Journal de Théorie des Nombres
de Bordeaux **11** (1999), 211–230

On a functional-differential equation related to Golomb's self-described sequence

par Y.-F.S. PÉTERMANN, J.-L. RÉMY, et I. VARDI

RÉSUMÉ. L'équation différentielle fonctionnelle $f'(t) = 1/f(f(t))$ a des liens étroits avec la suite auto-décrite F de Golomb,

$$\underbrace{1,}_{1,}\ \underbrace{2,2,}_{2,}\ \underbrace{3,3,}_{2,}\ \underbrace{4,4,4}_{3,}\ \underbrace{5,5,5,}_{3,}\ \underbrace{6,6,6,6,}_{4,}\cdots.$$

Nous décrivons les solutions croissantes de cette équation. Nous montrons qu'une telle solution possède nécessairement un point fixe non négatif, et que pour chaque nombre $p \geq 0$ il y a exactement une solution croissante ayant p pour point fixe. Nous montrons également qu'en général une condition initiale ne détermine pas une solution unique: les courbes représentatives de deux solutions croissantes distinctes se croisent en effet une infinité de fois. En fait, nous conjecturons que la différence de deux solutions croissantes se comporte de façon très similaire au terme d'erreur $E(n)$ dans l'expression asymptotique $F(n) = \phi^{2-\phi}n^{\phi-1} + E(n)$ (où ϕ est le nombre d'or).

ABSTRACT. The functional-differential equation $f'(t) = 1/f(f(t))$ is closely related to Golomb's self-described sequence F,

$$\underbrace{1,}_{1,}\ \underbrace{2,2,}_{2,}\ \underbrace{3,3,}_{2,}\ \underbrace{4,4,4}_{3,}\ \underbrace{5,5,5,}_{3,}\ \underbrace{6,6,6,6,}_{4,}\cdots.$$

We describe the increasing solutions of this equation. We show that such a solution must have a nonnegative fixed point, and that for every number $p \geq 0$ there is exactly one increasing solution with p as a fixed point. We also show that in general an initial condition doesn't determine a unique solution: indeed the graphs of two distinct increasing solutions cross each other infinitely many times. In fact we conjecture that the difference of two increasing solutions behaves very similarly as the error term $E(n)$ in the asymptotic expression $F(n) = \phi^{2-\phi}n^{\phi-1} + E(n)$ (where ϕ is the golden number).

1. Introduction.

For a real number p we are considering the functional differential equation

$$f'(t) = \frac{1}{f(f(t))} \tag{1}$$

with initial condition $f(p) = p$. We came across this equation while studying a self-described sequence of positive integers usually referred to as "Golomb's sequence", by the name of S.W. Golomb who rediscovered it in 1966 [Go]. We say that a sequence ("word") W of positive integers is *self-described* or *self-generating* if $\tau(W) = W$, where $\tau(W)$ is the sequence consisting of the numbers of consecutive equal entries of W. Golomb's sequence F is the sequence $\underbrace{1,}_{1,}\ \underbrace{2,2,}_{2,}\ \underbrace{3,3,}_{2,}\ \underbrace{4,4,4}_{3,}\ \underbrace{5,5,5,}_{3,}\ \underbrace{6,6,6,6,}_{4,}\cdots$. It is the only nondecreasing self-described sequence taking all positive integral values. In [Go] Golomb asks for a proof of the asymptotic equivalence

$$F(n) \sim \phi^{2-\phi} n^{\phi-1} \quad (n \to \infty), \tag{2}$$

where ϕ denotes the golden number. One such proof can be found in [Fi]. Another one, based on a heuristic argument of D. Marcus [Ma] and later completed in [Pé], establishes in particular the fact that

$$F(n) \sim f(n) \quad (n \to \infty),$$

where f is any positive solution of (1). In his argument Marcus mentions the solution of (1) $f_+(t) = \phi^{2-\phi} t^{\phi-1}$; note that this function has two fixed points, $p = 0$ and $p = \phi$.

It seems natural to ask whether Marcus' solution is the only one of (1) with initial condition $f(0) = 0$. The answer to this question is yes, and proving it is proposed as problem 10573 of the A.M.S. Monthly. But it also seems natural then to ask whether other solutions of (1) than Marcus' exist, and if so how they behave. We eventually realised that answers to these questions were too voluminous for a "problems section" such as in the A.M.S. Monthly, and that at least a large note was necessary to treat them.

In this paper we describe all the increasing solutions of (1). We prove that for each $p \geq 0$ there is exactly one (increasing) solution f of (1) with $f(p) = p$ (Theorem 1), and that there are no other increasing solutions than those having (at least) one nonnegative fixed point (Theorem 2). We long thought (and tried to prove) that solutions of (1) satisfied a more general unicity condition, i.e. that we could have $f(p) = q$ for at most one solution of (1), for every real numbers p and q, and not only for $q = p \geq 1$. This would have implied of course that two distinct solutions could never cross each other. But in fact the unicity condition is not satified in a very strong sense: indeed each pair of distinct increasing solutions cross each other infinitely often (Theorem 3).

But are there other solutions than increasing ones? We shall see below ("Preliminary remark 3.3") that a solution is necessarily strictly monotone on an interval on which it is defined. We first thought there were no decreasing solutions, until we discovered $f_-(t) = -\phi^{-\phi-1}(-t)^{-\phi}$; note that this function has the fixed point $p = -1/\phi$. It is not our purpose to also treat decreasing solutions; we shall nevertheless see that the existence of many other such solutions is not difficult to establish: see the last section of the paper.

Note added in May, 1999. Until well after the present paper was written we were unaware of the existence of [McK], which was brought to our attention by Berthold Schweizer, of the University of Massachusetts at Amherst. In this work M.A. McKiernan establishes the existence of a complex solution of (1) satisfying $f(p) = p$, for every complex number p with $|p| \geq \phi$, and announces the presentation, in a future article (which to our knowledge has not been published), of the extension of this result to all p with $|p| > 1$. He doesn't address the problem of the unicity of a solution with a given initial condition.

Last remark on Golomb's sequence. The sequence $F(n)$ was recently studied in several papers, and a much more precise asymptotic estimate than Golomb's (2) is now known. I. Vardi [Va] first proved that the error term associated with (2) is $\ll n^{\phi-1}/\log n$. He also proposed two conjectures, which were proved respectively by J.-L. Rémy and by Y.-F.S. Pétermann and J.-L. Rémy, and which can be summarized as follows (putting $F(t) = F([t])$ if t is not an integer).

Theorem 0 ([Ré] and [PéRé]). *We have for $t \geq 2$*

$$F(t) = \phi^{2-\phi}t^{\phi-1} + \frac{t^{\phi-1}}{\log t} h\left(\frac{\log\log t}{\log\phi}\right) + O\left(\frac{t^{\phi-1}}{\log^2 t}\log\log t\right), \tag{3}$$

where the real function h is continuous, not identically zero, and satisfies $h(x) = -h(x+1)$ for $x \geq 0$.

