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Substitution invariant Sturmian bisequences

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Résumé. Les suites sturmiennes indexées sur $\mathbb{Z}$, de pente α et d'intercept ρ, sont laissées fixes par une substitution non triviale si et seulement si α est un nombre de Sturm et ρ appartient à $\mathbb{Q}(\alpha)$. On remarque aussi que les suites de Beatty permettent de définir des partitions de l'ensemble des entiers relatifs.

abstract. We prove that a Sturmian bisequence, with slope α and intercept ρ, is fixed by some non-trivial substitution if and only if α is a Sturm number and ρ belongs to $\mathbb{Q}(\alpha)$. We also detail a complementary system of integers connected with Beatty bisequences.

1. Introduction

Beatty sequences $(\lfloor n\alpha+\rho\rfloor)_{n\in\mathbb{N}}$ and $(\lceil n\alpha+\rho\rceil)_{n\in\mathbb{N}}$ have been studied extensively. Many papers deal with the case $\rho = 0$, see [1, 9, 10, 14, 15, 28, 29]. The inhomogeneous case is also discussed from several points of view [6, 7, 16, 20, 21, 22]. By the way, this Note provides a new contribution about *complementary systems* of integers. This problem arose, in various forms, in the works of A. S. Fraenkel [13], R. L. Graham [17] and R. Tijdeman [30, 31].

A natural way to examine Beatty sequences is to consider the class of *Sturmian words* defined by G. A. Hedlund and M. Morse in the context of topological dynamics, see [25, 26]. For further details, both [3] and [8] contain extensive lists of references. Here we are especially interested in substitution invariant Sturmian words. In [27] we elicited properties about some right-sided infinite Sturmian words the *intercept* of which is a particular homography of the *slope*. We therefore obtained a partial generalization of Crisp *et al.*'s main Theorem concerning *cutting sequences* [12]. The aim of this Note is the full characterization of Sturmian bisequences which are fixed by some non-trivial substitution.

2. Definitions and notations

Let $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}^- = \{-1, -2, \dots\}$. Let $\mathbb{Z} = \mathbb{N}^- \cup \mathbb{N}$ and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. We consider the sets $\mathcal{Z}_{\beta,\delta} = \{\lfloor k\beta + \delta \rfloor \mid k \in \mathbb{Z}\}$ and $\mathcal{Z}'_{\beta,\delta} = \{\lceil k\beta + \delta \rceil \mid k \in \mathbb{Z}\}$, with β irrational and δ real. As usual $\lfloor x \rfloor$ is the integer part and $\lceil x \rceil$ the ceiling of any real number x. Let $r_{\beta,\delta}$ and $r'_{\beta,\delta}$ be the generating bisequences of $\mathcal{Z}_{\beta,\delta}$ and $\mathcal{Z}'_{\beta,\delta}$: we set

$$r_{\beta,\delta}(n) = \begin{cases} 1 \text{ if } n \in \mathcal{Z}_{\beta,\delta} \\ 0 \text{ otherwise} \end{cases} \quad \text{and} \quad r'_{\beta,\delta}(n) = \begin{cases} 1 \text{ if } n \in \mathcal{Z}'_{\beta,\delta} \\ 0 \text{ otherwise} \end{cases}$$

for each $n \in \mathbb{Z}$. We say that two subsets of $\mathbb{Z}$ are a complementary system if they form a partition of $\mathbb{Z}$.

Let $\mathcal{A}^*$ be the free monoid generated by the two-letter alphabet $\mathcal{A} = \{0, 1\}$. The set of right-sided infinite words is denoted by $\mathcal{A}^\omega$ and ${}^\omega\mathcal{A}$ is the set of left-sided infinite words. A bisequence is a doubly infinite word and ${}^\omega\mathcal{A}^\omega$ is the set of bisequences over $\mathcal{A}$. We say that the bisequences $\dots v_{-2}v_{-1}v_0v_1v_2\dots$ and $\dots v'_{-2}v'_{-1}v'_0v'_1v'_2\dots$ are equal if there exists an integer $k \in \mathbb{Z}$ such that $v_i = v'_{i+k}$ for each $i \in \mathbb{Z}$. In this event, we note $\dots v_{-2}v_{-1}v_0v_1v_2 \dots \simeq \dots v'_{-2}v'_{-1}v'_0v'_1v'_2\dots$

