Journal de Théorie des Nombres
de Bordeaux **11** (1999), 15–30

A C^*-dynamical system with Dedekind zeta partition function and spontaneous symmetry breaking

par Paula B. COHEN

Résumé. Dans cet article nous étendons une construction de Bost-Connes, au cas d'un corps de nombres quelconque, d'un C^*-système dynamique à brisure spontanée de symétrie et fonction de partition la fonction zeta de Riemann.

Abstract. In this paper we extend to arbitrary number fields a construction of Bost-Connes of a C^*-dynamical sytem with spontaneous symmetry breaking and partition function the Riemann zeta function.

1. Introduction

In [BC], J-B. Bost and A. Connes, motivated most notably by work of B. Julia (see for example [J]), develop the idea that by displaying the Riemann zeta function as the partition function of a dynamical system with spontaneous symmetry breaking at the pole of the zeta function, one can gain insight into the statistics of the primes of the field of rational numbers using the tools of quantum statistical mechanics. Their construction of such a dynamical system as a 1-parameter automorphism group on an appropriate Hecke algebra has done a lot to enrich the dictionary between concepts from number theory and concepts from quantum statistical mechanics. Moreover, it has been a motivation and guide for the considerations of the proposed approach to the Riemann Hypothesis in [C].

A generalization of the work of [BC] to the case of arbitrary global fields was proposed in [HaLe]. In the number field case, a Hecke algebra construction using semi-group crossed products was proposed in [ALR], see also [LR1] and [LR2]. A more general study developed in [L] applies in particular to the dynamics on this algebra for the class number 1 case. In the present article, which owes a great deal to both approaches, we construct a different generalization for number fields of the dynamical system of [BC] having the full Dedekind zeta function of the number field as partition function. In [HaLe] and [L] this is only achieved when the number field has class number 1. For class number greater than 1, the construction of [HaLe] is not canonical and the partition function recovered is the Dedekind zeta function with a finite number of Euler factors removed. The advantage of our treatment comes from viewing, by contrast to these other approaches, the ideals rather than just the principal ideals as playing the same role as the positive integers do in [BC].

The dynamical system we construct has a natural symmetry group which displays the phenomenon of spontaneous symmetry breaking at the pole of the Dedekind zeta function. In mechanical terms, this means that for inverse temperature β less than 1 the temperature is high enough to create disorder in the system, so that the equilibrium state is unique and invariant under the action of the symmetry group. At the critical temperature $\beta = 1$ a phase transition occurs, so that for $\beta > 1$, when the temperature is low enough, the particles of the system start to align and the symmetry is broken. The equilibrium states are then no longer unique and the symmetry group acts on the extremal points of the compact convex space of equilibrium states. These equilibrium states are, in the C^*-algebraic formulation, the KMS_β states. In [BC], §1, an overview of the C^*-algebraic approach to quantum statistical mechanics is given, including the definition of KMS_β states.

I would like to thank M. Laca and I. Raeburn of Newcastle University, Australia, for their hospitality in August 1996 and August 1997 and for introducing me to their work with J. Arledge and to the developments subsequently pursued by M. Laca. I also thank D. Harari and E. Leichtnam for their interest in my ideas.

2. Statement of the Theorem

We begin by introducing some notations and conventions. Let K be a number field of degree d over $\mathbb{Q}$ and with ring of integers $\mathcal{O}$. Let M_K be the set of places of K and M_K^o the subset of finite places. For $v \in M_K$ we choose a valuation $|\ |_v$ normalised as follows: let K_v be the completion of K at v and $d_v = [K_v : \mathbb{Q}_v]$ be the local degree. Then for $x \in K_v$ we set

$$|x|_v = \|x\|_v^{\frac{d_v}{d}}$$

where $\|\ \|_v$ is the unique valuation on K_v extending the usual p-adic or archimedean valuation on $\mathbb{Q}_v$. In particular, for this normalisation we have for all $x \in K, x \neq 0$ the product formula

$$\prod_{v \in M_K} |x|_v = 1.$$

For $v \in M_K^o$, let

$$\mathcal{O}_v = \{x \in K_v : |x|_v \leq 1\}$$

be the ring of integers of K_v with unit subgroup

$$\mathcal{O}_v^* = \{x \in K_v : |x|_v = 1\}.$$

For most of this article we shall work with the finite adèles A, that is the restricted product of the K_v with respect to the $\mathcal{O}_v$, $v \in M_K^o$. An element $a \in A$ is therefore an infinite vector $a = (a_v)_{v\in M_K^o}$ indexed by the set M_K^o with $a_v \in K_v$ and $a_v \in \mathcal{O}_v$ for all but finitely many $v \in M_K^o$. The finite adèles form a ring with respect to component-wise addition and multiplication. We have a natural embedding $K \hookrightarrow A$, whose image is called the principal adèles, which is just the diagonal embedding $x \mapsto (x)_{v\in M_K^o}$ induced by the embeddings of K into K_v for $v \in M_K^o$. The image under this embedding of an element $a \in \mathcal{O}$ is an element of $R = \prod_{v\in M_K^o} \mathcal{O}_v$, the maximal compact subring of A. Let $W = \prod_{v\in M_K^o} \mathcal{O}_v^*$ be the group of units of R. Let J be the group of finite idèles, that is the group of invertible elements of A consisting of the restricted product of $K_v^* = K_v \setminus \{0\}$ with respect to $\mathcal{O}_v^*$, $v \in M_K^o$. An element $j \in J$ is therefore an infinite vector $j = (j_v)_{v\in M_K^o}$ indexed by the set M_K^o with $j_v \in K_v^*$ and $j_v \in \mathcal{O}_v^*$ for all but finitely many $v \in M_K^o$. The module on J is defined by

$$| \ | : J \to \mathbb{R}_+^*$$

$$| \ | : j = (j_v)_v \mapsto \prod_{v\in M_K^o} |j_v|_v.$$

The image under the diagonal embedding of K^* into J is called the principal idèle group. The semigroup $I = J \cap R$ satisfies $I^{-1}I = J$. For $v \in M_K^o$, let $\mathfrak{P}_v$ be the prime ideal associated with v,

$$\mathfrak{P}_v = \{a \in \mathcal{O};\, |a|_v < 1\}.$$

The quotient $\mathcal{O}/\mathfrak{P}_v$ has finite cardinality $N(\mathfrak{P}_v)$ and for any $a_v \in \mathcal{O}_v$, there is a unique integer $\mathrm{ord}_v a_v$ such that $\mathrm{Card}(\mathcal{O}_v/a_v\mathcal{O}_v) = N(\mathfrak{P}_v)^{\mathrm{ord}_v a_v}$. For $a \in I$ we denote by $\mathfrak{A}$ the ideal

$$\mathfrak{A} = \prod_{v\in M_K^o} \mathfrak{P}_v^{\mathrm{ord}_v a_v}$$

and call it the ideal associated to a. It is well-defined as $\mathrm{ord}_v a_v = 0$ for almost all $v \in M_K^o$, and

$$N(\mathfrak{A}) = \mathrm{Card}(\mathcal{O}/\mathfrak{A}) = \prod_{v\in M_K^o} N(\mathfrak{P}_v)^{\mathrm{ord}_v a_v}.$$