It is interesting to note that, while the main term $\phi^{2-\phi}t^{\phi-1}$ is a solution of the functional differential equation (1), the error term associated with (2) (i.e. the second and third terms on the right of (3)) satisfies an approximate functional-integral equation

$$E(t) = -\phi^{1-\phi}t^{\phi-2}\int_2^{\phi^{2-\phi}t^{\phi-1}} E(u)du + O\left(\frac{t^{\phi-1}}{\log^2 t}\right).$$

As we mentioned, the function $f_+(t) = \phi^{2-\phi}t^{\phi-1}$, which is the main term on the right of (3), is an increasing solution of (1). We also announced Theorem 3 below, which states that the graphs of two distinct increasing solutions of (1) cross each other an infinity of times. We think these crossings occur very regularly, in a pattern that can be described similarly as the error term in (3).

Conjecture 0. *If f is an increasing solution of (1) which is different from $f_+(t) = \phi^{2-\phi}t^{\phi-1}$, then*

$$f(t) = \phi^{2-\phi}t^{\phi-1} + \frac{t^{\phi-1}}{\log t}\eta\left(\frac{\log\log t}{\log\phi}\right) + O\left(\frac{t^{\phi-1}}{\log^2 t}\right),$$

where the real function η is continuous, satisfies $\eta(x) = -\eta(x+1)$, and vanishes exactly once on $[0,1)$.

Acknowledgements. This work was partly done while the first author was visiting the second and supported by the Université Henri Poincaré Nancy 1 and by the INRIA Lorraine. The authors are very grateful to these institutions for their help.

2. Statement of the results.

We adopt the convention that f is a maximal solution (MS for short) of (1) above if the following conditions are satisfied.

(a) It is defined and continuous on some interval I containing p, but not only p.
(b) $f(I) \subset I$.
(c) Equation (1) is satisfied for every t in I (where by convention we write $f'(t) = \infty$ if $f(f(t)) = 0$).
(d) f cannot be extended to a larger interval J containing I and where Conditions (a), (b) and (c) are satisfied.

The main result of this paper is

Theorem 1. *For every real number $p \geq 0$ there is exactly one MS f of (1) with $f(p) = p$.*

The proof splits up into three cases depending on whether $p < 1$, $p = 1$, or $p > 1$. For $p > 1$ the general idea is fairly simple: First one notes that the functional differential equation (1) implies that all higher derivatives $f^{(k)}(t)$ of $f(t)$ can be expressed as rational functions of iterates of $f(t)$. It then follows that if p is a fixed point of $f(t)$, then $f^{(k)}(p)$ is a uniquely defined function of p. This then implies that the Taylor series of $f(t)$ at p is uniquely defined in terms of (computable) rational functions of p. To show that a solution exists and is unique one then has to prove

(1) The *formal* Taylor series defined in this way actually satisfies (1).

(2) The Taylor series defined in this way converges in some neighbourhood of p.

(3) A solution of (1) is analytic in this neighbourhood.

Lemma 2 below shows that the case $p < 1$ can be reduced to $p > 1$. For $p = 1$ we have not been able to prove (2) directly, so another method is required. It is not difficult to verify (see Preliminary remark 3.3 next section) that all the solutions of (1) with $p \geq 0$ are increasing. But are all the increasing solutions of the functional differential equation (1) the functions described by Theorem 1 and its proof? It is a priori conceivable that some other increasing solution (i.e. with no fixed point) might exist. We show that this is not the case.

Theorem 2. *If f is a solution of (1) which is increasing, and maximal (in the sense that its interval of definition I satisfies the conditions (b), (c), and (d)), then f has a (nonnegative) fixed point.*

We shall see in the proof of Theorem 1 that if $1 < q < q'$, and if $f_q(t)$ is the solution of (1) with $p = q$, with I_q its domain of definition, then for $0 \leq t \leq q'$ and $t \in I_q$ we have $f_q(t) < f_{q'}(t)$. But this doesn't remain true for all $t > q'$, In fact the graphs of two arbitrary distinct increasing solutions cross each other infinitely many times.

Theorem 3. *if f and g are increasing maximal solutions of (1), then there is a sequence of numbers $s_i \to \infty$ $(i \to \infty)$ with $f(s_i) = g(s_i)$.*

3. Preliminary remarks.

3.1. *A maximal solution of (1) with $p = 0$ or with $p = \phi$, where ϕ denotes the golden number and $I = [0, +\infty)$ is*

$$f_+(t) = \phi^{2-\phi} t^{\phi-1}.$$

A maximal solution of (1) with $p = -1/\phi$ and $I = (-\infty, 0)$ is

$$f_-(t) = -\phi^{-\phi-1}(-t)^{-\phi}.$$

The solution f_- is clearly maximal; to see that f_+ is maximal, see Remark 3.5.

3.2. *If f is a MS of (1) then $f'(t) \neq 0$ for every t in I.*

(Otherwise $f(f(t))$ would not be defined).

3.3. *If f is a MS of (1) then f' remains of the same sign on I, whence the function $f \circ f$ cannot change sign on I. It also follows that f is strictly monotone on I.*

Suppose on the contrary that f' takes positive and negative values on I. Then there must be three numbers t_1, t, t_2 with $t_1 < t < t_2$, and such that $f(t_1) - f(t)$ and $f(t_2) - f(t)$ are of the same sign. So for instance we have $f(t_1) > f(t)$ and $f(t_2) > f(t)$. Let T be in the interval $(f(t), \min\{f(t_1), f(t_2)\}]$, and define $I_- := f^{-1}(T) \cap [t_1, t)$ and $I_+ := f^{-1}(T) \cap (t, t_2]$. The sets I_- and I_+ are not empty, as f is continuous. And we can choose T with $f(T) \neq 0$ since f' cannot vanish by Remark 3.2. Now we put $t_1' := \sup I_-$ and $t_2' = \inf I_+$. Again by using the continuity of f we have $t_1' \in I_-$ and $t_2' \in I_+$. This means that $t_1' < t < t_2'$, that $f(t_1') = f(t_2') = T$, and that $f(s) < T$ for every s in the interval (t_1', t_2'). And, again since f' cannot vanish, we must have $f'(t_1') < 0$ and $f'(t_2') > 0$. But this is in contradiction with $f'(t_1') = f'(t_2') = 1/f(T)$, since $f(T) \neq 0$.

3.4. *If f is a MS of (1) and if $f'(t_0) = \infty$ for some real number t_0 in I, then t_0 is the largest lower bound or the smallest upper bound of the interval I.*

Indeed if a neighbourhood of t_0 is included in I, then with the help of Remark 3.3 we infer that a neighbourhood N of $T_0 = f(t_0)$ is included in I, that $f(T_0) = 0$, and that $f(T)$ has a constant sign in N. The relation $f'(T_0) = 0$, being excluded by Remark 3.2, we now reach a contradiction very similarly as in the proof of Remark 3.3.

3.5. *A MS f of (1) is indefinitely differentiable in I.*

This will follow from expression (4) in Section 4 below, where by convention we write $f^{(k)}(t) = \infty$ if $f_\ell(t) := f^{\circ\ell}(t) = 0$ for some ℓ with $2 \leq \ell \leq k+1$. With the help of Remarks 3.3 and 3.4 we see that this can in fact possibly occur only at a t_0 satisfying $f(f(t_0)) = 0$, that is at $t_0 = \inf I$ or $t_0 = \sup I$.

3.6. *If f is a MS of (1) and if $f(a) \geq 0$ and $f'(a) > 0$ for some a, then $f(t)$ is positive and strictly increasing for all t in I exceeding a, so that $f'(t)$ is positive and (strictly) decreasing for all t in I exceeding a.*

3.7. *If f is an increasing MS of (1) and if I is unbounded above, then so is $f(I)$.*

Indeed the assumption $0 < f(f(t)) \leq A$ for all t leads to a direct contradiction, as then $f'(t) \geq 1/A$ for all t.