Let α be irrational and ρ be real. Consider the bisequences $z_{\alpha,\rho}$ and $z'_{\alpha,\rho}$ defined by

$$z_{\alpha,\rho}(n) = \lfloor (n+1)\alpha + \rho \rfloor - \lfloor n\alpha + \rho \rfloor - \lfloor \alpha \rfloor$$

and

$$z'_{\alpha,\rho}(n) = \lceil (n+1)\alpha + \rho \rceil - \lceil n\alpha + \rho \rceil - \lfloor \alpha \rfloor$$

for each $n \in \mathbb{Z}$. A bisequence x is said to be Sturmian if $x \simeq z_{\alpha,\rho}$ or $x \simeq z'_{\alpha,\rho}$ for a suitable choice of α and ρ. It is clear that $z_{\alpha,\rho}(n) = z_{\alpha+1,\rho}(n)$ and $z'_{\alpha,\rho}(n) = z'_{\alpha+1,\rho}(n)$ for each $n \in \mathbb{Z}$, so without loss of generality, we may take $0 < \alpha < 1$. Finally, a right-sided infinite word y is Sturmian if there exist a Sturmian bisequence x and a left-sided infinite word y' such that $x \simeq y'y$. Noting that Sturmian words are intimately related to straight lines in the plane, the number α is the slope and ρ the intercept.

A substitution f is a map from $\mathcal{A}^*$ into itself such that $f(uu') = f(u)f(u')$ for all finite words u and u'. Let $w = w_0w_1w_2\dots$ be a right-sided infinite word. Let Inv be the operator defined by $Inv(w) = \dots w_2w_1w_0$ and $Inv(Inv(w)) = w$. As usual, we set $f(w) = f(w_0)f(w_1)f(w_2)\dots$ and $f(Inv(w)) = \dots f(w_2)f(w_1)f(w_0)$. The image of $\dots v_{-2}v_{-1}v_0v_1v_2\dots$ under f is $\dots f(v_{-2})f(v_{-1})f(v_0)f(v_1)f(v_2)\dots$ A one-sided infinite word y is fixed by f if $f(y) = y$, and a bisequence x is fixed by f if $f(x) \simeq x$.

Moreover a substitution f is *Sturmian* if $f(w)$ is a right-sided infinite Sturmian word whenever w is. F. Mignosi and P. Séébold proved that a substitution f is Sturmian if and only if f is a composition of the three substitutions $E: \begin{array}{l} 0 \mapsto 1 \\ 1 \mapsto 0 \end{array}$, $\varphi: \begin{array}{l} 0 \mapsto 01 \\ 1 \mapsto 0 \end{array}$ and $\tilde{\varphi}: \begin{array}{l} 0 \mapsto 10 \\ 1 \mapsto 0 \end{array}$ in any order and number [24]. A substitution f is *locally Sturmian* if there exists a right-sided infinite Sturmian word w such that $f(w)$ is Sturmian. J. Berstel and P. Séébold stated that any locally Sturmian substitution is actually Sturmian [4, 5].

Furthermore a substitution is non-trivial if it differs from the identical transformation over $\mathcal{A}$. In [27] we proved that if a right-sided infinite Sturmian word is fixed by some non-trivial substitution then its slope α, with $0 < \alpha < 1$, is a *Sturm number*, that is, there exists an integer $n \geq 2$ such that:

$$\alpha = [0, 1 + k_n,\ \overline{k_{n-1}, \ldots, k_2, k_1 + k_n}\ ] \text{ with } (k_1, k_n) \in \mathbb{N}^2 \setminus \{(0,0)\}$$

or

$$\alpha = [0, 1, k_n,\ \overline{k_{n-1}, \ldots, k_2, k_1 + k_n}\ ] \text{ with } (k_1, k_n) \in \mathbb{N}^{*\,2}$$

where the partial quotients $k_2, \ldots, k_{n-1}$ belong to $\mathbb{N}^*$. Remark that these numbers were introduced, in a slightly different way, by Crisp *et al.* [12].

