 is to obtain a quantitative result of Romanoff's inequality, i.e. a numerical constant in the upper bound in (3.1).

Proof of Lemma 4. Let $s(N)$ be the number of solutions of the equation

$$p_1 - p_2 = 2^{m_2} - 2^{m_1} \tag{3.2}$$

satisfying

$$p_j \le N \text{ and } m_j \le L,\ j = 1, 2. \tag{3.3}$$

Then we have

$$\int_0^1 |S(\alpha, N)G(\alpha)|^2 d\alpha = s(N). \tag{3.4}$$

By Lemma 3 one has, uniformly for even integer h,

$$|\{p_j \le N:\ \ p_1 - p_2 = h\}| < C_0 C_2 g(h) \frac{N}{\log^2 N},$$

where C_0 is as in (2.6),

$$C_2 = 7.8342 \prod_{p>2} \left(1 - \frac{2}{p(p-1)}\right)^{-1}, \quad g(h) = \prod_{\substack{p|h \\ p>2}} \left(1 + \frac{1}{p}\right). \tag{3.5}$$

Thus for fixed m_1, m_2 with $m_1 \ne m_2$, one sees that

$$|\{p_j \le N:\ \ p_1 - p_2 = 2^{m_2} - 2^{m_1}\}| \le C_0 C_2 g(2^{m_2} - 2^{m_1}) \frac{N}{\log^2 N}.$$

If $m_2 > m_1$, then

$$\begin{aligned} g(2^{m_2} - 2^{m_1}) &= g(2^{m_1}(2^{m_2 - m_1} - 1)) \\ &= g(2^{m_1}) g(2^{m_2 - m_1} - 1) \\ &= g(2^{m_2 - m_1} - 1). \end{aligned}$$

Similarly, $g(2^{m_2} - 2^{m_1}) = g(2^{|m_2 - m_1|} - 1)$ for $m_2 < m_1$. Since for fixed $\ell \ge 1$,

$$|\{m_j \le L:\ \ |m_2 - m_1| = \ell\}| < 2L,$$

one has

$$\begin{aligned} &|\{p_j \le N,\ m_j \le L:\ p_1 - p_2 = 2^{m_2} - 2^{m_1}, m_1 \ne m_2\}| \\ &\qquad < C_0 C_2 \frac{N}{\log^2 N} \sum_{\substack{m_j \le L \\ m_1 \ne m_2}} g(2^{m_2} - 2^{m_1}) \\ &\qquad < \frac{2}{\log 2} C_0 C_2 \frac{N}{\log N} \sum_{1 \le \ell \le L} g(2^\ell - 1). \end{aligned} \tag{3.6}$$

Also, by the prime number theorem, the number of solutions of (3.2) with $m_1 = m_2$ in (3.3) is

$$\pi(N) L \le \left(\frac{1}{\log 2} + \varepsilon\right) N$$

if N is sufficiently large. This in combination with (3.6) gives

$$s(N) < \frac{2}{\log 2} C_0 C_2 \frac{N}{\log N} \sum_{1\le \ell\le L} g(2^\ell - 1) + \left(\frac{1}{\log 2} + \varepsilon\right) N. \tag{3.7}$$

The sum on the right hand side of (3.7) can be transformed as

$$\sum_{1\le\ell\le L} g(2^\ell-1) = \sum_{1\le\ell\le L} \sum_{\substack{d|2^\ell-1 \\ 2\nmid d}} \frac{\mu^2(d)}{d} = \sum_{\substack{d\le N \\ 2\nmid d}} \frac{\mu^2(d)}{d} \sum_{\substack{1\le\ell\le L \\ \varrho(d)|\ell}} 1$$

$$\le L \sum_{\substack{d\le N \\ 2\nmid d}} \frac{\mu^2(d)}{d\varrho(d)} \le \frac{\log N}{\log 2} \sum_{\substack{d=1 \\ 2\nmid d}}^{\infty} \frac{\mu^2(d)}{d\varrho(d)} =: \frac{\log N}{\log 2} C_3, \quad \text{say,}$$

where for odd d, $\varrho(d)$ denotes the least integer $\varrho \ge 1$ for which $2^\varrho \equiv 1 (\bmod\ d)$. Hence (3.7) becomes

$$s(N) \le \left(\frac{2}{\log^2 2} C_0 C_2 C_3 + \frac{1}{\log 2} + \varepsilon\right) N. \tag{3.8}$$

By [HR,p.128], $C_0 > 0.6601$. Also by (3.5), a straightforward computation gives $C_2 < 7.8342 \times 1.8998$.