3.8. *Let a solution f of (1) have a fixed point p. Then we have the following.*

(i) *If $p \neq 0$ then f can have no fixed point of the other sign.*
Indeed the derivatives of f at fixed points of opposite signs are of opposite signs.

(ii) *If $p < 0$, then $f'(p) = 1/p < 0$, f is decreasing on I, and thus f can have no other fixed point.*

(iii) *If $p \geq 1$, then $f'(p) = 1/p \leq 1$, $f'(t)$ is strictly decreasing for $t \geq p$, and thus f can have no fixed point larger than p.*

(iv) *If $0 \leq p < 1$ and if f has a larger fixed point p_2, then $p_2 > 1$.*
Indeed since $f'(p) = 1/p > 1$, if we suppose that p_2 is the next fixed point of f we must have $f(t) > t$ for $p < t < p_2$, whence $f'(p_2) \leq 1$ and $p_2 \geq 1$. And $p_2 = 1$ is ruled out since then $f'(1) = 1$ and by Remark 3.6 $f'(t) > 1$ for $p < t < 1$, which is in contradiction with $f(p) = p$ and $f(1) = 1$.

(v) *If $p = 0$ then f can have no negative fixed point.*
Note that $f'(0) = \infty$, so that by Remark 3.4 $f(t)$ is defined either on some nonnegative, or on some nonpositive, values of t. If it is on nonnegative values (like f_+ of Remark 3.1), then f can have no negative fixed point. If it is on nonpositive values, then by Condition (b) $f(t)$ is always negative, and so is $f'(t)$ ($f'(t) = \infty$ (or "$-\infty$") being allowed), so that f is strictly decreasing and negative. But this is not compatible with $f(0) = 0$.

(vi) *From all this it follows that if p is negative or if $p = 1$, then f has no other fixed point, that if $0 \leq p < 1$ then f can have at most one other fixed point p_2, which must be larger than 1, and that if $p > 1$, then f can have at most one other fixed point p_1, which must satisfy $0 \leq p_1 < 1$.*

3.9. *If f is a MS of the differential equation (1), then (by Remark 3.3) the inverse function f^{-1} is defined on the interval $f(I)$. So for $t \in f(I)$ the function f satisfies the indefinite integral equation*

$$\int f(t)dt = f^{-1}(t). \tag{1'}$$

To see this, note that in general, if a and b belong to $f(I)$, we have

$$\int_a^b k(t)f(t)dt = \int_{f^{-1}(a)}^{f^{-1}(b)} k(f(t))f(f(t))df(t) = \int_{f^{-1}(a)}^{f^{-1}(b)} k(f(t))dt\,,$$

where $k(t)$ is an integrable function. Equation (1') follows by putting $k \equiv 1$. But also note that (1') is not quite equivalent to (1), since in general, for a solution f of (1) the inverse function f^{-1} is not necessarily defined everywhere in I (i.e $f(I)$ can be strictly included in I).

4. Proof of Theorem 1.

With the two first lemmas we begin by showing that a MS of (1) with some p satisfying $0 \leq p < 1$ is also a MS of (1) for some $p_2 > 1$. First we show that I contains no $t < p$, which by Condition (a) implies that it contains some $t > p$.

Lemma 1. *If f is a MS of (1) with $0 \leq p \leq 1$, then I contains no number $t < p$.*

Proof. If $p = 0$ this follows from Remark 3.8 (see the proof of (v)). If $0 < p < 1$ and $t_0 < p$ belongs to I, then $f(t_0) < t_0$ since by Remark 3.8 f has no fixed point smaller than p. So if we note $a_0 = t_0$ and $a_{i+1} = f(a_i)$ for $i \geq 0$, the sequence a_i is decreasing and entirely in I. If it eventually becomes negative, then there is a $T_0 > 0$ with $f(T_0) = 0$, and thus there is also a T_1 with $T_0 < T_1 < p$, $f(T_1) = T_0$, and $f'(T_1) = \infty$, which is in contradiction with Remark 3.4. So $a_i > 0$ for all i, and $a := \lim_{i\to\infty} a_i \geq 0$ exists. If a is in I then $f(a) = \lim_{i\to\infty} f(a_i) = a$, and a is a fixed point smaller than p, a contradiction. If a is not in I we define $f(a) = \lim_{i\to\infty} f(a_i)$, and on the one hand we have $a = f(a) = f(f(a))$, and on the other hand, since f' is decreasing on (a, ∞), $f'(a)$ exists and is equal to $\lim_{i\to\infty} f'(a_i)$. It follows that we can extend the definition of f at a in such a way that Conditions (a), (b) and (c) be satisfied on $J := I \cup \{a\}$, a contradiction to Condition (d). □

We are now in position to prove the following.

Lemma 2. *If f is a MS of (1) with $0 \leq p < 1$ there is exactly one other fixed point p_2 of f, which is larger than 1.*

Proof. By Remark 3.8 above it is sufficient to show that f has another fixed point. By Lemma 1 there is some $t_0 > p$ such that f is defined on $[p, t_0]$. We put as before $a_0 = t_0$ and $a_{i+1} = f(a_i)$ for $i \geq 0$. Then I contains $[p, a_i]$ for every i. If no such interval contains a fixed point larger than p, then $a_{i+1} = f(a_i) > a_i$ for every i. This implies that the sequence of a_i is bounded (if it weren't, then $f'(a_i)$ would tend to 0, and $f(a_i)/a_i$ could not remain larger than 1). Thus $a := \lim_{i\to\infty} a_i$ exists. If a is in I, then $f(a) = a$ and we are done. And the assumption that a is not in I leads to a contradiction as in Lemma 1: we *define* $f(a) = \lim_{i\to\infty} a_i$ and we infer that f satisfies (1) at $t = a$. This concludes the proof. □

Now we prove Theorem 1, considering separately the three cases $p > 1$, $0 \leq p < 1$, and $p = 1$.

<u>**Case 1:**</u> $p > 1$. In the next three lemmas we first establish the existence of a solution and describe its domain of definition I.

Lemma 3. *If $p > 1$ then there is a function $f(t)$ defined in the interval $(p - p^2(p-1)/2, p + p^2(p-1)/2)$ and satisfying (1) there.*

Proof. Let $f(t)$ satisfy the equation (1) and define $f_1(t) = f(t)$ and $f_{k+1}(t) = f(f_k(t))$, $k \geq 1$. Then, by a simple induction

$$[f_k(t)]' = \frac{1}{f_{k+1}(t) f_k(t) \cdots f_2(t)}\,.$$

Also, one has

$$f''(t) = \left[\frac{1}{f_2(t)}\right]' = -\frac{[f_2(t)]'}{[f_2(t)]^2} = -\frac{1}{f_3(t)[f_2(t)]^3}\,,$$

and similarly for any $k \geq 1$

$$f^{(k)}(t) = (-1)^{k+1} \sum_{S_k} \frac{C(a_2, \cdots, a_{k+1})}{f_2(t)^{a_2} \cdots f_{k+1}(t)^{a_{k+1}}}\,, \tag{4}$$

where the sum runs over a finite collection S_k of sets of exponents $a_2, \cdots, a_{k+1}$, and where the $C(a_2, \cdots, a_{k+1})$ are positive constants. Now, provided t is such that $f_k(t) > 0$ for every $k \geq 2$ (it follows from Remark 3.5 that this will be the case if $f_2(t) > 0$), we have

$$\begin{aligned} |f^{(k+1)}(t)| &= \sum_{S_k} \frac{C(a_2, \cdots, a_{k+1})}{f_2(t)^{a_2} \cdots f_{k+1}(t)^{a_{k+1}}} \sum_{i=2}^{k+1} \frac{a_i}{f_{i+1}(t) f_i^2(t) f_{i-1}(t) \cdots f_2(t)} \\ &\leq \max a_i \sum_{i=2}^{k+1} \frac{1}{f_{i+1}(t) f_i^2(t) f_{i-1}(t) \cdots f_2(t)} |f^{(k)}(t)|. \end{aligned}$$