3. Results

As usual, for any quadratic irrational α, let $\mathbb{Q}(\alpha) = \{p + q\alpha \mid (p,q) \in \mathbb{Q}^2\}$ be the splitting field of α over $\mathbb{Q}$. The main result of this Note is the full characterization of Sturmian bisequences which are invariant under some non-trivial substitution:

Theorem 1. *Let x be a Sturmian bisequence with slope $0 < \alpha < 1$. The word x is fixed by some non-trivial substitution if and only if α is a Sturm number and ρ belongs to $\mathbb{Q}(\alpha)$.*

In [27], we computed the slope and the intercept of $f(x)$ for any Sturmian substitution f and any right-sided infinite Sturmian word x. Lemma 2 is a translation of these formulas for Sturmian bisequences:

Lemma 2. *Let α be irrational with $0 < \alpha < 1$ and let ρ be real. Then*

$$E(z_{\alpha,\rho}) \simeq z'_{1-\alpha,1-\rho} \text{ and } \varphi(z_{\alpha,\rho}) \simeq z'_{\frac{1-\alpha}{2-\alpha},\frac{1-\rho}{2-\alpha}} \simeq \tilde{\varphi}(z_{\alpha,\rho}).$$

Moreover

$$E(z'_{\alpha,\rho}) \simeq z_{1-\alpha,1-\rho} \text{ and } \varphi(z'_{\alpha,\rho}) \simeq z_{\frac{1-\alpha}{2-\alpha},\frac{1-\rho}{2-\alpha}} \simeq \tilde{\varphi}(z'_{\alpha,\rho}).$$

The proof of these properties requires a careful study of generating bisequences of Beatty bisequences:

Lemma 3. *Let $\beta > 1$ be irrational and δ be real. For each $n \in \mathbb{Z}$, we have*

$$r_{\beta,\delta}(n) = z'_{\frac{1}{\beta},\frac{-\delta}{\beta}}(n) \text{ and } r'_{\beta,\delta-1}(n) = z_{\frac{1}{\beta},\frac{-\delta}{\beta}}(n).$$

As an immediate corollary, we can characterize the occurrences of a letter in any Sturmian bisequence. More precisely we remark that

$$\{n \in \mathbb{Z} \mid z_{\gamma,\nu}(n) = 1\} = \mathcal{Z}'_{\frac{1}{\gamma},\frac{-\nu}{\gamma}-1} \text{ and } \{n \in \mathbb{Z} \mid z'_{\gamma,\nu}(n) = 1\} = \mathcal{Z}_{\frac{1}{\gamma},\frac{-\nu}{\gamma}}$$

for each γ irrational with $0 < \gamma < 1$ and ν real. This result is a generalization of earlier work of A. S. Fraenkel, M. Mushkin and U. Tassa dealing with the homogeneous case [15]. From Lemma 3 we also obtain a property about complementary systems of integers:

Proposition 4. *Let $\beta > 1$ be irrational and δ be real. Then $\mathcal{Z}_{\beta,\delta}$ and $\mathcal{Z}'_{\frac{\beta}{\beta-1},\frac{-\delta}{\beta-1}-1}$, as well as $\mathcal{Z}'_{\beta,\delta}$ and $\mathcal{Z}_{\frac{\beta}{\beta-1},\frac{-\delta}{\beta-1}+1}$, are complementary systems of integers.*

4. Proofs

First of all, we examine the generating bisequences of Beatty bisequences:

Proof of Lemma 3. Let $n \in \mathbb{Z}$. If $z'_{\frac{1}{\beta},\frac{-\delta}{\beta}}(n) = 1$ we state that

$$\frac{n}{\beta} - \frac{\delta}{\beta} \leq \left\lceil \frac{n}{\beta} - \frac{\delta}{\beta} \right\rceil = \left\lceil \frac{n+1}{\beta} - \frac{\delta}{\beta} \right\rceil - 1 < \frac{n+1}{\beta} - \frac{\delta}{\beta}$$

thus $n \leq \left\lceil \frac{n}{\beta} - \frac{\delta}{\beta} \right\rceil \beta + \delta < n+1$. Next comes $\left\lfloor \left\lceil \frac{n}{\beta} - \frac{\delta}{\beta} \right\rceil \beta + \delta \right\rfloor = n$ and $r_{\beta,\delta}(n) = 1$.
Conversely, if $r_{\beta,\delta}(n) = 1$ there exists an integer $k \in \mathbb{Z}$ such that $\lfloor k\beta + \delta \rfloor = n$. We therefore observe that

$$\left\lceil \frac{n}{\beta} - \frac{\delta}{\beta} \right\rceil - 1 < \frac{n}{\beta} - \frac{\delta}{\beta} \leq k < \frac{n+1}{\beta} - \frac{\delta}{\beta} \leq \left\lceil \frac{n+1}{\beta} - \frac{\delta}{\beta} \right\rceil .$$