Now we proceed to estimate C_3. For positive integers x, put

$$c(x) = \sum_{m\le x} \sum_{\varrho(d)=m} \frac{\mu^2(d)}{d}.$$

Then for $x = 1, 2, ..., 9$, we have

x	1	2	3	4	5	6	7	8	9
$c(x) \le$	1	4/3	31/21	183/105	1.7752	1.8228	1.8307	1.9248	1. 9405

To bound $c(x)$ for $x \ge 10$, we let $X = \prod_{d\le x}(2^d - 1)$, and prove that when $x \ge 9$,

$$\frac{2X}{\varphi(2X)} \le 2e^\gamma \log x \tag{3.9}$$

where γ is Euler's constant (so $1.7810 < e^\gamma < 1.7811$). In fact, according to [RS,(3.42)], for every $d \ge 3$ we have

$$\frac{d}{\varphi(d)} < e^\gamma \log\log d + \frac{2.5064}{\log\log d} =: u(d).$$

Since the function $u(d)$ increases as $d \ge 30$,

$$\frac{2X}{\varphi(2X)} \le u(2X) \le u(2^{x^2+1}).$$

If $x \geq 9$, then we have

$$\begin{aligned} u(2^{x^2+1}) &\leq e^\gamma \log\{(x^2+1)\log 2\} + \frac{2.5064}{\log\log 2^{82}} \\ &< e^\gamma \log\left\{\frac{82}{81}x^2 \log 2\right\} + 0.6204 < 2e^\gamma \log x, \end{aligned}$$

and (3.9) follows. Thus if $x \geq 10$, then

$$\begin{aligned} c(x) &\leq \sum_{\substack{2\nmid q \\ q|X}} \frac{\mu^2(q)}{q} = \prod_{p|X}\left(1+\frac{1}{p}\right) = \frac{2}{3}\frac{2X}{\varphi(2X)}\prod_{p|2X}\left(1-\frac{1}{p^2}\right) \\ &\leq \frac{4}{3}e^\gamma(\log x)\prod_{p|2X}\left(1-\frac{1}{p^2}\right) \\ &\leq \frac{4}{3}(1-2^{-2})(1-7^{-2})(1-3^{-2})(1-5^{-2})(1-31^{-2})(1-127^{-2}) \\ &\quad \times(1-17^{-2})(1-73^{-2})e^\gamma \log x \\ &\leq 1.4818\log x. \end{aligned}$$

Obviously, this estimate also holds for non-integral $x \geq 10$.

We therefore conclude that

$$\begin{aligned} C_3 &= \sum_{m=1}^{\infty}\frac{1}{m}\sum_{\varrho(d)=m}\frac{\mu^2(d)}{d} = \int_1^{10}\frac{c(x)}{x^2}dx + \int_{10}^{\infty}\frac{c(x)}{x^2}dx \\ &\leq \sum_{j=1}^{9} c(j)\int_j^{j+1}\frac{1}{x^2}dx + 1.4818\int_{10}^{\infty}\frac{\log x}{x^2}dx \\ &\leq 1.1160 + \frac{1.4818(1+\log 10)}{10} < 1.6054. \end{aligned}$$

Consequently, (3.8) becomes

$$s(N) \leq \frac{2}{\log^2 2}C_0C_1N$$

with

$$C_1 < C_2C_3 + \frac{\log 2}{2C_0} + \varepsilon < 24.4189$$

as stated in the lemma. This in combination with (3.4) gives (3.1). The proof of Lemma 4 is completed.

4. Proof of Theorem 1

Lemma 5. *Assume the GRH. Let P, Q be defined as in (2.1). Then for $\alpha \in C(\mathcal{M})$,*

$$S(\alpha, N) \ll \left(\frac{N}{P} + N^{1/2}Q^{1/2} + \frac{N}{Q^{1/2}}\right) \log^2 N.$$

Proof. This is a consequence of [LLW1, Lemma 1].

Lemma 6. *Let $t_k(n)$ be the number of solutions of the equation $n = 2^{\nu_1} + 2^{\nu_2} + \ldots + 2^{\nu_k}$. Then as $N \to \infty$,*

$$\sum_{n \le N} t_k(n) \sim L^k, \qquad \sum_{NL^{-2} < n \le N} t_k(N-n)n \ge (1 - o(1))NL^k.$$

Proof. The first sum under consideration is the number of k-dimensional vectors $(\nu_1, \ldots, \nu_k)$ such that $2^{\nu_1} + 2^{\nu_2} + \ldots + 2^{\nu_k} \le N$. Thus,

$$\log_2^k(N/k) \le \sum_{n \le N} t_k(n) \le \log_2^k N,$$

and the first estimate follows. The second estimate can be established as follows:

$$\begin{aligned}
\sum_{NL^{-2} < n \le N} t_k(N-n)n &= \sum_{\substack{\nu_1 \le L \\ 2^{\nu_1} + \cdots + 2^{\nu_k} \le N - NL^{-2}}} \cdots \sum_{\nu_k \le L} (N - 2^{\nu_1} - \ldots - 2^{\nu_k}) \\
&\ge \sum_{\nu_1 \le \log_2(\frac{N}{kL})} \cdots \sum_{\nu_k \le \log_2(\frac{N}{kL})} \left(N - \frac{N}{L}\right) \\
&\ge (1 - o(1))NL^k.
\end{aligned}$$

This completes the proof of Lemma 6.

