Journal de Théorie des Nombres
de Bordeaux **11** (1999), 247–262

One Special Class of Modular Forms and Group Representations

par Galina V. VOSKRESENSKAYA

Résumé. On étudie une famille de formes modulaires qui sont des produits de fonctions η de Dedekind. On s'intéresse aussi aux liens entre ces fonctions et les représentations des groupes finis.

Abstract. In this article we consider one special class of modular forms which are products of Dedekind η-functions and the relationships between these functions and representations of finite groups.

1. Modular forms with divisors in cusps

The study of relations between finite groups and modular forms is an interesting topic of modern mathematical investigations. We shall study from this point of view one special class of modular forms which is described by the following theorem.

Theorem 1. *There are only 28 functions determined by the following conditions: these functions are 1) cusp functions of integral weight with characters; 2) eigenforms of the Hecke algebra; 3) they have no zeroes outside of the cusps. We can describe them completely by the following formula:*

$$f(z) = \prod_{k=1}^{s} \eta^{t_k}(a_k z),\ a_k,\ t_k \in \mathbb{N},\ \sum_{k=1}^{s} t_k a_k = 24,$$

$$2 \mid \sum_{k=1}^{s} t_k,\ a_k \mid a_s,\ a_1 a_s = a_{s+1-k} a_k,\ 1 \le k \le s,$$

where $\eta(z)$ is Dedekind's η-function defined by the formula

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),\ q = e^{2\pi i z},$$

where z belong to the upper complex half-plane.

Proof. All necessary definitions and notations of the theory of modular forms can be found in Shimura's book [10].

Let $f(z)$ be a cusp form satisfying the conditions of the theorem 1 . We shall denote by k_f its weight and by N its level. Let $\mathbb{H}^*$ be the complex half-plane with the point of infinity. Let us consider the Riemann surfaces $\Gamma_0(N) \setminus \mathbb{H}^*$ and $\Gamma_\chi \setminus \mathbb{H}^*$, where $\Gamma_\chi = \{\sigma \in \Gamma_0(N) : \chi(\sigma) = 1\}$. Our functions belong to the space $S_{k_f}(\Gamma_\chi)$.

It is well-known that the cusp form which is an eigenform of Hecke algebra has a zero with multiplicity 1 in every cusp of the group Γ_χ. Let us consider the differential form $\omega = f(z)(dz)^{\frac{k_f}{2}}$ on the Riemann surface $\Gamma_\chi \setminus \mathbb{H}^*$. The degree of the divisor of this form is equal to

$$deg(div(\omega)) = \frac{k_f}{2}(2g-2) = k_f(g-1),$$

where g is the genus of the Riemann surface $\Gamma_\chi \setminus \mathbb{H}^*$.

The differential form ω in every elliptic point of the order 2 has a polar with multiplicity $\frac{k_f}{4}$ and in every elliptic point of the order 3 it has a polar with multiplicity $\frac{k_f}{3}$. In every cusp form the form ω has a polar with multiplicity $\frac{k_f}{2} - 1$. Hence

$$deg(div(\omega)) = -\frac{\nu_2 k_f}{4} - \frac{\nu_3 k_f}{3} - \frac{\nu_\infty k_f}{2} + \nu_\infty,$$

where ν_2, ν_3 is the number of Γ_χ- nonequivalent elliptic points of the orders 2 and 3 correspondingly, ν_∞ is the number of Γ_χ- nonequivalent cusps. Then

$$k_f(g-1) = -\frac{\nu_2 k_f}{4} - \frac{\nu_3 k_f}{3} - \frac{\nu_\infty k_f}{2} + \nu_\infty.$$

It follows from the theorem 1.40 of the book [10] that

$$g - 1 = \frac{\mu}{12} - \frac{\nu_2}{4} - \frac{\nu_3}{3} - \frac{\nu_\infty}{2},$$

where

$$\mu = \mid \bar{\Gamma}(1) : \bar{\Gamma_\chi} \mid, \bar{\Gamma}(1) = SL_2(\mathbb{Z})/\{\pm E\}, \bar{\Gamma_\chi} = \Gamma_\chi/\{\pm E\}.$$

We have

$$k_f(\frac{\mu}{12} - \frac{\nu_2}{4} - \frac{\nu_3}{3} - \frac{\nu_\infty}{2}) = -\frac{\nu_2 k_f}{4} - \frac{\nu_3 k_f}{3} - \frac{\nu_\infty k_f}{2} + \nu_\infty,$$

$$\frac{k_f \mu}{12} = \nu_\infty \tag{1.1}$$

Let Γ_χ has the index n in the group $\Gamma_0(N)$ then $\mu = n\mu_0$, where

$$\mu_0 = \mid \bar{\Gamma}(1) : \bar{\Gamma_0}(N) \mid$$

and can be calculated by the formula:

$$\mu_0 = N \prod_p (1 + \frac{1}{p})$$

Let us consider the covering

$$\phi_0 : \bar{\Gamma_\chi} \setminus \mathbb{H}^* \to \bar{\Gamma_0}(N) \setminus \mathbb{H}^*.$$

If s is a cusp of $\Gamma_0(N)$ such that its prototypes under the mapping ϕ are regular cusps of Γ_χ then the full preimage of the point s consists of n points. If s is a cusp of $\Gamma_0(N)$ such that its prototypes under the mapping ϕ are irregular cusps of Γ_χ then the full preimage of the point s consists of $\frac{n}{2}$ points.