It follows that $\left\lceil \frac{n}{\beta} - \frac{\delta}{\beta} \right\rceil \leq k < \left\lceil \frac{n+1}{\beta} - \frac{\delta}{\beta} \right\rceil$ and $z'_{\frac{1}{\beta},\frac{-\delta}{\beta}}(n) = 1$.
The truth of the first statement is now clear, and we turn to the second part of the proof. Let $n \in \mathbb{Z}$. If $z_{\frac{1}{\beta},\frac{-\delta}{\beta}}(n) = 1$ then

$$\frac{n}{\beta} - \frac{\delta}{\beta} < \left\lfloor \frac{n}{\beta} - \frac{\delta}{\beta} \right\rfloor + 1 = \left\lfloor \frac{n+1}{\beta} - \frac{\delta}{\beta} \right\rfloor \leq \frac{n+1}{\beta} - \frac{\delta}{\beta}$$

hence we have

$$n < \left\lfloor \frac{n+1}{\beta} - \frac{\delta}{\beta} \right\rfloor \beta + \delta \leq n+1$$

that is $\left\lceil \left\lfloor \frac{n+1}{\beta} - \frac{\delta}{\beta} \right\rfloor \beta + \delta - 1 \right\rceil = n$. This implies that $r'_{\beta,\delta-1}(n) = 1$.

Conversely, if $r'_{\beta,\delta-1}(n) = 1$ then there exists $k \in \mathbb{Z}$ such that $\lceil k\beta + \delta - 1 \rceil = n$. Thus we check $z_{\frac{1}{\beta}, \frac{-\delta}{\beta}}(n) = 1$ since

$$\left\lfloor \frac{n}{\beta} - \frac{\delta}{\beta} \right\rfloor \leq \frac{n}{\beta} - \frac{\delta}{\beta} < k \leq \left\lfloor \frac{n+1}{\beta} - \frac{\delta}{\beta} \right\rfloor .$$

□

In order to describe the complementary system of integers, connected with a Beatty bisequence, we need to introduce the following Lemma:

Lemma 5. *Let* $0 < \alpha < 1$ *be irrationnal and* ρ *be real. Then* $E(z_{\alpha,\rho}) \simeq z'_{1-\alpha,1-\rho}$ *and* $E(z'_{\alpha,\rho}) \simeq z_{1-\alpha,1-\rho}$.

Proof. We only detail the proof concerning the first result. Let $n \in \mathbb{Z}$. Since the relation $\lfloor a \rfloor = -\lceil -a \rceil$ holds for each real number a, we verify

$$z'_{1-\alpha,1-\rho}(n) = 1 - (\lceil -n\alpha - \rho \rceil - \lceil -(n+1)\alpha - \rho \rceil) = 1 - z_{\alpha,\rho}(n) = E(z_{\alpha,\rho}(n)).$$

□

Proof of Proposition 4. Let $n \in \mathbb{Z}$. From Lemma 3, we remark that

$$n \in \mathcal{Z}_{\beta,\delta} \Leftrightarrow r_{\beta,\delta}(n) = 1 \Leftrightarrow z'_{\frac{1}{\beta}, \frac{-\delta}{\beta}}(n) = 1.$$

Then Lemma 5 implies that

$$n \in \mathcal{Z}_{\beta,\delta} \Leftrightarrow z_{\frac{\beta-1}{\beta}, \frac{\beta+\delta}{\beta}}(n) = 0 \Leftrightarrow r'_{\frac{\beta}{\beta-1}, -\frac{\beta+\delta}{\beta-1}-1}(n) = 0 \Leftrightarrow r'_{\frac{\beta}{\beta-1}, -\frac{\delta}{\beta-1}-1}(n) = 0.$$