□

Now note that when a term under the sum of (4) is differentiated, the exponent b_i of $f_i(t)$ appearing in one of the resulting terms always satisfies $a_i \leq b_i \leq a_i + 2$: it follows that $\max a_i \leq 2k$. Hence if we put $t = p$ in the expression above we obtain

$$|f^{(k+1)}(p)| \leq 2k \sum_{i \geq 2} \frac{1}{p^{i+1}} |f^{(k)}(p)| \leq \frac{2k}{p^2(p-1)} |f^{(k)}(p)|, \tag{5}$$

that is

$$\limsup \frac{|f^{(k+1)}(p)|/(k+1)!}{|f^{(k)}(p)|/k!} \leq \frac{2}{p^2(p-1)}.$$

So the Taylor series for f has a radius of convergence at least $p^2(p-1)/2$.

Now consider

$$h_{(p)}(p+s) = \sum_{k \geq 0} A_k s^k,$$

the formal power series solution of $h'_{(p)}(p+s) = 1/h_{(p)}(h_{(p)}(p+s))$, with $h_{(p)}(p) = p$. Then $h_{(p)}(p+s)$ is also its own formal Taylor series at $s = 0$. Note that the above computation is valid at $t = p$ if we replace f by $h_{(p)}$. It follows that this Taylor series has a radius of convergence at least $p^2(p-1)/2$, and thus represents a solution of (1) within that radius.

Lemma 4. *If $p > 1$ then there is a function $f(t)$ defined in the interval $[p, \infty)$ and satisfying (1) there.*

Proof. Consider the function f of Lemma 2, defined and satisfying (1) on $K_0 := [p, P_0]$, where $P_0 < p + p^2(p-1)/2$, and put for t in K_0

$$g(t) = p + \int_p^t f(u)du. \tag{6}$$

Since f is strictly increasing its inverse function f^{-1} is well defined on $f(K_0)$. So for t in $f(K_0)$ we have

$$g(t) = p + \int_p^{f^{-1}(t)} f(f(v))d(f(v)) = p + \int_p^{f^{-1}(t)} dv = f^{-1}(t).$$

The inverse function g^{-1} is thus well defined on K_0, and $g^{-1}(v) = f(v)$ there. But since $f(t) > 1$ for $t \geq p$ we see from (6) that in fact the inverse function g^{-1} is well defined on $g(K_0) = f^{-1}(K_0) =: K_1$, where K_1 is an interval containing K_0 and larger than K_0. Moreover, since for t in K_0 we infer from (6) that $g'(v) = f(v)$, we have for t in K_1

$$(g^{-1})'(t) = \frac{1}{g'(g^{-1}(t))} = \frac{1}{g^{-1}(g^{-1}(t))}.$$

Thus the function g^{-1} coincides with f on K_0, and satisfies (1) on K_1. We write $f(t) := g^{-1}(t)$ for t in K_1.

Continuing in this way we obtain a sequence of intervals $K_i = [p, P_i]$ $(i = 0, 1, 2, \cdots)$, with $P_{i+1} > P_i$, on each of which the function f of Lemma 3 can be extended and still satisfy (1). And we have $f(P_{i+1}) = P_i$. To conclude the proof of the lemma we have to ensure that the sequence of P_i is unbounded. If it is bounded we can do as in Lemmas 3 and 4, that is put $P = \lim_{i\to\infty} P_i$ and *define* $f(P) = \lim_{i\to\infty} f(P_i) = \lim_{i\to\infty} P_i = P$. We then easily see that f still satisfies (1) at $t = P$. But $f(P) = P$, a contradiction to Remark 3.8. □

Lemma 5. *If $p > 1$ then there is a function $f(t)$ defined in an interval $[Q, p]$ and satisfying (1) there, where either Q is a fixed point of f with $0 \leq Q < 1$, or $Q < 0$ satisfies $f(Q) = Z$ and $f(f(Q)) = f(Z) = 0$ for some Z with $Q < Z < 0$.*

Proof. This time we use the fact that the function f of Lemma 3 is defined and satisfies (1) on $J_0 := [Q_0, p]$, where $Q_0 > p - p^2(p-1)/2$. As above we define $g(t)$ by (6) for t in J_0. Similarly as before we obtain recursively a sequence of intervals $J_i = [Q_i, p]$ $(i = 0, 1, 2, \cdots)$, with $Q_{i+1} < Q_i$, on each of which the function f of Lemma 2 can be extended and still satisfy (1), and with $f(Q_{i+1}) = Q_i$. But here the construction might end after a finite number of steps: indeed if for some i we obtain $f(Q_i) < 0$ the expression (6) does not produce anymore a well defined function on J_i. So we consider two cases.

I. At each step we have $f(Q_i) > 0$, so that the construction never ends. Since $f'(p) = 1/p < 1$ we may suppose that $f(Q_0) > Q_0$. Then if for some $i \geq 0$ $f(Q_i) > Q_i$, we have $f^{-1}(Q_i) = Q_{i+1} < Q_i$ whence $f(Q_{i+1}) = Q_i > Q_{i+1}$. Thus $f(Q_i) > Q_i$ for each i. (Note that this will also be true for each $i \leq i_0$ in the case where the construction ends at J_{i_0}). Now since $f''(t) = -(f_3(t)f_2^3(t))^{-1}$ is negative at least for all t for which $f(t) > 0$, that is in particular on every J_i, f' is decreasing on every J_i, and thus the sequence of Q_i cannot diverge. So put $Q := \lim_{i\to\infty} Q_i$. Again we define $f(Q) = \lim_{i\to} f(Q_i)$. We then easily see that $f(Q) = Q$ and that f still satisfies (1) at $t = Q$. And by Remark 3.8 we must have $Q \geq 0$.

II. At the i-th step of the construction we reach a Q_i with $f(Q_i) \leq 0$. So there is a $Z \geq Q_i$ with $f(Z) = 0$. Note that Z is negative since $Q_i \leq Z < Q_{i-1} = f(Q_i) \leq 0$. Since f is well defined and nonnegative on $[Z, p]$, expression (6) produces a function g well defined on $[Z, p]$, and whose inverse g^{-1}, coinciding with f on $[Z, p]$, is well defined on $[Q := g(Z), p]$ and satisfies (1) there. So we may extend the definition of f on $[Q, p]$ by putting $f = g^{-1}$ there. And we have $f(Q) = Z$, as claimed.