Moreover, we have in this way a short exact sequence, with arrows semigroup homomorphisms,

$$1 \to W \to I \to \mathcal{I} \to 1,$$

where $\mathcal{I}$ denotes the semigroup of integral ideals of $\mathcal{O}$. By the Strong Approximation Theorem, there are additive isomorphisms between $A/R \simeq \oplus_{v\in M_K^o} K_v/\mathcal{O}_v$ and $K/\mathcal{O}$. The semigroup I acts on A by multiplication and preserves R, so that multiplication by $a \in I$ induces an endomorphism of A/R

$$x \mapsto a\cdot x, \quad x \in A/R.$$

If $a\in I$ and $y \in A/R$ the equation $a\cdot x = y$ has $N(\mathfrak{A})$ solutions in A/R where $\mathfrak{A}$ is the ideal associated to a. We denote the set of these solutions by $[x : a\cdot x = y]$. Let $\mathbb{C}(A/R) =: \operatorname{span}\{\delta_x : x \in A/R\}$. The formula

$$\alpha_a(\delta_y) = \frac{1}{N(\mathfrak{A})}\sum_{[x:\, a\cdot x=y]} \delta_x$$

where $a \in I$ and $\mathfrak{A}\in\mathcal{I}$ is the associated ideal, defines an action of I by endomorphisms of the associated C^*-algebra $C^*(A/R)$.

Let $+ : \mathcal{I} \to I$ with $+ : j \mapsto j^+$ denote any splitting semigroup homomorphism of the above exact sequence such that $\mathcal{O}^+ = (1,1,\dots)$, the identity in I, and such that for any principal prime ideal $\pi\mathcal{O}$ with generator π (so that we could replace π by $u\pi$ for any unit u of $\mathcal{O}$) we have $(\pi\mathcal{O})^+ = (\pi,\pi,\dots)$, the image of the natural embedding of π as a principal idèle. This condition is essential to ensure that there is sufficient interaction between the different primes to exhibit the phenomenon of spontaneous symmetry breaking. We call such a splitting of the short exact sequence an interactive splitting. Let I_+ be the sub-semigroup of I given by the image of $\mathcal{I}$ under a fixed map $+$. We take as our basic algebra the crossed product associated to the triple $(C^*(A/R), I_+, \alpha)$ in the sense of [ALR], see also §4 of the present paper. This is the universal object for covariant representations of this triple, namely pairs (π, V) where π is a unital representation of $C^*(A/R)$ on a Hilbert space $\mathcal{H}$ and V is an isometric representation of I on $\mathcal{H}$ satisfying

$$\pi(\alpha_a(f)) = V_a\pi(f)V_a^*, \qquad a\in I,\ f\in C^*(A/R).$$

We denote this semigroup crossed product by $C_K = C^*(A/R) \rtimes_\alpha I_+$. It is the universal C^*-algebra generated by $\{e(x) : x \in A/R\}$ and $\{\mu_a : a \in I_+\}$ subject to the relations

$$\begin{aligned}
&\mu_a^*\mu_a = 1, \quad a\in I_+\\
&\mu_a\mu_b = \mu_{ab}, \quad a,b\in I_+\\
&e(0) = 1,\ e(x)^* = e(-x),\ e(x)e(y) = e(x+y), \quad x,y\in A/R\\
&\frac{1}{N(\mathfrak{A})}\sum_{[x:\, a\cdot x=y]} e(x) = \mu_a e(y)\mu_a^*, \quad a\in I_+,\ y\in A/R
\end{aligned}$$

where $\mathfrak{A}$ is the ideal associated to a.

Let $\{\sigma_t; t \in \mathbb{R}\}$ be the 1-parameter automorphism group of C_K given by the following action on the symbols $e(x)$, $x \in A/R$ and μ_a, $a \in I_+$,

$$\sigma_t(e(x)) = e(x), \quad \sigma_t(\mu_a) = N(\mathfrak{A})^{it}\mu_a, \quad x \in A/R,\, a \in I_+,\, t \in \mathbb{R},$$

where $\mathfrak{A}$ is the ideal associated to a. Consider the Hilbert space $l^2(I_+)$ and let $(\varepsilon_a)_{a\in I_+}$ be the standard orthonormal basis. Define an unbounded positive operator H on $l^2(I_+)$ by

$$H\varepsilon_a = \log(N(\mathfrak{A}))\varepsilon_a, \quad a \in I_+.$$

Notice that $H\varepsilon_1 = 0$ where $1 = (1)_{v\in M_K^\circ}$. Consider, for a fixed admissible character χ (see §5) on A/R and any $u \in W$, the involutive representation ρ_u of C_K on $l^2(I_+)$ which is the unique extension of the representation defined by

$$\rho_u(\mu_a)\varepsilon_b = \varepsilon_{ab}, \quad a, b \in I_+$$
$$\rho_u(e(y))\varepsilon_b = \chi((ub)\cdot y)\varepsilon_b, \quad b \in I_+,\ \ y \in A/R.$$

We have

$$\rho_u(\sigma_t(x)) = e^{itH}\rho_u(x)e^{-itH}, \quad x \in C_K.$$

The main result of this paper is the following generalization of Theorem 5 of [BC] and Théorème 0.1 of [HaLe], see also Proposition 46 of [L]. Its proof follows closely the treatments of [BC] and [HaLe], although differences do arise and we content ourselves in §5 with an explanation of how to handle them, leaving the remaining details to the reader.