Let ν'_∞ is the number of cusps of $\Gamma_0(N)$ such that its prototypes under the mapping ϕ are regular cusps of Γ_χ and let $\nu"_\infty$ is the number of cusps of $\Gamma_0(N)$ such that its prototypes under the mapping ϕ are irregular cusps of Γ_χ . The sum $\nu'_\infty + \nu"_\infty$ is equal to the number ν^0_∞ of unequivalent cusps of $\Gamma_0(N)$, which can be calculated by the formula:

$$\nu^0_\infty = \sum_{d|N, d>0} \phi((d, \frac{N}{d})),$$

where ϕ - the Euler function. We have

$$\frac{k_f \mu_0}{12} = \nu'_\infty + \frac{\nu"_\infty}{2}$$

$$\frac{1}{2}\nu^0_\infty < \frac{k_f \mu_0}{12} \le \nu^0_\infty \Longrightarrow 6 < \frac{k_f \mu_0}{\nu^0_\infty} \le 12.$$

In the left part we have the strict inequality because all cusps of Γ_χ cannot be irregular if there are nonzero cusp forms of the weight k of Γ_χ. In fact $(-E) \notin \Gamma_\chi$ and

$$\Gamma_\chi \bigcap \Gamma_\infty = \left\{ \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \right\},$$

where Γ_∞ is stabilizator of the point ∞ in $SL_2(\mathbb{Z})$. Hence ∞ is a regular cusp of Γ_χ.

Let represent the ratio $\frac{k_f \mu_0}{\nu^0_\infty}$ in the suitable for calculations form. It easy to show that the function $\nu^0_\infty(N)$ is multiplicative. Let p be a prime. Then

$$\nu_\infty(p^{2l+1}) = 2(\phi(1) + \phi(p) + \cdots + \phi(p^l)) = 2\sum_{d|p^{k_f}} \phi(d) = 2p^l$$

$$\nu_\infty(p^{2l}) = 2(\phi(1)+\phi(p)+\cdots+\phi(p^{l-1}))+\phi(p^l) = 2p^{l-1}+p^l-p^{l-1} = p^l+p^{l-1}.$$

Let $N = nm^2$, where n is a squarefree number.

$$\nu_\infty(nm^2) = \prod_{p^{2l_p+1}\|n} (2p^{l_p}) \prod_{p^{l_p}\|m} (p^{l_p} + p^{l_p-1}).$$

we have

$$\begin{aligned}\frac{k_f\mu}{\nu_\infty} &= \frac{k_f nm^2 \prod_{p|N}(1+\frac{1}{p})}{\prod_{p^{2l_p+1}\|n}(2p^{l_p})\prod_{p^{l_p}\|m}(p^{l_p}+p^{l_p-1})} \\ &= k_f nm^2 \frac{\prod_{p|n}(1+\frac{1}{p})\prod_{p^{l_p}\|m}(p^{l_p}+p^{l_p-1})}{\prod_{p^{2l_p+1}\|n}(2p^{l_p})\prod_{p^{l_p}\|m}(p^{l_p}+p^{l_p-1})} \\ &= k_f nm \prod_{p^{2l_p+1}\|n} \frac{p+1}{2p^{l_p+1}} \\ &= k_f m \prod_{p^{2l_p+1}\|n} \frac{p^{l_p}(p+1)}{2}.\end{aligned}$$

At first we shall consider the case when all cusps of Γ_χ are regular. In this case we have

$$k_f m \prod_{p^{2l_p+1}\|n} \frac{p^{l_p}(p+1)}{2} = 12.$$

We find all numbers k_f and $N = nm^2$ satisfying this conditions. We have the following result:

$k_f = 1,\ N = 23, 33, 35, 42, 44, 56, 60, 63, 80, 96, 108, 128, 144.$

$k_f = 2,\ N = 11, 14, 15, 20, 24, 27, 32, 36.$

$k_f = 3,\ N = 7, 12, 16.$

$k_f = 4,\ N = 5, 6, 8, 9.$

$k_f = 6,\ N = 3, 4.$

$k_f = 8,\ N = 2.$

$k_f = 12,\ N = 1.$

We shall use the following proposition from the article [11] for finding the cusp forms with the known weights k_f and levels N.