$\square$

This concludes the proof of Lemma 5. So if $p > 1$ we established the existence of a function f satisfying (1) in an interval $[Q, \infty)$ where Q is either a fixed point for f with $0 \leq Q < 1$, or is such that $f'(Q) = \infty$. With Lemma 1 and Remark 3.4 this shows that f is a MS of (1).

To prove the unicity, we first note that the proof of Lemma 2 shows that any solution $h(t)$ of (1) must have at $t = p$ the same Taylor series as f. If h is analytic in some interval $[Q_0, P_0]$ with $Q_0 < p < P_0$ then $h = f$ there. In order to verify this, note in addition that there is some p_0 with $p > p - p_0 > 1$, such that if $p > t > p - p_0$ and $t \in I$, then $h(t) > t > 0$ and $h'(t) > 0$: this follows from the facts that $0 < h'(p) = 1/p < 1$, that h' is continuous, and that h has no other fixed point $p' \geq 1$. It follows that if $h_{k-1}(t) > t > 0$ for $p > t > p - p_0$ then $h_k(t) = h(h_{k-1}(t)) > h(t) > t > 0$ for $p > t > p - p_0$. Thus, for some $p_0 > 0$, the functions h_k all satisfy $h_k(t) > t$ if $p > t > p - p_0$. It follows in turn that estimate (5) in the proof of Lemma 3 remains true if we replace f by h and p by some t_0 with $1 < t_0 \leq p$. Thus

$$|h^{(k+1)}(t_0)| \leq k! \left(\frac{2}{t_0^2(t_0 - 1)} \right)^k h'(t_0). \tag{7}$$

Also note that from estimate (4) in the same proof $h^{(k)}(t)$ is positive and decreasing if k is odd, and negative and increasing if k is even, whence $|h^{(k)}(t)|$ is decreasing for every $k \geq 1$. It follows that the remainder term in modulus of the Taylor expansion of order k of $h(t)$ at p does not exceed

$$\frac{|h^{k+1}(t_0)|}{(k+1)!}|t-p|^{k+1} \quad \text{where} \quad t_0 := \min\{t,p\}.$$

And estimate (7) shows that if t is close enough to p the last expression tends to 0 as $k \to \infty$. Thus h is analytic in some $[Q_0, P_0]$ as claimed.

Now we show that h is f, with $I = [Q, \infty)$ its domain of definition. Suppose it is not the case and let $I' = [Q', \omega)$ or $I' = [Q', \omega]$ be the domain of definition of h. Suppose first that $h \equiv f$ on $I \cap I'$. Then $I = I'$ would contradict the hypothesis, $I' \subset I$ or $Q' < Q$ would contradict the maximality of f, and $I \subset I'$ or $\omega < \infty$ would contradict the maximality of h. To see this recall that $I_0 \subset I'$, and recall the proofs of Lemmas 4 and 5. So there must be some $t \in I \cap I'$ with $f(t) \neq h(t)$. Now by Remark 3.8 and Lemma 1, for every t in I which is not a fixed point of f we have $0 < (f(t)-p)/(t-p) < 1$, and for every t in I' which is not a fixed point of h we have $0 < (h(t)-p)/(t-p) < 1$. Suppose for instance there is some $t < p$ with $f(t) \neq h(t)$ (the other case is treated similarly). If we put $t_0 := \sup\{t < p, f(t) \neq h(t)\}$, we have $h \equiv f$ on $[t_0, p]$ and $t_0 \leq Q_0$. Note that t_0 cannot be a fixed point of f or h, as it then would be the infimum of I or I' (by Remark 3.8), which would contradict the maximality of f or of h, f and h being defined (and distinct) on some $t < t_0$. Thus $h(t_0) > t_0$ and $f(t_0) > t_0$. Now by hypothesis $f(t)$ and $h(t)$ are defined for values of $t < t_0$; moreover f and h are strictly increasing. So the inverse functions $f^{-1}(t)$ and $h^{-1}(t)$ are defined for values of $t < h(t_0) = f(t_0)$. It follows that there is some s_0 with $t_0 \leq s_0 < h(t_0) = f(t_0)$, and such that f^{-1} and h^{-1} are both defined on $[s_0, p]$. Thus by Remark 3.9 we have, for $t \geq s_0$,

$$h^{-1}(t) - p = \int_p^t h(u)du = \int_p^t f(u)du = f^{-1}(t) - p,$$

and $f^{-1} \equiv h^{-1}$ on $[s_0, p]$. The strict monotonicity of f and h now imply that $f \equiv h$ on $[t_1, p]$ for some $t_1 < t_0$ (satisfying $f(t_1) = h(t_1) = s_0$), contradicting the definition of t_0. This concludes the proof of Theorem 1 in case $p > 1$.

Case 2: $0 \leq p < 1$. We first prove the unicity. Consider two MS of (1) with $0 \leq p < 1$ f and h. Then by Theorem 1 f has another fixed point $p_2 > 1$, and h has another fixed point $p_3 > 1$. Suppose they are unequal, say $p_2 < p_3$. Let c be the largest number not exceeding p_3 such that $f(c) = h(c)$. This number exists since $f(p) = h(p)$ and since f and h are continuous. Note that $c < p_2$ and that $f(t) < h(t)$ for $t \in (c, p_3]$, and recall that $f(t) > t$ and $h(t) > t$ for $t \in (p, p_2)$. It follows that $f'(c) \leq h'(c)$. It also follows that if $c > p$ then $f(f(c)) < h(f(c)) = h(h(c))$, a contradiction.

So $c = p$. But then

$$0 < \int_p^{p_2} (h(t) - f(t))dt = h^{-1}(p_2) - p_2 < 0.$$

So $p_2 = p_3$, and by Case 1 $h = f$ on $[p, \infty)$.

Now we prove the existence. For $q = p_2 > 1$ denote by f_q the MS of (1) and by $I_q = [Q_q, \infty)$ its domain of definition. Recall that the fixed points of f_ϕ are ϕ and 0. So from Remark 3.1 and the argument used to prove the unicity we infer that for every q with $1 < q < \phi$ we have $f_q(t) < f_\phi(t)$ for $t \in [\max\{0, Q_q\}, \phi]$, and that for every $q > \phi$ we have $f_q(t) > f_\phi(t)$ for $t \in [\max\{0, Q_q\}, q]$. It follows that each f_q with $1 < q < \phi$ has a fixed point $p = p(q)$ with $0 < p < 1$ (and that no f_q with $q > \phi$ has such a fixed point). Moreover the same argument shows that if $1 < q < q' < \phi$, then $p(q') < p(q)$ and $f_q(t) < f_{q'}(t)$ for $t \in [p(q), q']$. So the function $p : (1, \phi] \to [0, 1)$ is decreasing. What we need to show is that it is continuous, and that $p(q) \to 1$ as $q \to 1$.