Theorem. *Let K be a number field. The C^*-dynamical system (C_K, σ_t) has symmetry group W, with the action $[u] \in \mathrm{Aut}(C_K)$ of $u \in W$ given on $e(x)$, $x \in A/R$ and μ_a, $a \in I_+$ by*

$$[u] : e(x) \mapsto e(u \cdot x),\, x \in A/R \quad , \quad [u] : \mu_a \mapsto \mu_a,\, a \in I_+.$$

This action commutes with σ, so that $[u] \circ \sigma_t = \sigma_t \circ [u]$ for $u \in W, t \in \mathbb{R}$. Moreover,

(1) for $0 < \beta \le 1$, there is a unique KMS_β state Φ_β. It is a factorial state of Type III_1 and the associated factor is the Araki-Woods factor R_∞.

(2) for $\beta > 1$ and $u \in W$, the state

$$\Phi_{\beta,u}(x) = \zeta_K(\beta)^{-1}\, Trace(\rho_u(x)e^{-\beta H}), \ \ x \in C_K$$

is a KMS_β state on (C_K, σ_t) which is factorial of Type I_∞ where

$$\zeta_K(\beta) = \sum_{\mathfrak{A}\in\mathcal{I}} \frac{1}{N(\mathfrak{A})^\beta} = Trace(e^{-\beta H})$$

is the Dedekind zeta function (at β) of the number field K. The action of W on C_K induces an action on these KMS_β states which permutes them transitively and the map $u \mapsto \Phi_{\beta,u}$ is a homeomorphism of the compact group W into the space $\mathcal{E}(K_\beta)$ of extreme points of the convex compact Choquet simplex K_β of KMS_β states on (C_K, σ_t).

(3) *the Dedekind zeta function* ζ_K *of the number field* K *is the partition function of* (C_K, σ_t).

3. A system without interaction

Generalising §2 of [BC], we can construct a non-interactive system, which will be useful in the sequel, as follows. Let $\mathcal{P}$ be the set of prime ideals of $\mathcal{O}$ and S be the second quantisation functor (as in [BC], p416).

Proposition 1. (a) *For every prime ideal* $\mathfrak{P}$, *let* $\mu_{\mathfrak{P}}$ *be the isometry in* $\mathrm{S}l^2(\mathcal{P}) = l^2(\mathcal{I})$ *given by the polar decomposition of the creation operator associated to the unit vector* $\varepsilon_{\mathfrak{P}} \in \mathrm{S}l^2(\mathcal{P})$, *where* $\{\varepsilon_{\mathfrak{B}}, \mathfrak{B} \in \mathcal{I}\}$ *denotes the standard orthonormal basis of* $l^2(\mathcal{I})$. *The* C^**-algebra* $C^*(\mathcal{I})$ *generated by the* $\mu_{\mathfrak{P}}$ *with* $\mathfrak{P}$ *prime is the same as that generated by the isometries* $\mu_{\mathfrak{A}}, \mathfrak{A} \in \mathcal{I}$ *defined by,*

$$\mu_{\mathfrak{A}}\varepsilon_{\mathfrak{B}} = \varepsilon_{\mathfrak{A}\mathfrak{B}}, \quad \mathfrak{B} \in \mathcal{I}.$$

The C^**-algebra* $C^*(I_+)$ *generated by the* μ_a, $a \in I_+$ *is isomorphic to* $C^*(\mathcal{I})$.

(b) *Let* $\tau_{\mathfrak{P}}$ *be the Toeplitz* C^**-algebra generated by* $\mu_{\mathfrak{P}}$. *Then* $C^*(\mathcal{I})$ *is the infinite tensor product*

$$C^*(\mathcal{I}) = \otimes_{\mathfrak{P}\in\mathcal{P}}\tau_{\mathfrak{P}}.$$

(c) *Let* H *be the operator in* $l^2(\mathcal{I})$ *given by*

$$H\varepsilon_{\mathfrak{B}} = \log(N(\mathfrak{B}))\varepsilon_{\mathfrak{B}}, \quad \mathfrak{B} \in \mathcal{I},$$

then the equality

$$\sigma_t(x) = e^{itH}xe^{-itH}, \quad x \in C^*(\mathcal{I})$$

defines a 1-parameter group of automorphisms of $C^*(\mathcal{I})$ *which may be factorised as*

$$\sigma_t = \otimes_{\mathfrak{P}\in\mathcal{P}}\sigma_{t,\mathfrak{P}}$$

where

$$\sigma_{t,\mathfrak{P}}(\mu_{\mathfrak{P}}) = N(\mathfrak{P})^{it}\mu_{\mathfrak{P}}.$$

Proof. By analogy with the proof of Proposition 7 of [BC]. □

Notice that the dynamical system σ_t, once transported to $C^*(I_+)$, is the restriction to that algebra of the dynamical system σ_t on C_K defined in §2.

Similar arguments to that given in [BC], Proposition 8 for the case $K = \mathbb{Q}$ give the following result which underlines the fact that, as the above system is an infinite tensor product of non-interacting systems, it displays no phenomenon of spontaneous symmetry breaking at a critical temperature.

Proposition 2. *(a) For every $\beta > 0$, there is a unique KMS_β state on $(C^*(\mathcal{I}), \sigma_t)$. It is the infinite tensor product*

$$\Phi_\beta = \otimes_{\mathfrak{P}\in\mathcal{P}} \Phi_{\beta,\mathfrak{P}}$$

where $\Phi_{\beta,\mathfrak{P}}$ is the unique KMS_β state on $(\tau_{\mathfrak{P}}, \sigma_{t,\mathfrak{P}})$. The eigenvalue list of $\Phi_{\beta,\mathfrak{P}}$ is

$$\{(1 - N(\mathfrak{P})^{-\beta})N(\mathfrak{P})^{-n\beta} : n \in \mathbb{N}\}.$$

(b) For $\beta > 1$, the state Φ_β is of Type I_∞ and is given by

$$\Phi_\beta(x) = \zeta_K(\beta)^{-1}\, \mathit{Trace}(xe^{-\beta H}), \quad x \in C^*(\mathcal{I}).$$

(c) For $0 < \beta \leq 1$ the state Φ_β is of Type III_1 and the associated factor is the Araki-Woods factor R_∞.