Proposition 1. *Let us consider the product*

$$\pi = \prod_{k=1}^{s} a_k^{t_k},$$

where $a_k \in \mathbb{N}, t_k \in \mathbb{Z}$ with only finite $t_k \neq 0$. Let the following conditions be fulfilled:

1- $wt(\pi) := \frac{1}{2}\sum_{k=1}^{s} t_k$ *is nonzero integer,*

2- $deg(\pi) = \sum_{k=1}^{s} t_k a_k \equiv 0 \pmod{24}$,

Let N and f be positive integers satisfying

3- $a_k|N$, *if* $t_k \neq 0$,

4- $\displaystyle\sum_{k=1}^{s} \frac{Nt_k}{a_k} \equiv 0 \pmod{24}$,

5- f *is squarefree and the number* $f^{-1}\displaystyle\prod_{k=1}^{s} a_k^{t_k}$ *is a square of a rational number,*

6- $N \equiv 0 \pmod 4$, *if* $(-1)^{wt(\pi)}f \equiv -1 \pmod 4$, $N \equiv 0 \pmod 4$, *if* $f \equiv 2 \pmod 4$.

Let

$$\eta_\pi(z) = \prod_{k=1}^{s} \eta^{t_k}(a_k),$$

Then

$$\eta_\pi(\sigma(z)) = \chi(d)(cNz + d)^{wt(\pi)}\eta_\pi(z),$$

where

$$\sigma = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in \Gamma_0(N)$$

and $\chi(d)$ *are the character of square field* $\mathbb{Q}(\sqrt{(\epsilon f)})$*,*
where $\epsilon = (-1)^{wt(\pi)}$*, defined modulo N. Further let*

7- $\forall e \in \mathbb{N} \quad \displaystyle\sum_{k=1}^{s} (a_k, e)^2 \frac{t_k}{a_k} \geq 0.$

Then the function $\eta_\pi(z)$ *is holomorphic in every cusp of* $\Gamma_0(N)$ *and, if in the condition 7 the sign* $\geq$ *will be changed to* $>$*, then* $\eta_\pi(z)$ *is a cusp form.*

Let us show that for N = 33,35,42,56,60,96 there are no cusp forms satisfying the conditions of the theorem 1. We shall find for every N, except N = 42, the cusp form of the weight 2 which has in every cusp zero with multiplicity 2. We also find the cusp form $g_{42}(z)$ of the weight 3 which has in every cusp zero with multiplicity 3. Every such function is defined uniquely up to the multiplication on a constant because due to the Riemann-Roch theorem two holomorphic functions with equal divisors differ only on a constant.

These functions are:

$g_{33}(z) = \eta(33z)\eta(11z)\eta(3z)\eta(z)$,
$g_{35}(z) = \eta(35z)\eta(7z)\eta(5z)\eta(z)$,
$g_{56}(z) = \eta(28z)\eta(14z)\eta(4z)\eta(2z)$,
$g_{60}(z) = \eta(20z)\eta(15z)\eta(12z)\eta(z)$,
$g_{96}(z) = \eta(24z)\eta(12z)\eta(8z)\eta(4z)$,
$g_{42}(z) = \eta^3(21z)\eta^3(3z)$.

Since N satisfies to the condition (1.1), then the function $g_N(z)$ has no zeros on the upper half-plane. If there is a cusp form $f_N(z)$ of the level N and of the weight 1 which has in every cusp a zero with multiplicity 1 and no zeros on upper half-plane then $g_N = cf_N^2(z)$ if $N \neq 42$ and $g_{42} = cf_{42}^3(z)$. If both functions are normalized then $c = 1$. The direct inspection of the Fourier coefficients of the functions $g_N(z)$ when $N \neq 42$ shows that it cannot be a square of a cusp form. g_{42} is the cube of the cusp form of the level 63 and the weight 1. So we have no required forms in these cases.

For other values k_f and N the spaces $S_k(N, \chi)$ are one-dimensional as it pointed out in the article [3]. Using the proposition we find the required cusp forms. Since the corresponding spaces $S_k(N, \chi)$ are one-dimentional these cusp forms are eigenforms of Hecke algebra.

Further we shall consider the case when there are irregular cusps on Γ_χ. By the definition of the irregular point its stabilizator is generated by the element with $\sigma \in \Gamma_\chi$ with the trace equal to (-2). If χ is the trivial character then $\Gamma_\chi = \Gamma_0(N)$ and k_f must be even. In this case our argument is like previous. If χ is nontrivial character then it must have nontrivial kernel. It follows from the Dirichlet character's properties that it is possible only if $4|N$.

We find the values k_f and $N = nm^2$ satisfying the conditions

$$6 < k_f m \prod_{p^{2l_p+1}\|n} \frac{p^{l_p}(p+1)}{2} < 11, \quad 4|N.$$

We get

k_f =1, N = 28,40, 48, 64, 72.

k_f =2, N = 8.

k_f =3, N = 4.