In other words, we get

$$n \in \mathcal{Z}_{\beta,\delta} \Leftrightarrow n \notin \mathcal{Z}'_{\frac{\beta}{\beta-1}, -\frac{\delta}{\beta-1}-1}.$$

Furthermore, since $\beta > 1$ we can affirm that any integer occurs at most one time in $\mathcal{Z}_{\beta,\delta}$. Clearly this property also holds for $\mathcal{Z}'_{\frac{\beta}{\beta-1}, -\frac{\delta}{\beta-1}-1}$. In short, the sets $\mathcal{Z}_{\beta,\delta}$ and $\mathcal{Z}'_{\frac{\beta}{\beta-1}, -\frac{\delta}{\beta-1}-1}$ are a complementary system of integers. The part of proof concerning $\mathcal{Z}'_{\beta,\delta}$ and $\mathcal{Z}_{\frac{\beta}{\beta-1}, \frac{-\delta}{\beta-1}+1}$ is similar in all respects. □

From now on we study properties of substitution invariant Sturmian bisequences.

Proof of Lemma 2. Assume first that $0 \leq \rho < 1$. We split the bisequence $z_{\alpha,\rho}$ into the words

$$w = z_{\alpha,\rho}(0) z_{\alpha,\rho}(1) \ldots z_{\alpha,\rho}(m) \cdots \in \mathcal{A}^{\omega}$$

and

$$w' = \ldots z_{\alpha,\rho}(-m) \ldots z_{\alpha,\rho}(-2) z_{\alpha,\rho}(-1) \in {}^{\omega}\mathcal{A}.$$

Let $\varphi(w) = y_0y_1\dots$ with $y_j \in \mathcal{A}$ for $j = 0, 1, \dots$ We observe that $y_0 = 0 = z'_{\frac{1-\alpha}{2-\alpha},\frac{1-\rho}{2-\alpha}}(0)$. Let n_{q+1} be the $(q+1)$-th occurrence of the letter 0 in the word $\varphi(w)$ for each $q \geq 1$. We easily check:

$$n_{q+1} = (q + \sum_{i=0}^{q-1}(1 - z_{\alpha,\rho}(i)) + 1) - 1 = 2q - \lfloor q\alpha + \rho \rfloor = \lceil q(2-\alpha) - \rho \rceil.$$

For each $n \geq 1$ we state that:

$$\begin{aligned} y_n = 0 &\Leftrightarrow \exists\, q \in \mathbb{N}^* \;\; n = \lceil q(2-\alpha) - \rho \rceil \\ &\Leftrightarrow \exists\, q \in \mathbb{Z} \;\; n = \lceil q(2-\alpha) - \rho \rceil \\ &\Leftrightarrow r'_{2-\alpha,-\rho}(n) = 1 \\ &\Leftrightarrow z_{\frac{1}{2-\alpha},\frac{\rho-1}{2-\alpha}}(n) = 1. \end{aligned}$$

From Lemma 5, we prove that $y_n = 0$ if and only if $z'_{\frac{1-\alpha}{2-\alpha},\frac{1-\rho}{2-\alpha}}(n) = 0$. In short we obtain $\varphi(w) = (z'_{\frac{1-\alpha}{2-\alpha},\frac{1-\rho}{2-\alpha}}(n))_{n\in\mathbb{N}}$. To compute $\varphi(w')$, we remark that

$$w' = \dots z'_{\alpha,1-\rho}(m) \dots z'_{\alpha,1-\rho}(1) z'_{\alpha,1-\rho}(0).$$

Indeed, for each $n \in \mathbb{N}^*$ it is clear that

$$\begin{aligned} z_{\alpha,\rho}(-n) &= \lfloor (-n+1)\alpha + \rho \rfloor - \lfloor -n\alpha + \rho \rfloor - \lfloor \alpha \rfloor \\ &= -\lceil (n-1)\alpha - \rho \rceil + \lceil n\alpha - \rho \rceil - \lfloor \alpha \rfloor \end{aligned}$$