4. Semigroup crossed products and a system with interaction

In this section, we recall some basic facts on semigroup crossed product C^*-algebras needed for the construction of C_K. We use as references the articles [LR1], [LR2] and [ALR] (see [LR1] and [L] for some historical background including the relation to work of Nica [Ni]). Our semigroups will all be abelian without zero divisors. A semigroup system is a triple $(\mathcal{A}, S, \alpha)$ consisting of a separable unital C^*-algebra $\mathcal{A}$, a semigroup S and an action α of S by endomorphisms on $\mathcal{A}$. These endomorphisms need not be unital. Define a covariant representation of $(\mathcal{A}, S, \alpha)$ to be a pair (π, V) consisting of a unital representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ and an isometric representation V of S on $\mathcal{H}$ such that

$$\pi(\alpha_a(x)) = V_a\pi(x)V_a^*, \quad a \in S,\ x \in \mathcal{A}.$$

Lemma 1. *Suppose $(\mathcal{A}, S, \alpha)$ is a semigroup system which has a non-trivial covariant representation. Then there is a triple $(\mathcal{B}, \iota_{\mathcal{A}}, \iota_S)$ consisting of a C^*-algebra $\mathcal{B}$, a unital homomorphism $\iota_{\mathcal{A}} : \mathcal{A} \to \mathcal{B}$ and a semigroup homomorphism ι_S of S into the isometries of $\mathcal{B}$ such that*

(1) *$\iota_{\mathcal{A}}(\alpha_a(x)) = \iota_S(a)\iota_{\mathcal{A}}(x)\iota_S(a)^*$ for $a \in S$ and $x \in \mathcal{A}$,*

(2) *for every covariant representation (π, V) of $(\mathcal{A}, S, \alpha)$ there is a unital representation $\pi \times V$ of $\mathcal{B}$ with $(\pi \times V)\circ\iota_{\mathcal{A}} = \pi$ and $(\pi \times V)\circ\iota_S = V$,*

(3) *the C^*-algebra $\mathcal{B}$ is generated by $\{\iota_{\mathcal{A}}(x) : x \in \mathcal{A}\} \cup \{\iota_S(a) : a \in S\}$.*

The triple $(\mathcal{B}, \iota_{\mathcal{A}}, \iota_S)$ is unique up to isomorphism.

Proof. This is a direct application of Proposition 2.1 of [LR1]. □

We define the crossed product of $\mathcal{A}$ by S to be the unital C^*-algebra $\mathcal{B}$ together with the pair $(\iota_{\mathcal{A}}, \iota_S)$. We denote this crossed product by $\mathcal{A} \rtimes_\alpha S$, or by $\mathcal{A} \rtimes S$ when α is understood.

Returning to the situation of §2, the semigroup $I = J \cap R$ acts on $C^*(A/R)$ via C^*-endomorphisms defined by

$$\gamma_a : \delta_x \mapsto \delta_{ax}, \quad a \in I,\ x \in A/R.$$

Right inverses for these endomorphisms are given by the action α of the semigroup I on $C^*(A/R)$ defined by

$$\alpha_a(\delta_y) = \frac{1}{N(\mathfrak{A})} \sum_{[x:\, a\cdot x = y]} \delta_x, \quad a \in I,\ y \in A/R,$$

where $\mathfrak{A}$ is the ideal associated to a. The α_a, $a \in I$ are indeed C^*-endomorphisms of $C^*(A/R)$, as one checks by a straightforward computation.

A non-trivial covariant representation of $(C^*(A/R), I, \alpha)$ on $l^2(A/R)$ is given by (λ, L) , with λ the left regular representation of $C^*(A/R)$ on $l^2(A/R)$ and

$$L_a \varepsilon_y = \frac{1}{\sqrt{N(\mathfrak{A})}} \sum_{[x:\, a\cdot x = y]} \varepsilon_x, \quad a \in I, y \in A/R,$$

where $\mathfrak{A}$ is the ideal associated to a. Here $\{\varepsilon_y;\, y \in A/R\}$ is the usual orthonormal basis of $l^2(A/R)$. Hence the system $(C^*(A/R), I, \alpha)$ has a crossed product $C^*(A/R) \rtimes_\alpha I$.

The C^*-algebra $C^*(K/\mathcal{O}) \rtimes \mathcal{O}^\times$ of [ALR], §1, where $\mathcal{O}^\times$ is the semigroup of non-zero integers of K, is obtained by embedding $\mathcal{O}^\times$ in I diagonally by $a \mapsto (a)_{v \in M_K^o}$, $a \in \mathcal{O}^\times$ and considering the restriction of α to an action of $\mathcal{O}^\times$.

Let $\mathcal{T}$ be any sub-semigroup of I and let $C^*(A/R) \rtimes_\alpha \mathcal{T}$ be the semigroup crossed product obtained by restricting α to an action of $\mathcal{T}$. We have the following generalization of Proposition 2.1 of [ALR] which is a reformulation of the universal property of $C^*(A/R) \rtimes_\alpha \mathcal{T}$ in terms of generators and relations.

Lemma 2. *The semigroup crossed product $C^*(A/R) \rtimes_\alpha \mathcal{T}$ is the universal C^*-algebra generated by elements $\{\mu_a : a \in \mathcal{T}\}$ and $\{e(x); x \in A/R\}$ subject to the relations*

$$\mu_a^* \mu_a = 1, \quad a \in \mathcal{T}$$
$$\mu_a \mu_b = \mu_{ab}, \quad a, b \in \mathcal{T}$$
$$e(0) = 1,\ e(x)^* = e(-x),\ e(x)e(y) = e(x+y), \quad x, y \in A/R$$
$$\frac{1}{N(\mathfrak{A})} \sum_{[x:\, a\cdot x = y]} e(x) = \mu_a e(y) \mu_a^*, \quad a \in \mathcal{T}, y \in A/R$$

where $\mathfrak{A}$ is the ideal associated to a.