To prove the continuity we consider some $q = p_2$ with $1 < q \leq \phi$ and let $q' \to q$ with $1 < q' < \phi$. Without loss of generality we may suppose that $q' - q$ remains of the same sign. Suppose for instance that q' increases to q; the other case is treated similarly. We put f for the MS of (1) with fixed point q, and h for the MS of (1) with fixed point q'. Let s be a real number. For some s' and s'' between 0 and s we have

$$h(q' + s) - f(q + s) = \sum_{k<K} \frac{h^{(k)}(q') - f^{(k)}(q)}{k!} s^k + \frac{h^{(K)}(q' + s') - f^{(K)}(q + s'')}{K!} s^K$$

If, for some negative real number s_0 we have $s_0 \leq s \leq -s_0$, and if $q' \geq \bar{q} > 1$ (this will hold for instance if $\bar{q} := (q + 1)/2$ and $q - q' < (q - 1)/2$) we see with the help of estimate (7) that the last term does not exceed in modulus

$$\frac{|s_0|^K}{K} \left(\frac{2}{(s_0 + \bar{q})^2 (s_0 + \bar{q} - 1)} \right)^{K-1} (h'(s_0 + \bar{q}) + f'(s_0 + \bar{q})),$$

provided $s_0 + \bar{q} > 1$. The last expression tends to 0 as $K \to \infty$ if $-s_0$ is small enough. Now from (2) it is clear that for every k, $h^{(k)}(q') - f^{(k)}(q) \to 0$ as $q' \to q$. So the sum tends to 0 as $q' \to q$ and $h(q' + s)$ converges uniformly to $f(q + s)$ in $[s_0, -s_0]$. It easily follows that $h(t)$ converges uniformly to $f(t)$ in $[Q_0, q]$ for some $Q_0 < q$. Now we shall essentially follow the construction of Lemma 5, after the following.

Remark. Let $p = p(q)$ and $p' = p(q')$ be the other fixed points of f and h, and let $\epsilon > 0$. Note that p' decreases if q' increases. There is a $\beta = \beta(\epsilon) > 0$

such that the derivatives of $f(t)$ in $[p+\epsilon, q]$ and of the functions $h(t)$ in $[p'+\epsilon, q]$ are all within the interval $[1/\phi, 1/\beta]$.

Now from

$$f^{-1}(t) - h^{-1}(t) = (q-q') + \int_q^{q'} h(u)du + \int_q^t (f(u) - h(u))du$$

for $t \in [Q_0, q]$, we see that $h^{-1}(t)$ converges uniformly to $f^{-1}(t)$ for $t \in [Q_0, q]$. For $\epsilon > 0$ as in the Remark put $p_\epsilon = \inf\{p'\} + \epsilon$. Then if $\delta > 0$, $h(t)$ converges uniformly to $f(t)$ for $t \in [f^{-1}(Q_0) + \delta := Q_1, q]$, provided $Q_1 > p_\epsilon$. And we can as well fix $\delta = \epsilon$. Recursively we see that if $h(t)$ converges uniformly to $f(t)$ for $t \in [Q_i, q]$, then it does also for $t \in [f^{-1}(Q_i) + \epsilon := Q_{i+1}, q]$, provided $Q_{i+1} > p_\epsilon$.

So for $t \in [Q', q]$, and for every $Q' > \max\{\liminf_{i\to\infty} Q_i =: Q; p_\epsilon\}$, $h(t)$ converges uniformly to $f(t)$. We have $f(Q) = \limsup_{i\to\infty} f(f^{-1}(Q_i) + \epsilon) \le Q + \epsilon/p \le Q + \epsilon$. So $h(t)$ converges uniformly to $f(t)$ for $t \in [\max\{Q_\epsilon, p_\epsilon\}, q]$, where $[p, Q_\epsilon]$ is the largest interval beginning at p where $0 \le f(t) - t \le \epsilon$. It now easily follows that $\inf p' = p$.

Finally, if $p_2 = q = 1 + \epsilon$ where $\epsilon < 1/9$, we show that $p(q) \ge q - 9\epsilon$. If not it is easy to verify that $f'(t) > q - 2\epsilon$ if $t \le q$ and that $f'(t) \ge q + \epsilon$ if $t \le q - 3\epsilon$, whence $f(q - 9\epsilon) \le q - 9\epsilon$, a contradiction. This shows that $p(q) \to 1$ as $q \to 1$ and concludes the proof of Theorem 1 in Case 2.

Case 3: $p = 1$. We first prove the existence of a solution. Let $\{f_n\}_{n\ge 0}$ and $\{g_n\}_{n\ge 0}$ be the two sequences of functions $[1, \infty) \to [1, \infty)$ defined as follows.

$$f_0(t) = t, \qquad g_0(t) = 1 + \int_1^t f_0(u)du,$$

$$f_n(t) = g_{n-1}^{-1}(t), \qquad g_n(t) = 1 + \int_1^t f_n(u)du \quad (n \ge 1).$$

We see recursively that they are well defined. Indeed, if f_{n-1} is well defined with $f_{n-1}(t) \ge 1$ for every $t \ge 1$, then g_{n-1} is strictly increasing and unbounded, whence f_n is well defined with $f_n(t) \ge 1$ for every $t \ge 1$. Now we have, for every $t \ge 1$,

$$f_0(t) \ge f_2(t) \ge f_1(t). \tag{8}$$

And if h_+ and h_- are two increasing and unbounded functions $[1, \infty) \to [1, \infty)$ with $h_+(1) = h_-(1) = 1$ and $h_+(t) \ge h_-(t)$ for every $t \ge 1$, then we have

$$k_+(t) := 1 + \int_1^t h_+(u)du \ge k_-(t) := 1 + \int_1^t h_-(u)du,$$

whence $k_+^{-1}(t) \le k_-^{-1}(t)$ for every $t \ge 1$. Thus an induction argument starting with (8) shows that the sequences $\{f_{2k}(t)\}_{k\ge 0}$ and $\{f_{2k+1}(t)\}_{k\ge 0}$ are for every $t \ge 1$ respectively decreasing and increasing (strictly, when

$t > 1$), and that in addition $f_{2k}(t) \geq f_{2k+1}(t)$ for every $k \geq 0$. Now we put, for $t \geq 1$, $F(t) = \lim_{k\to\infty} f_{2k}(t)$ and $G(t) = \lim_{k\to\infty} f_{2k+1}(t)$, and we shall prove that $F \equiv G$ and that F is a (maximal) solution of (1) for $p = 1$. First we verify that F and G are strictly increasing and unbounded, and that their inverse functions satisfy

$$G^{-1}(t) = 1 + \int_1^t F(u)du \quad \text{and} \quad F^{-1}(t) = 1 + \int_1^t G(u)du, \tag{9}$$

whence in particular they are indefinitely differentiable functions. We prove the first equation in (9); the second one is treated similarly. Since $f_{2k}(t) \geq 1$ and $\{f_{2k}(t)\}_{k\geq 0}$ is decreasing, F is integrable and

$$H(t) := 1 + \int_1^t F(u)du = \lim_{k\to\infty} \left(1 + \int_1^t f_{2k}(u)du\right) = \lim_{k\to\infty} g_{2k}(t).$$

Thus we see that the inverse function H^{-1} exists. And we know that the function g_{2k} is increasing for every $k \geq 0$, and that the sequence $\{g_{2k}(t)\}_{k\geq 0}$ is decreasing for every $t \geq 1$. Now if $\epsilon > 0$ let $\eta = H(y) - H(y - \epsilon)$. Then for k large enough we have

$$g_{2k}(y - \epsilon) < H(y - \epsilon) + \eta = H(y) < g_{2k}(y),$$

whence $t := H(y) \in (g_{2k}(y - \epsilon), g_{2k}(y))$ and $g_{2k}^{-1}(t) \in (y - \epsilon, y)$. It follows that $g_{2k}^{-1}(t)$ converges to $H^{-1}(t)$. But $g_{2k}^{-1}(t)$ also converges to $G(t)$, and the first equality of (9) is proved.