hence

$$z_{\alpha,\rho}(-n) = z'_{\alpha,-\rho}(n-1) = z'_{\alpha,1-\rho}(n-1).$$

If we write $w' = \dots a_m \dots a_1 a_0$ over ${}^{\omega}\mathcal{A}$, we get

$$\varphi(w') = \dots 01^{1-a_m} \dots 01^{1-a_1} 01^{1-a_0}$$

because $\varphi(0) = 01$ and $\varphi(1) = 0$. We can deduce that

$$Inv(\varphi(w')) = 1^{1-a_0}01^{1-a_1}0 \dots 1^{1-a_m}0 \dots = \tilde{\varphi}(a_0a_1 \dots a_m \dots)$$

and $\varphi(w') = Inv(\tilde{\varphi}((z'_{\alpha,1-\rho}(n))_{n\in\mathbb{N}}))$. Much as above, we verify

$$\varphi((z'_{\alpha,1-\rho}(n))_{n\in\mathbb{N}}) = (z_{\frac{1-\alpha}{2-\alpha},\frac{\rho}{2-\alpha}}(n))_{n\in\mathbb{N}}.$$

Moreover we observe that $\tilde{\varphi}(a) = 1^{1-a}0$ and $\varphi(a) = 01^{1-a}$ for each $a \in \{0,1\}$. Next comes $\varphi(u) = 0\tilde{\varphi}(u)$ for any $u \in \mathcal{A}^{\omega}$, and consequently $\varphi(w') = Inv((z_{\frac{1-\alpha}{2-\alpha},\frac{1-\alpha+\rho}{2-\alpha}}(n))_{n\in\mathbb{N}})$. Bearing in mind that $z_{\alpha,\rho} \simeq w'w$, and noting that $z_{\frac{1-\alpha}{2-\alpha},\frac{1-\alpha+\rho}{2-\alpha}}(n) = z'_{\frac{1-\alpha}{2-\alpha},\frac{1-\rho}{2-\alpha}}(-n-1)$ for each $n \in \mathbb{N}$, we finally obtain $\varphi(z_{\alpha,\rho}) \simeq z'_{\frac{1-\alpha}{2-\alpha},\frac{1-\rho}{2-\alpha}}$. To conclude, we must prove that the relation

$\varphi(z_{\alpha,\rho+k}) \simeq z'_{\frac{1-\alpha}{2-\alpha},\frac{1-(\rho+k)}{2-\alpha}}$ holds for each $k \in \mathbb{Z}$. Since $z'_{\beta,\delta+1} \simeq z'_{\beta,\delta} \simeq z'_{\beta,\delta+\beta}$ for arbitrarily β irrational and δ real, we directly claim:

$$z'_{\frac{1-\alpha}{2-\alpha},\frac{1-(\rho+k)}{2-\alpha}} \simeq z'_{\frac{1-\alpha}{2-\alpha},\frac{1-(\rho+k)}{2-\alpha}+k-k\frac{1-\alpha}{2-\alpha}} \simeq z'_{\frac{1-\alpha}{2-\alpha},\frac{1-\rho}{2-\alpha}} \simeq \varphi(z_{\alpha,\rho}) \simeq \varphi(z_{\alpha,\rho+k}).$$

The computation of $\tilde{\varphi}(z_{\alpha,\rho+k})$ becomes trivial because we have $\tilde{\varphi}(v) \simeq \varphi(v)$ for each $v \in {}^{\omega}\mathcal{A}^{\omega}$. Finally, the part of proof concerning $z'_{\alpha,\rho}$ is similar in all respects. □

For each Sturmian substitution f it is therefore clear that $f(x)$ is a Sturmian bisequence whenever x is. Now we turn to the proof of Theorem 1. Some preliminaries are required. Let x and y be two Sturmian bisequences. Let f be a substitution such that $f(x) \simeq y$. There exist a word $x' \in {}^{\omega}\mathcal{A}$ and a right-sided infinite Sturmian word x'' such that $x \simeq x'x''$. Since we have $y \simeq f(x')f(x'')$, the word $f(x'')$ is a right-sided infinite Sturmian word. Thus f is locally Sturmian and consequently f belongs to the monoid $\{E, \varphi, \tilde{\varphi}\}^*$.