Now consider, as in §2, the semigroup crossed product obtained by restricting α to an action of the image $I_+ \subset I$ of a fixed interactive splitting of the short exact sequence

$$1 \to W \to I \to \mathcal{I} \to 1.$$

We form the corresponding semigroup crossed product $C_K = C^*(A/R) \rtimes_\alpha I_+$. The presentation of this algebra as the universal C^*-algebra generated by elements $\{\mu_a : a \in I_+\}$ and $\{e(x); x \in A/R\}$ subject to the relations of Lemma 2, with $\mathcal{T} = I_+$, was given in §2. One can deduce from these relations two further ones, given by

$$\begin{aligned} \mu_a \mu_b^* &= \mu_b^* \mu_a, \quad a, b \in I_+, \ \ \mathfrak{A} + \mathfrak{B} = \mathcal{O} \\ e(x)\mu_a &= \mu_a e(a \cdot x), \quad a \in I_+, \ \ x \in A/R \end{aligned}$$

where $\mathfrak{A}$ and $\mathfrak{B}$ are the ideals associated to a and b. The analogous observation for $K = \mathbb{Q}$ was made in [ALR]. Compare with Proposition 18 of [BC]. Moreover, arguments similar to those of [BC], p. 433, show that the universal involutive algebra $\mathcal{C}$ generated by the $\{\mu_a : a \in I_+\}$ and $\{e(x); x \in A/R\}$ subject to the relations of Lemma 2, with $\mathcal{T} = I_+$, is a dense sub-algebra of C_K spanned linearly by the (independent) monomials of the form $\mu_a e(x) \mu_b^*$ where $a, b \in I_+$ have coprime associated ideals and $x \in A/R$.

It is now easy to see how the action of the symmetry group W on C_K arises. The group W acts by outer automorphisms on C_K. To compute this action, consider C_K as a subalgebra of $C^*(A/R) \rtimes_\alpha I$ and let $u \in W$ act by $[u] \in \mathrm{Aut}(C_K)$ where

$$[u](x) = \mu_u^* x \mu_u, \quad x \in C_K.$$

The right hand side is computed within the algebra $C^*(A/R) \rtimes_\alpha I$. That this is the action of W on C_K described in the Theorem is immediate.

Lemma 3. *The fixed point algebra C_K^W of the action of W on C_K is the C^*-algebra $C^*(I_+)$ generated by the μ_a, $a \in I_+$.*

Proof. One adapts easily the proof of [BC], Proposition 21 (b). □

Let $J_+ = I_+^{-1} I_+$. The C^*-algebra C_K is isomorphic to $C_r^*(P_A^+, P_R^+)$ in the sense of [BC], the C^*-Hecke algebra associated to the almost normal inclusion $P_R^+ \subset P_A^+$ where

$$P_R^+ = \begin{pmatrix} 1 & R \\ 0 & 1 \end{pmatrix}, \qquad P_A^+ = \begin{pmatrix} 1 & A \\ 0 & J_+ \end{pmatrix}.$$

The dense involutive Hecke algebra associated to this inclusion as defined in [BC], §1, is isomorphic to the algebra $\mathcal{C}$ introduced above.

We can use the identification of C_K as a C^*-Hecke algebra and define the dynamical system in the same way as in [BC], Proposition 4. This gives the same dynamical system (C_K, σ_t) as that of §2. It is the 1-parameter group $\{\sigma_t : t \in \mathbb{R}\}$ of automorphisms of C_K determined by their values on the spanning monomials,

$$\sigma_t(\mu_a e(x)\mu_b^*) = N(\mathfrak{A})^{it} N(\mathfrak{B})^{-it} \mu_a e(x) \mu_b^*$$

for $a, b \in I_+$ with associated coprime ideals $\mathfrak{A}$, $\mathfrak{B}$ and for $x \in A/R$. Compare with [L], Proposition 7.

Notice that the algebra $C^*(A/R) \rtimes_\alpha I$ is not the one giving rise to interaction. It is a restricted product over the primes. It is the splitting $+ : \mathcal{I} \to I$ with the appropriate properties which gives rise to interaction, as we shall see in §5.

5. Outline of the proof of the Theorem

It remains to comment on parts (1) and (2) of the Theorem. Let us suppose first that $\beta > 1$. Recall from [HaLe], §5 the following facts about characters of A/R. A character χ_v of $K_v/\mathcal{O}_v$, $v \in M_K^o$ is said to be admissible if it is non-trivial on $\pi_v^{-1}\mathcal{O}_v/\mathcal{O}_v$, where π_v is a local uniformiser at v. An admissible character always exists for all $v \in M_K^o$. A character χ of A/R is said to be admissible if there exists for all $v \in M_K^o$ an admissible character χ_v such that, for all $y = (y_v) \in A/R$,

$$\chi(y) = \prod_{v \in M_K^o} \chi_v(y_v).$$

The group of characters of A/R is isomorphic to R. Indeed, on fixing an admissible character χ this isomorphism is given by

$$y \mapsto \chi(y\,\cdot\,), \quad y \in R,$$

and the application

$$u \mapsto \chi(u\,\cdot\,), \quad u \in W$$

of W into the characters of A/R is injective.

One verifies easily and in a similar way to Proposition 23 of [BC] that the maps ρ_u of C_K on $l^2(I_+)$ given in §2 are indeed representations and that the operator H implements the dynamical system σ_t,

$$\rho_u(\sigma_t(x)) = e^{itH}\rho_u(x)e^{-itH}, \quad x \in C_K.$$

It is clear that the states $\Phi_{\beta,u}$ of part (2) of the Theorem are KMS_β-states for $\beta > 1$.

We now want to study the map $u \mapsto \Phi_{\beta,u}$, $u \in W$. For this, we adapt the arguments of [HaLe] §5.3. Consider the representation

$$\rho_{\beta,u} : C_K \to \mathcal{L}(l^2(I_+) \otimes l^2(I_+))$$

$$\rho_{\beta,u}(x)(\xi \otimes \eta) = \rho_u(x)(\xi) \otimes \eta.$$

Let $\Omega_{\beta,u}$ be the unit vector of $l^2(I_+) \otimes l^2(I_+)$ defined by

$$\Omega_{\beta,u} = \zeta_K(\beta)^{-1/2} \sum_{\mathfrak{A} \in \mathcal{I}} N(\mathfrak{A})^{-\beta/2} \varepsilon_a \otimes \varepsilon_a$$

where $\mathfrak{A}$ is the ideal associated to a. For each $x \in C_K$ we have

$$\Phi_{\beta,u}(x) = \langle \rho_{\beta,u}(x)\Omega_{\beta,u}, \Omega_{\beta,u} \rangle.$$