Now we show that $F \equiv G$.

Lemma 6. *Let F and G be two increasing and unbounded functions $[1, \infty) \to [1, \infty)$, indefinitely differentiable, and satisfying*

$$G(1) = F(1) = 1, \quad \textit{and} \quad G(t) \leq F(t) < t \ (t > 1).$$

Then if the equations (9) hold we have $F \equiv G$.

Proof. First note that F and G must be strictly increasing, unbounded functions, and that their inverse functions F^{-1} and G^{-1} are well-defined. From

$$t = 1 + \int_1^{G(t)} F(u)du = 1 + \int_1^{F(t)} G(u)du$$

we see that

$$G'(t) = \frac{1}{F(G(t))} \quad \text{and} \quad F'(t) = \frac{1}{G(F(t))}.$$

It follows that

$$\begin{aligned} F(t) - G(t) = \int_1^t (F'(u) - G'(u))du &= \int_1^t \left(\frac{1}{G(F(u))} - \frac{1}{F(G(u))}\right) du \\ &= \int_1^t \frac{F(G(u)) - G(F(u))}{G(F(u))F(G(u))} du. \end{aligned}$$

Now let $H(u) := \max\{F(G(u)) - G(F(u)), 0\}$. Then from the above expression and since $F(v) \geq 1$ and $G(v) \geq 1$ for all $v \geq 1$ me may write

$$F(t) - G(t) \leq \int_1^t H(u)du.$$

Since on the other hand $F(F(u)) \geq F(G(u))$ and $F(F(u)) - G(F(u)) \geq 0$, we have $F(F(u)) - G(F(u)) \geq H(u)$ whence

$$F(t) - G(t) \leq \int_1^t (F(F(u)) - G(F(u)))du.$$

There is some (necessarily nonnegative) constant C such that $F(t) - G(t) \leq C$ for every t with $1 \leq t \leq 3/2$. Let C be the smallest such constant. Then since $G(t) \leq F(t) \leq t$ we have

$$F(t) - G(t) \leq \int_1^t C du = (t-1)C \leq \frac{C}{2}$$

if $1 \leq t \leq 3/2$. Thus $C = 0$ and $F \equiv G$ on $[1, 3/2]$. In order to see that $F \equiv G$ on $[1, \infty)$ we recursively use the expression

$$F^{-1}(t) = 1 + \int_1^t F(u)du, \tag{10}$$

which we just proved holds when t belongs to the interval $[1, 3/2]$. If we put $t_0 = 3/2$ and $t_i = F^{-1}(t_{i-1})$ $(i > 0)$, then the sequence $\{t_i\}$ is strictly increasing since $F^{-1}(t) > t$ for every $t > 1$ by hypothesis. And if (10) holds for $t \leq t_i$, then $F^{-1}(t) = G^{-1}(t)$ for $t \leq t_i$, whence $F(t) = G(t)$ for $t \leq t_{i+1}$ and (10) holds for $t \leq t_{i+1}$. So we only need to ensure that the sequence $\{t_i\}$ cannot converge. If it did, to some $T < \infty$, say, then we would have $T = \lim_{i\to\infty} t_i = \lim_{i\to\infty} F^{-1}(t_i) = F^{-1}(T)$, in contradiction to one of the hypotheses. This concludes the proof of the Lemma. $\square$

Now by differentiating (10) and putting $v = F(t)$ we see that F is a solution of (1) for $p = 1$ (and it is a MS by Lemma 1).

Finally we show that the solution is unique. First note that by Lemma 1 a MS f of (1) for $p = 1$ is defined on an interval I whose lower bound is 1, that by Remark 3.3 f is strictly increasing, and that we thus have $f(u) \geq 1$, whence $f'(u) \leq 1$, for every u in I. In particular we also have $f(u) \leq u$ for every u in I. Also note that similarly as we proved above that $F \equiv G$ on $[1, \infty)$, we can put $t_i = f^{-1}(t_{i-1})$ $(i \geq 1)$ where $t_0 > 1$ is some number in I, and see that in fact $I = [1, \infty)$. If F is the solution we constructed above, then we have

$$F(t) - f(t) = \int_1^t (F'(u) - f'(u))du = \int_1^t \frac{f(f(u)) - F(F(u))}{f(f(u))F(F(u))}du.$$

Now there is some nonnegative constant C such that $|F(t) - f(t)| \le C$ for every t in $[1, 5/4]$. Let C be minimal with this property. Then for some function $v = v(u)$ where $v(u)$ is between $f(u)$ and $F(u)$, we have

$$\begin{aligned}
|F(t) - f(t)| &\le \int_1^t |F(F(u)) - f(f(u))| du \\
&\le \int_1^t |F(F(u)) - F(f(u))| du + \int_1^t |F(f(u)) - f(f(u))| du \\
&= \int_1^t |F'(v)||F(u) - f(u)| du + \int_1^t |F(f(u)) - f(f(u))| du \\
&\le 2 \int_1^t C du \le \frac{C}{2}.
\end{aligned}$$

Thus $C = 0$ and $F \equiv f$ on $[1, 5/4]$. And in fact $F \equiv f$ on $[1, \infty)$: again this can be seen similarly as in the proof of $F \equiv G$ above. This concludes the proof of Theorem 1.

5. Proofs of Theorems 2 and 3.

We now show that there are no other increasing MS of (1) than those described by Theorem 1.

Theorem 2. *If f is a solution of $f'(t) = 1/f(f(t))$ which is increasing, and maximal (in the sense that its interval of definition I satisfies the conditions (b), (c), and (d)), then f has a (nonnegative) fixed point.*

Proof. Let A be the largest lower bound of I. If $A \ge 0$, then considering limits we see that $f(A)$ exists and $A \in I$; and we must have $f(A) \ge A$. If $A < 0$ then since f takes positive values and I is an interval, f is defined on $t = 0$. Suppose $f(0) < 0$. Then since f is increasing, $f(t) < 0$ if $t < 0$, and thus $f'(0) = 1/f(f(0)) < 0$, which is not true. So we must have $f(0) \ge 0$.

Hence there is a $t_0 \ge 0$ with $f(t_0) \ge t_0$. If $f(t_0) = t_0$ the theorem is proved. So suppose $f(t_0) > t_0$. Then f is defined at least on the interval $[t_0, f(t_0) =: t_1]$. Now define $t_n = f(t_{n-1})$ $(n = 1, 2, \cdots)$. Then either $f(t_n) > t_n$ for every n, or $f(t_n) \le t_n$ for some n. In the second case there is some $t \ge 0$ with $f(t) = t$, and again the theorem is proved. In the first case, we first note that the increasing sequence t_n cannot diverge: if it did then we would have $f'(t_n) \to 0$, which is incompatible with the assumption that $f(t_n) \le t_n$ for every n. Thus t_n converges to some number T as $n \to \infty$. Again considering limits we see that $f(T)$ exists and $T \in I$. We have $f(T) = \lim_{n\to\infty} f(t_n) = \lim_{n\to\infty} t_n = T$, and again the theorem is proved. □

Remark. It is conceivable that some solution f of (1) with $f(t) > 0$ if $t < 0$ $f(t) < 0$ if $t > 0$ might exist. Such a function, however, would have derivatives of opposite signs for arguments of opposite signs, and could thus not be continuous on an interval containing 0. In other words, it could not be an MS on any interval I.