Let us recall some basic properties about Sturmian bisequences. For any irrational α we set $\mathbb{Z} + \mathbb{Z}\alpha = \{a + b\alpha \mid (a, b) \in \mathbb{Z}^2\}$. Let Δ be the set of couples (β, δ) with $0 < \beta < 1$ irrational and δ real. We also set $\mathcal{U} = \{(\beta, \delta) \in \Delta \mid \forall\, k \in \mathbb{Z} \;\; k\beta + \delta \notin \mathbb{Z}\}$. Let $(\alpha, \rho) \in \Delta$ and $(\alpha', \rho') \in \Delta$. We have $z_{\alpha,\rho} \simeq z_{\alpha',\rho'}$ if and only if $\alpha = \alpha'$ and $\rho - \rho' \in \mathbb{Z} + \mathbb{Z}\alpha$, see [26]. A similar result can be stated from the relation $z'_{\alpha,\rho} \simeq z'_{\alpha',\rho'}$. Furthermore, if $z_{\alpha,\rho} \simeq z'_{\alpha',\rho'}$ then (α, ρ) belongs to $\mathcal{U}$ and $z_{\alpha,\rho} \simeq z'_{\alpha,\rho}$. In short, if two Sturmian bisequences are equal then they have the same slope in $]0, 1[$. Bearing these remarks in mind, we therefore obtain:

Lemma 6. *Let x be a Sturmian bisequence with slope $0 < \alpha < 1$. If x is invariant under some non-trivial substitution then α is a Sturm number.*

Proof (Sketch). Assume that there exists a non-trivial substitution f such that $f(x) \simeq x$. Then f belongs to $\{E, \varphi, \tilde{\varphi}\}^*$. Let $\beta \in]0, 1[$ be the slope of $f(x)$ which is obtained by Lemma 2. Clearly this computation can be done regardless of intercepts, and there exists an homography h, with integer coefficients, such that $\beta = h(\alpha)$. Therefore it only remains to solve the equation $\alpha = h(\alpha)$. In this context, we have yet observed that α is a Sturm number: for a full characterization of the homographies connected with Sturmian substitutions, see the proof of Theorem 1 in [27]. □

In order to prove our main result, we add here a new necessary condition of invariance:

Lemma 7. *Let x be a Sturmian bisequence, with slope α and intercept ρ. If x is invariant under some non-trivial substitution then ρ belongs to $\mathbb{Q}(\alpha)$.*

Proof. Assume, without loss of generality, that $0 < \alpha < 1$. Let f be a non-trivial substitution such that $f(x) \simeq x$. Lemma 6 implies that α is a Sturm number. Since α is a quadratic irrational, the image of α under any homography, with integer coefficients, belongs to $\mathbb{Q}(\alpha)$. Using Lemma 2, we compute the image of x under f. Let β be the slope and δ be the intercept we obtain. It is clear that $\beta \in \mathbb{Q}(\alpha)$ and $0 < \beta < 1$. We also remark that $\delta \in \mathbb{Q}(\alpha)+\rho\mathbb{Q}(\alpha)$. Since $f(x) \simeq x$, we must check $\beta = \alpha$ and $\delta - \rho \in \mathbb{Z}+\mathbb{Z}\alpha$. Next comes $\rho \in \mathbb{Q}(\alpha)$. $\square$

Combining Lemmas 6 and 7, we establish the "only if part" of Theorem 1. Now we turn to the proof of the "if part": the idea is to use some properties that we reported in [27]. First of all, a technical result concerning Sturmian continuations is required [26].

Definition 8 (cf. [26])**.** Let y be a right-sided infinite Sturmian word. A Sturmian continuation of y is a left-sided infinite word y' such that $y'y$ is a Sturmian bisequence.

Lemma 9 (cf. [26])**.** *Let α be irrational with $0 < \alpha < 1$ and ρ be real. Each right-sided infinite Sturmian word y, with slope α and intercept ρ, admits at least one and at most two Sturmian continuations. In the case where y admits different Sturmian continuations there exist two integers $k_1 \in \mathbb{Z}$ and $k_2 \in \mathbb{N}^*$ such that $\rho = k_1 + k_2\alpha$.*

Definition 10 (cf. [27])**.** For each $m \geq 1$, we set

$$\mathcal{C}'(m) = \{(a,b) \in \mathbb{Z}^2 \mid 0 \leq a+b \leq m,\ 0 \leq a \leq m\} \setminus \{(m,0)\}.$$

A right-sided infinite Sturmian word y is said to be permitted if there exist an irrational α with $0 < \alpha < 1$, an integer $m \geq 1$ and a couple of integers $(a,b) \in \mathcal{C}'(m)$ such that $y = (z_{\alpha,\frac{a}{m}+\frac{b}{m}\alpha}(n))_{n\in\mathbb{N}}$ or $y = (z'_{\alpha,\frac{a}{m}+\frac{b}{m}\alpha}(n))_{n\in\mathbb{N}}$.