Let $C^*(I_+)$ denote the C^*-algebra generated by the μ_a, $a \in I_+$. Each vector $\varepsilon_a \otimes \varepsilon_b$ of the basis of $l^2(I_+) \otimes l^2(I_+)$ belongs to the closure of $\rho_{\beta,u}(C^*(I_+))(\Omega_{\beta,u})$. Therefore $\rho_{\beta,u}(C_K)(\Omega_{\beta,u})$ is a dense sub-vector space of $l^2(I_+) \otimes l^2(I_+)$ and $(\rho_{\beta,u}, \Omega_{\beta,u})$ defines the GNS representation of $\Phi_{\beta,u}$. The representation ρ_u is irreducible, so that the commutant of $\rho_{\beta,u}(C_K)$ is $\mathrm{Id} \otimes \mathcal{L}(l^2(I_+))$. The von Neumann algebra M generated by $\rho_{\beta,u}(C_K)$ is thus $\mathcal{L}(l^2(I_+)) \otimes \mathrm{Id}$ and $M \cap M'$ has trivial centre. Therefore $\Phi_{\beta,u}$ is a factor state of Type I_∞, with list of eigenvalues $\{\zeta_K(\beta)^{-1} N(\mathfrak{A})^{-\beta}, \mathfrak{A} \in \mathcal{I}\}$. As $\Phi_{\beta,u}$ is factorial, it is an extremal KMS_β state.

Notice that $\Phi_{\beta,u}$ determines a unique state $\widetilde{\Phi}_{\beta,u}$ on the von-Neumann algebra $M = \mathcal{L}(l^2(I_+)) \otimes \mathrm{Id}$ generated by $\rho_{\beta,u}(C_K)$,

$$\widetilde{\Phi}_{\beta,u}(X \otimes \mathrm{Id}) = \langle X(\Omega_{\beta,u}), \Omega_{\beta,u} \rangle, \quad X \otimes \mathrm{Id} \in M.$$

For each $s > 0$, the operator $e^{-sH} \in M$. Hence given $\Phi_{\beta,u}$ we can determine, for each $x \in A/R$, the value of

$$\begin{aligned} \lim_{s \to \infty} \widetilde{\Phi}_{\beta,u}(\rho_u(e(x))e^{-sH} \otimes \mathrm{Id}) &= \zeta_K(\beta)^{-1} \langle \rho_u(e(x))\varepsilon_1, \varepsilon_1 \rangle \\ &= \zeta_K(\beta)^{-1} \chi(u \cdot x). \end{aligned}$$

Therefore $\Phi_{\beta,u}$ determines uniquely the character $x \mapsto \chi(u \cdot x)$ of A/R and as we remarked already, this map from W to the characters of A/R is injective. It follows that the map $u \mapsto \Phi_{\beta,u}$ is an injective continuous map of the compact group W into the space $\mathcal{E}(K_\beta)$ of extreme points of the convex compact Choquet simplex K_β of KMS_β states on (C_K, σ_t).

For any $u \in W$, we have

$$\Phi_{\beta,1} \circ [u] = \Phi_{\beta,u}.$$

Let Ψ be an extremal KMS_β state on (C_K, σ_t). The following two KMS_β states

$$\int_W \Psi \circ [u] du, \quad \int_W \Phi_{\beta,u} du$$

are invariant under the action of W. They are therefore completely determined by their restriction to $C_K^W = C^*(I_+)$. By Proposition 2 of §3, the system $(C^*(I_+), \sigma_t)$ is without interaction and has a unique KMS_β state denoted by Φ_β. Therefore,

$$\int_W \Psi \circ [u] du = \int_W \Phi_{\beta,u} du = \int_W \Phi_{\beta,1} \circ [u] du.$$

As Ψ is extremal, this gives two decompositions of the same state as a barycenter of measures over $\mathcal{E}(K_\beta)$, and as K_β is a Choquet simplex,

$$\Psi \circ [u] \in \{\Phi_{\beta,v} : v \in W\}$$

for almost all $u \in W$. Hence, for some $u, v \in W$ we have $\Psi = \Phi_{\beta,v} \circ [u^{-1}]$ and Ψ is in the image of $u \mapsto \Phi_{\beta,u}$. Since $u \mapsto \Phi_{\beta,u}$ is continuous and bijective and W is compact, it is a homeomorphism with range $\mathcal{E}(K_\beta)$. This concludes our treatment of part (2) of the Theorem.

Now suppose $0 < \beta \leq 1$. The key steps in the proof of part (1) of the Theorem are the generalizations of Lemma 27 and Corollary 29 of [BC] to the dynamical system (C_K, σ_t). It is here that the assumptions on the interactive splitting $+ : \mathcal{I} \to I_+$ play a crucial role. We shall again make use of the C^*- dynamical system $(C^*(I_+), \sigma_t)$ with its unique KMS_β state Φ_β. We consider as in [BC] for the case $K = \mathbb{Q}$, the spectral subspaces $C_{K,\chi}$ for each character χ of the abelian compact group W,

$$C_{K,\chi} = \{x \in C_K : [u](x) = \chi(u)(x), \text{ all } u \in W\}.$$

Therefore $C_{K,1} = C^*(I_+)$ by Lemma 3.

Lemma 4. *Let $0 < \beta \leq 1$ and let Ψ be a KMS_β state on (C_K, σ_t). Then:*
(1) *The restriction of Ψ to $C^*(I_+)$ is Φ_β.*
(2) *The restriction of Ψ to the spectral subspace $C_{K,\chi}$ is zero when χ is non-trivial.*

Proof. Part (1) is clear. For part (2), we need to generalise the proof of Lemma 27, (b) and (c) of [BC]. We say $V \in C^*(A/R) = C(R)$ is localised in a finite subset F of finite places if

$$V \in (\otimes_{v \in F} C(\mathcal{O}_v)) \otimes 1 \subset C(R).$$

Similarly, given a character χ of W, we say that it is localised in F if it factors through the projection $W \to \prod_{v \in F} K_v^*$. Let $w \in M_K^o \setminus F$ and let $\mathfrak{P}_w$ be the corresponding prime ideal of $\mathcal{O}$. Let $p_w = (p_{w,v})_v = \mathfrak{P}_w^+$ be the image of $\mathfrak{P}_w$ under the given interactive splitting $+$. To p_w we associate the following element $g_w \in W$. Writing $g_w = (g_{w,v})_v$ we let

$$g_{w,v} = p_w \in \mathcal{O}_w^* \quad \text{if } v \neq w, \quad g_{w,w} = 1.$$