Now we give

Proof of Theorem 1. Let $t_k(N - M)$ be as in Lemma 6, $\mathcal{E}$ as in Lemma 2 and $\mathcal{M}$ as in (2.3) with P, Q determined by (2.4). Then we have

$$\begin{aligned}
r_k(N) &= \int_0^1 S^2(\alpha, N) G^k(\alpha) e(-N\alpha) d\alpha \\
&= \sum_M t_k(N-M) \int_0^1 S^2(\alpha, M) e(-M\alpha) d\alpha \\
&= \sum_M t_k(N-M) \left\{ \int_{\mathcal{M}} + \int_{C(\mathcal{M}) \cap \mathcal{E}} + \int_{C(\mathcal{M}) \cap C(\mathcal{E})} \right\} \\
&\qquad\qquad S^2(\alpha, M) e(-M\alpha) d\alpha \\
&=: J_1 + J_2 + J_3, \qquad \text{say.}
\end{aligned} \tag{4.1}$$

Now we estimate J_1, J_2, J_3 in (4.1) respectively.

To estimate J_1, we bound the contribution from $M \le NL^{-2}$ as follows:

$$\begin{aligned}
&\sum_{M \le NL^{-2}} t_k(N-M) \int_{\mathcal{M}} S^2(\alpha, M) e(-M\alpha) d\alpha \\
&\ll \sum_{M \le NL^{-2}} t_k(N-M) \int_0^1 |S(\alpha, M)|^2 d\alpha \\
&\ll \sum_{M \le NL^{-2}} t_k(N-M) \frac{M}{\log M} \\
&\ll NL^{-3} \sum_{M \le N} t_k(N-M) \ll NL^{k-3},
\end{aligned}$$

on using the first estimate of Lemma 6. The contribution from other M can be estimated by Theorem 2 and the second estimate in Lemma 6, which give

$$\begin{aligned}
&\sum_{NL^{-2} < M \le N} t_k(N-M) \int_{\mathcal{M}} S^2(\alpha, M) e(-M\alpha) d\alpha \\
&\ge \sum_{NL^{-2} < M \le N} t_k(N-M) \mathfrak{S}(M) \frac{M}{\log^2 M} \left\{1 - O\left(\frac{1}{\log M}\right)\right\} \\
&\ge 2C_0 \frac{1}{\log^2 N} \left\{1 - O\left(\frac{1}{\log N}\right)\right\} \sum_{NL^{-2} < M \le N} t_k(N-M) M \\
&\ge 2(1-\varepsilon) C_0 \frac{NL^k}{\log^2 N},
\end{aligned}$$

provided $k \ge 2$ and $N \ge N_{k,\varepsilon}$. Thus if $k \ge 2$ and $N \ge N_{k,\varepsilon}$, then

$$J_1 \ge 2(1-2\varepsilon) C_0 \frac{NL^k}{\log^2 N}. \tag{4.2}$$

To estimate J_2, one notes that

$$J_2 = \int_{C(\mathcal{M}) \cap \mathcal{E}} S^2(\alpha, N) G^k(\alpha) e(-N\alpha) d\alpha.$$

By Lemma 5, we have $S(\alpha, N) \ll N^{3/4} \log^2 N$ for $\alpha \in C(\mathcal{M})$. Let $\eta = 0.0161$ so that the definition of Θ in Lemma 2 gives $\Theta < 0.4998 < 1/2$. Thus Lemma 2 gives

$$J_2 \ll \int_{C(\mathcal{M}) \cap \mathcal{E}} |S(\alpha, N)|^2 |G(\alpha)|^k d\alpha \ll N^{\Theta - 1} N^{3/2} L^{k+7} \ll NL^{k-3}. \tag{4.3}$$

On using Lemmas 2 and 4, the last integral J_3 can be estimated as

$$\begin{aligned} J_3 &= \int_{C(\mathcal{M})\cap C(\mathcal{E})} S^2(\alpha,N)G^k(\alpha)e(-N\alpha)d\alpha \\ &\le \{(1-\eta)L\}^{k-2}\int_0^1 |S(\alpha,N)G(\alpha)|^2 d\alpha \le 2C_0C_1(1-\eta)^{k-2}\frac{NL^k}{\log^2 N}. \end{aligned} \tag{4.4}$$

Inserting (4.2), (4.3) and (4.4) into (4.1), we get

$$r_k(N) \ge 2C_0\frac{NL^k}{\log^2 N}\{1 - C_1(1-\eta)^{k-2} - 3\varepsilon\} \tag{4.5}$$

if $k \ge 2$ and $N \ge N_{k,\varepsilon}$. Also when $k \ge 200$ and $\varepsilon = 10^{-6}$, one has $C_1(1-\eta)^{k-2} + 3\varepsilon < 0.9818$. Consequently if $k \ge 200$ and $N \ge N_k$, then (4.5) becomes

$$r_k(N) \ge 0.0240\frac{NL^k}{\log^2 N},$$

on recalling that $C_0 > 0.6601$. This proves (1.3) and hence Theorem 1.

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