Proposition 11 (cf. [27])**.** *Let α be a Sturm number. Each permitted word y, with slope α, is invariant under some non-trivial substitution.*

Proof of Theorem 1. Let α be a Sturm number and $\rho \in \mathbb{Q}(\alpha)$. Let x be a Sturmian bisequence such that $x \simeq z_{\alpha,\rho}$. Clearly there exists $(a,b,n) \in \mathbb{Z}^3$ with $n \geq 1$ such that $\rho = \frac{a+b\alpha}{n}$. Moreover, since $z_{\alpha,\delta+1} \simeq z_{\alpha,\delta} \simeq z_{\alpha,\delta+\alpha}$ for each real δ, we actually have $x \simeq z_{\alpha,\frac{a \pmod n + (b \pmod n)\alpha}{n}}$. As usual, the residue $i \pmod n$ is the integer j, with $0 \leq j < n$, such that there exists an integer $k \in \mathbb{Z}$ satisfying $j = i + kn$. For each real δ we set

$$z^+_{\alpha,\delta} = z_{\alpha,\delta}(0)z_{\alpha,\delta}(1)\ldots \quad \text{and} \quad \ldots z_{\alpha,\delta}(-2)z_{\alpha,\delta}(-1) = z^-_{\alpha,\delta}.$$

We first assume that $a \pmod n + b \pmod n \leq n$. Then

$$y = z^+_{\alpha,\frac{a \pmod n + (b \pmod n)\alpha}{n}}$$

is a right-sided infinite permitted word. From Proposition 11, it follows that there exists a non-trivial Sturmian substitution f such that $f(y) = y$. Noting that

$$\begin{aligned} x &\simeq z_{\alpha,\frac{a \pmod n + (b \pmod n)\alpha}{n}} \\ &\simeq z^-_{\alpha,\frac{a \pmod n + (b \pmod n)\alpha}{n}} z^+_{\alpha,\frac{a \pmod n + (b \pmod n)\alpha}{n}} \end{aligned}$$

we have

$$f(x) \simeq f(z^-_{\alpha,\frac{a \pmod n + (b \pmod n)\alpha}{n}}) z^+_{\alpha,\frac{a \pmod n + (b \pmod n)\alpha}{n}}.$$

Hence the word y admits $z^-_{\alpha,\frac{a \pmod n + (b \pmod n)\alpha}{n}}$ and $f(z^-_{\alpha,\frac{a \pmod n + (b \pmod n)\alpha}{n}})$ as Sturmian continuations. If the relation

$$f(z^-_{\alpha,\frac{a \pmod n + (b \pmod n)\alpha}{n}}) = z^-_{\alpha,\frac{a \pmod n + (b \pmod n)\alpha}{n}}$$

is not valid then Lemma 9 implies that there exists $(k_1, k_2) \in \mathbb{Z}^2$ with $k_2 \geq 1$ such that

$$\frac{a \pmod n + (b \pmod n)\alpha}{n} = k_1 + k_2\alpha.$$

In this event, since α is irrational we observe that $k_2 = 0$, which leads to a contradiction. We therefore obtain $f(x) \simeq x$.

If $n+1 \leq a \pmod n + b \pmod n$ we state that $(a \pmod n, (b \pmod n)) - n)$ belongs to $\mathcal{C}'(n)$. Since $x \simeq z_{\alpha,\frac{a \pmod n + ((b \pmod n)) - n)\alpha}{n}}$ we easily verify that there exists a non-trivial substitution g such that $g(x) \simeq x$.

There are no other possibilities and the truth of the claim is now clear for the word $z_{\alpha,\rho}$. The proof concerning $z'_{\alpha,\rho}$ is similar in all respects. □

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