Our assumptions on the interactive splitting ensure that, at least on the images under $+$ of the principal prime ideals coprime to F, the map $p_w \mapsto g_w$ is not trivial. By Dirichlet's Density Theorem (see [N], Theorem 6.2, p131), we know that there are infinitely many principal prime ideals coprime to F. For any $f \in C(R)$ and $a \in I_+$, we have (see §4) $f\mu_a = \mu_a f_a$ where $f_a(x) = f(ax)$ for $x \in R$. If f is localised in F and $w \notin F$ we have $[g_w](f) = f_{p_w}$ so that

$$f\mu_{p_w} = \mu_{p_w}[g_w](f).$$

Let V be a partial isometry in $C^*(A/R)$ and χ be a non-trivial character of W both localised in F and such that

$$[g](V) = \chi(g)V, \quad \text{for all } g \in W.$$

Then, letting $f = V$ we have

$$V\mu_{p_w} = \chi(g_w)\mu_{p_w}V$$

which gives the analogue of equation (5) of the Proof of Lemma 27 in [BC], p. 445. By continuity, we may view χ as a character of $G = \prod_{v\in F}\mathcal{O}_v^*/(1+\pi_v^{n_v}\mathcal{O}_v)$ for certain minimal integers $n_v > 0$, $v \in F$. Let $\mathfrak{C}$ be the (finite) cycle $\prod_{v\in F}\mathfrak{P}_v^{n_v}$. Then $G \simeq (\mathcal{O}/\mathfrak{C}\mathcal{O})^*$. An interactive splitting $+$ can be extended multiplicatively to a group homomorphism from the fractional ideals $\mathcal{F} = \mathcal{I}^{-1}\mathcal{I}$ of K to J. Let $\mathcal{F}(\mathfrak{C})$ be the group of fractional ideals prime to $\mathfrak{C}$. Let $P(\mathfrak{C})$ be the group of principal ideals prime to $\mathfrak{C}$ and $P_{\mathfrak{C}}$ be the subgroup of principal ideals generated by elements $\alpha \in K^*$ with $\alpha \equiv 1$ modulo $\mathfrak{P}_v^{n_v}$ for all $v \in F$. There are natural projections $p_1 : J \to \prod_{v\in F}K_v^*$ and $p_2 : \prod_{v\in F}\mathcal{O}_v^* \to G$ with $(p_1 \circ +)(\mathcal{F}(\mathfrak{C}))$ contained in $\prod_{v\in F}\mathcal{O}_v^*$. By our assumptions on the interactive splitting, any prime ideal in $P(\mathfrak{C})$ which is non-trivial mod $P_{\mathfrak{C}}$ has non-zero image under the map $p_2 \circ p_1 \circ +$. Let $h = \mathrm{Card}(\mathcal{F}(\mathfrak{C})/\mathfrak{P}(\mathfrak{C}))$ and $h_{\mathfrak{C}} = \mathrm{Card}(\mathcal{F}(\mathfrak{C})/\mathfrak{P}_{\mathfrak{C}})$. As $\mathfrak{C}$ is non-trivial the quotient group $P(\mathfrak{C})/\mathfrak{P}_{\mathfrak{C}}$ is non-trivial and we have $h_{\mathfrak{C}} > h$. The Dirichlet density ([Lg], p167) of each class in $\mathcal{F}(\mathfrak{C})/\mathfrak{P}_{\mathfrak{C}}$ is $\frac{1}{h_{\mathfrak{C}}}$, so that there are infinitely many prime ideals in each class of $P(\mathfrak{C})/\mathfrak{P}_{\mathfrak{C}}$. Having checked these points, the rest of the proof of Lemma 27 of [BC] adapts easily to our situation. □

Part (1) of the Theorem is now an immediate consequence of Lemma 4. One can develop an analogous discussion to that of [BC], §3 relating the C^* Hecke algebra C_K to products of trees. This enables one to compute explicitly the unique KMS_β state for $0 < \beta \le 1$. To define the generalization of the function Ψ_β of [BC], Theorem 5, write $y \in A/R$ as $y = a/b$ with $a = (a_v)_v$, $b = (b_v)_v \in R$. For all v with $y_v \neq 0$ take $a_v, b_v \in \mathcal{O}_v$ non-zero with either a_v or b_v in $\mathcal{O}_v^*$. If $y_v = 0$, let $a_v = 0$ and $b_v = 1$. Then, if $\mathfrak{B}$ is the ideal associated to b, write its prime factorisation as $\mathfrak{B} = \prod_{\mathfrak{P}}\mathfrak{P}^{n_{\mathfrak{P}}}$. Set

$$\Psi_b(y) = \prod_{\mathfrak{P}, n_{\mathfrak{P}}\neq 0} N(\mathfrak{P})^{-n_{\mathfrak{P}}\beta}(1 - N(\mathfrak{P})^{\beta-1})(1 - N(\mathfrak{P})^{-1})^{-1}.$$

The discussion of [BC], §3 goes through with $P_{\mathbb{Q}}^+$ replaced by P_A^+ and $P_{\mathbb{Z}}^+$ replaced by P_R^+ (defined in §4 of this paper), the Hilbert space $\mathcal{H}_\beta$ as in [BC], Proposition 32 having now a natural basis indexed by P_A^+/P_R^+ with an inner product invariant under left translation by P_A^+ and given by

$$\langle \begin{pmatrix} 1 & y \\ 0 & j \end{pmatrix} \varepsilon_0,\, \varepsilon_0\rangle = 0, \quad j \neq 1; \quad \langle \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \varepsilon_0,\, \varepsilon_0\rangle = \Psi_\beta(y).$$

Here, the vector ε_0 is the class of P_R^+. Let δ be the module on P_A^+ defined by

$$\delta \begin{pmatrix} 1 & y \\ 0 & j \end{pmatrix} = |j|.$$

One shows that the opposite C^*-algebra C_K^o admits a representation ρ in $\mathcal{H}_\beta$ given by the right convolution with $\delta^{\beta/2} f$ for any P_R^+-bi-invariant function f on P_A^+, using the description of C_K as a Hecke algebra. Finally, one checks that the vector ε_0 defines a KMS$_\beta$ state on (C_K, σ_t), which gives therefore the unique one,

$$\Phi_\beta(x) = \langle \rho(x)\varepsilon_0,\, \varepsilon_0\rangle.$$

The proofs of all these statements are straightforward generalizations of those for $K = \mathbb{Q}$ given in [BC].