Now we prove that the graphs of two arbitrary distinct increasing solutions cross each other infinitely many times.

Theorem 3. *If f and g are increasing maximal solutions of (1), then there is a sequence of numbers $s_i \to \infty$ $(i \to \infty)$ with $f(s_i) = g(s_i)$.*

Proof. First note that any increasing solution of $f'(t) = 1/f(f(t))$ satisfies the asymptotic equivalence $f(t) \sim \phi^{2-\phi}t^{\phi-1}$ $(t \to \infty)$: this follows from Proposition 2 in [Pé] (The condition $f : [0, \infty) \to [0, \infty)$ of this general result can clearly be removed for the solutions of (1) we consider).

Let $[\tau, \infty)$ be the interval of nonnegative arguments on which both functions f and g are defined. Now first assume that f and g never cross each other, for instance that $f(t) > g(t)$ for all $t \geq \tau$. It follows that $f'(t) < g'(t)$ for all these t, and that if we put $f(t_0) - g(t_0) =: C$ for some $t_0 > \tau$ to be fixed later on, then $f(t) - g(t) \geq C$ for $\tau \leq t \leq t_0$. Now for $\tau \leq t \leq t_0$ let $t_1 = t_1(t) = f^{-1}(t)$. Then we have, if t_0 is large enough,

(11)
$$\begin{aligned} g'(t_1) - f'(t_1) &= \frac{f_2(t_1) - g_2(t_1)}{f_2(t_1)g_2(t_1)} \\ &\geq \frac{f(f(t_1)) - g(f(t_1)) + g(f(t_1)) - g(g(t_1))}{(f(t))^2} \\ &\geq \frac{f(t_0) - g(t_0) + g(f(t_1)) - g(g(t_1))}{(f(t_0))^2} \geq \frac{C}{(f(t_0))^2} \geq \frac{C'}{t_0^{2\phi-2}}, \end{aligned}$$

where $C' = C\phi^{2\phi-4}/2$. Note that we have $t_0 = t_1(f(t_0))$, and choose t_0 large enough to ensure that $f(t_0) \geq \tau$. Then by integrating we get from (11)

$$g(t_1) - g(t_0) \geq f(t_1) - f(t_0) + \frac{C'}{t_0^{2\phi-2}}(t_1 - t_0), \tag{12}$$

that is

$$g(t_1) - f(t_1) \geq -C + \frac{C' t_1}{t_0^{2\phi-2}} + O(t_0^{3-2\phi}). \tag{13}$$

But (13) holds in particular for $t_1(t_0) = f^{-1}(t_0) \sim \phi^{1-\phi}t_0^{\phi}$ as $t_0 \to \infty$, and it follows that $g(t_1(t)) - f(t_1(t))$ cannot remain negative for all $t \leq t_0$.

So there is at least one crossing. Now suppose the set S of crossings is bounded. Then $s = \sup S$ is itself a crossing. (Note in passing that the assumption $f \equiv g$ on an interval leads to a contradiction: i.e. crossings are crossings in the strictest sense). Suppose that for instance $f(t) > g(t)$ for all $t \geq s =: T_0$. Let T_1 be such that $g(T_1) = T_0$; then for $t \geq T_1$ we have $f(f(t)) > f(g(t)) > g(g(t))$, whence $g'(t) > f'(t)$. So if we put, for some $t_0 > T_1$, $f(t_0) - g(t_0) =: C$, then $f(t) - g(t) \geq C$ for $T_1 \leq t \leq t_0$.

Now for $T_1 \leq t \leq t_0$ let $t_1 = t_1(t) = f^{-1}(t)$. We may suppose that T_1 is large enough to ensure that $t_1 > t$; and also that t_0 is large enough to ensure that $f(t_0)$ be larger than T_1. Then the estimates (11), (12) and (13) hold if t_0 is large enough, and it follows as before that $g(t_1(t)) - f(t_1(t))$ cannot remain negative for all $t \leq t_0$. This concludes the proof of the theorem. $\square$

6. Last remark on decreasing solutions.

Although it is not our purpose to also treat decreasing solutions of (1) (like $f_-(t) = -\phi^{-\phi-1}(-t)^{-\phi}$), we note that it is now fairly easy to establish the existence of infinitely many such functions.

Indeed, for every $p < -1$ the argument in the proof of Case 1 of Theorem 1 helps verify that there is exactly one MS f_p of (1) with $f_p(p) = p$. Consider the formal power series solution $h_{(p)}(t)$ of (1) with fixed point p. From (4) it is clear that at $t = p$ this series has a radius of convergence at least as large as that of the power series expansion of $f_{|p|}(t)$, the solution of (1) with fixed point $|p|$, at $t = |p|$, which is positive by Lemma 3. Thus $h_{(p)}(t)$ represents within that radius a solution of (1), which we may call $f_p(t)$. We can then extend f_p to a MS of (1) by using

$$f^{-1}(t) = p + \int_p^t f(u)du, \quad \text{respectively} \quad f(t) = f'(f^{-1}(t)),$$

if the fixed point p is stable (as in Case 1 of Theorem 1), respectively unstable. (In view of the function f_- it seems likely that p is unstable). Unicity is obtained similarly as in Case 1 of Theorem 1, and by using the analyticity of $f_{|p|}(t)$.

We mention the fact that, by comparing the coefficients of the (formal) solution $h_{(p)}(t)$ written as a (formal) Taylor series at $t = p$ with those of the solution $f_{-1/\phi}(t) := f_-(t)$ (also at $t = p$), one can in fact establish the existence of a solution for every $p \leq -1/\phi$.

We have no opinion to offer regarding the existence of solutions when $-1/\phi < p < 0$.

References

[Fi] N.J. Fine. *Solution to problem 5407.* Amer. Math. Monthly **74** (1967), 740–743.

[Go] S.W. Golomb. *Problem 5407.* Amer. Math. Monthly **73** (1966), 674.

[Ma] Daniel Marcus. *Solution to problem 5407.* Amer. Math. Monthly **74** (1967), 740.

[McK] M.A. McKiernan. *The functional differential equation $Df = 1/ff$.* Proc. Amer. Math. Soc. **8** (1957), 230–233.

[Pé] Y.-F.S. Pétermann. *On Golomb's self describing sequence II.* Arch. Math. **67** (1996), 473–477.

[PéRé] Y.-F.S. Pétermann and Jean-Luc Rémy. *Golomb's self-described sequence and functional differential equations.* Illinois J. Math. **42** (1998), 420–440.

[Ré] Jean-Luc Rémy. *Sur la suite autoconstruite de Golomb.* J. Number Theory **66** (1997), 1–28.

[Va] Ilan Vardi. *The error term in Golomb's sequence.* J. Number Theory **40** (1992), 1–11.

Y.-F.S. Pétermann
Université de Genève
Section de Mathématiques
2-4, rue du Lièvre, C.P. 240
CH-1211 Genève 24
SUISSE
E-mail : peterman@sc2a.unige.ch

Jean-Luc Rémy
Centre de Recherche en Informatique de Nancy
CNRS et INRIA Lorraine
B.P. 239
F-54506 Vandœuvre-lès-Nancy Cedex
FRANCE
E-mail : Jean-Luc.Remy@loria.fr

Ilan Vardi
IHES
35 route de Chartres
F-91440 Bures-sur-Yvette
FRANCE
E-mail : ilan@ihes.fr