6. Concluding remarks

As we said in the Introduction, our treatment differs from those of [HaLe] and [ALR] in that we view the ideals rather than just the principal ideals as playing the role of the positive integers in [BC]. By choosing our interactive splitting $+$ in such a way that for $K = \mathbb{Q}$ we have $I_+ = \mathbb{N}_{>0}$, the positive integers, we recover the C^*-Hecke algebra dynamical system $(C^*(P_{\mathbb{Q}}^+, P_{\mathbb{Z}}^+), \sigma_t)$ of [BC]. Let $\mathcal{S}_+$ denote the semigroup given by the image of the semigroup $\mathcal{S}$ of principal integral ideals of $\mathcal{O}$ under an interactive splitting $+ : \mathcal{I} \to I$. Notice that when $\mathcal{O}$ is not principal $\mathcal{S}_+$ need not be a sub-semigroup of the principal idèles. This is because the map $+$ is built up multiplicatively from its value on the prime ideals, and not all of the prime ideals are principal. What happens when we consider the algebra $C'_K = C^*(K/\mathcal{O}) \rtimes_\alpha \mathcal{S}_+$ and restrict σ and the action of W to form the C^*-dynamical system (C'_K, σ_t)? The group W acts on C'_K by symmetries commuting with the action of the σ_t. Recall that if $\mathcal{F}$ denotes the group of fractional ideals of K and P denotes the subgroup of principal ideals, then the cardinality h of the ideal class group $\mathcal{F}/P$ is called the class number of K. We denote by $\mathcal{R}_1, \dots, \mathcal{R}_h$ the ideal classes of $\mathcal{F}$ modulo P, with $\mathcal{R}_1 = P = \mathcal{S}^{-1}\mathcal{S}$. The Hilbert space $l^2(I_+)$ is the direct sum of the Hilbert spaces $\mathcal{H}_i = l^2((\mathcal{I} \cap \mathcal{R}_i)_+)$, $i = 1, \dots h$, and each $\mathcal{H}_i$ is a cyclic representation space for C'_K. It is easy to see, for example when $\beta > 1$, that one has on (C'_K, σ_t) a larger family of extremal Type I_∞ KMS$_\beta$-states, given by

$$\Phi_{\beta,u,i}(x) = \zeta_i(\beta)^{-1}\mathrm{Trace}_{\mathcal{H}_i}(\rho_u(x)e^{-\beta H_i}), \quad x \in C_K, \quad u \in W, \quad i = 1, \dots, h,$$

where H_i is the restriction of H to $\mathcal{H}_i$ and

$$\zeta_i(\beta) = \sum_{\mathfrak{A} \in \mathcal{I} \cap \mathcal{R}_i} \frac{1}{N(\mathfrak{A})^\beta} = \mathrm{Trace}_{\mathcal{H}_i}(e^{-\beta H_i})$$

is the partial Dedekind zeta function (at β) of the number field K, associated to $\mathcal{R}_i$. For $\beta > 1$, a similar family of KMS$_\beta$ states indexed by the ideal class group occurs even for the analogue for number fields of the non-interactive system of Proposition 8 of [BC] as defined in §3 of the present paper (see also [L], Remark 47).

The question as to what extent W can be interpreted as a Galois group is treated in [HaLe], once appropriate modifications are made to account for the fact that we work here with the full set of finite places of K. The conclusion of that discussion is that only in the case $K = \mathbb{Q}$ can one identify as in [BC] the symmetry group W with a true Galois group, in that case the Galois group of the maximal abelian extension of $\mathbb{Q}$. As pointed out by R. Langlands, one can view the Artin correspondence in class field theory as an equality between appropriate Artin and Dirichlet L-functions (see [N], Chapter V, §5). One can set up these L-functions (for the case of an abelian extension, where they coincide) using the language of second quantisation inherent in the setting up of the non-interactive system $(C^*(\mathcal{I}), \sigma_t)$ (compare with [BC], §2). It would be interesting to construct more general types of Euler products using second quantisation.

References

[ALR] J. Arledge, M. Laca and I. Raeburn, *Semigroup crossed products and Hecke algebras arising from number fields*, Doc. Mathematica **2** (1997) 115–138.

[BC] J-B. Bost and A. Connes, *Hecke algebras, Type III factors and phase transitions with spontaneous symmetry breaking in number theory*, Selecta Math. (New Series), 1 (1995), 411–457.

[C] A. Connes, *Formule de trace en géométrie non commutative et hypothèse de Riemann*, C. R. Acad. Sci. Paris **t.323**, Série 1 (Analyse), (1996) 1231–1236.

[HaLe] D. Harari and E. Leichtnam, *Extension du phénomène de brisure spontanée de symétrie de Bost-Connes au cas des corps globaux quelconques*, Selecta Mathematica, New Series 3 (1997), 205–243.

[J] B. Julia, *Statistical Theory of Numbers*, in Number Theory and Physics, Les Houches Winter School, J-M. Luck, P. Moussa, M. Waldschmidt eds., Springer Proceedings in Physics 47 (1990), 276–293.

[L] M. Laca, *Semigroups of *-endomorphisms, Dirichlet series and Phase Transitions*, J. Functional Analysis, to appear.

[LR1] M. Laca and I. Raeburn, *Semigroup crossed products and the Toeplitz algebras of non-abelian groups*, J. Functional Analysis **139** (1996), 415–440.

[LR2] M. Laca and I. Raeburn, *A semigroup crossed product arising in number theory*, J. London Math. Soc., to appear.

[Lg] S. Lang, Algebraic Number Theory, Second Edition, Springer-Verlag, Berlin Heidelberg New York Tokyo, 1994.

[N] J. Neukirch, Class Field Theory, Grund. der math. Wissen. **280**, Springer-Verlag, Berlin Heidelberg New York Tokyo, 1980.

[Ni] A. Nica, *C^* - algebras generated by isometries and Wiener-Hopf operators*, J. Operator Theory **27** (1992), 17–52.

Paula B. Cohen
UMR Arithmétique - Géométrie - Analyse - Topologie
Université des Sciences et Technologies de Lille
F-59655 Villeneuve d'Ascq cedex
E-mail : Paula.Cohen@univ-lille1.fr