We shall find for every N, the cusp form $g_N(z)$ of the weight 2 which has in every cusp of Γ_χ zero with multiplicity 2.

$g_{28}(z) = \eta(28z)\eta(14z)\eta(4z)\eta(2z)$,

$g_{48}(z) = \eta(24z)\eta(12z)\eta(8z)\eta(4z)$,

$g_{40}(z) = \eta^2(20z)\eta^2(4z)$.

$g_{64}(z) = \eta^2(16z)\eta^2(8z)$.

$g_{72}(z) = \eta^4(12z)$.

Since N satisfies to the condition (1.1), then the function $g_N(z)$ has no zeros on the upper half-plane. If there is a cusp form $f_N(z)$ of the level N and of the weight 1 which has in every cusp a zero with multiplicity 1 and no zeros on upper half-plane then $g_N = cf_N^2(z)$. If both functions are normalized then $c = 1$. The direct inspection of the Fourier coefficients of the functions $g_{28}(z)$ and when $g_{48}(z)$ shows that they cannot be squares of cusp forms. g_{40}, g_{64}, g_{72} are the squares of the cusp forms of the weights 1 and of the levels 80, 128, 144 correspondingly. So we have no required forms in these cases.

The spaces $S_3(8, \chi_1)$ and $S_5(4, \chi_2)$ where $\chi_1 = \left(\frac{-2}{d}\right)$ and $\chi_2 = \left(\frac{-1}{d}\right)$ correspondingly are one dimensional [3]. We note that χ_1 and χ_2 are the unique nontrivial characters modulo 8 and 4 correspondingly such that there are two numbers a and d in the kernel of the character such that $a + d = -2$. Using the proposition we find $f_8(z) = \eta^2(8z)\eta(4z)\eta(2z)\eta^2(z)$, $f_4(z) = \eta^4(4z)\eta^2(2z)\eta^4(z)$. Since the corresponding spaces $S_k(N, \chi)$ are one-dimentional these cusp forms are eigenforms of Hecke algebra. The theorem 1 is proved.

Remark. If we omit the first condition the theorem 1 we can add to these functions two cusp forms of half-integral weight: $\eta(24z)$ and $\eta^3(8z)$. The Fourier coefficients of these 30 functions are multiplicative. In what follows for brevity and convinience we shall call them multiplicative η-products. Dummit, Kisilevsky and MacKay have received the same list of cusp forms from another point of view: they have shown that among functions of the kind

$$f(z) = \prod_{k=1}^{s} \eta^{t_k}(a_k z)$$

where a_k and $t_k \in \mathbb{N}$, only these 30 functions have multiplicative coefficients. They have checked it by the calculations on the computer [3].

2. Representations of finite groups and modular forms

There are different ways of assigning modular forms to the elements of a group. One of these mappings is as follows: let G be finite group, let g be an element of G, let Φ be a unimodular representation of the group G in the space V whose dimension is a multiple of 24, and let

$$P_g(x) = \prod_{k=1}^{s} (x^{a_k} - 1)^{t_k}$$

be a characteristic polynomial of the operator $\Phi(g)$. Then we can assign the function

$$\eta_g(z) = \prod_{k=1}^{s} \eta^{t_k}(a_k z)$$

with each element $g \in G$. The function $\eta_g(z)$ is a cusp form of a certain level $N(g)$ and of the weight

$$k(g) = \frac{1}{2} \sum_{k=1}^{s} t_k$$

and its character is equal to the character of the quadratic field

$$\mathbb{Q}\sqrt{\prod_{k=1}^{s}(ia_k)^{t_k}}.$$

Using the modular form $\eta_g(z)$, we can define, on an arbitrary finite group G, a function $a_n(g)$, for any n, so that the value of a_n is equal to the n-th coefficient of the Fourier expansion of $\eta_g(z)$ in the neighbourhood of the point $z = \infty$ $(q = 0)$ and, for any prime p, the function

$$\psi_p(g) = \begin{cases} p^{k(g)-1}\chi_g(p), & \text{if } (ord(g), p) = 1 \\ 0, & \text{if } (ord(g), p) = p \end{cases}$$

where $ord(g)$ is the order of the element g, $k(g)$ and $\chi_g(p)$ are the weight and the character of $\eta_g(z)$,

For a modular form $\eta_g(z)$ that is an eigenform of all Hecke operators, the functions $a_n(g)$ and $\psi_p(g)$ appear in the expansion of its Mellin transform in the Euler product

$$L_g(z) = \sum_{n=1}^{\infty}\frac{a_n(g)}{n^s} = \prod_p (1 - \frac{a_p(g)}{p^s} + \frac{\psi_p(g)}{p^{2s}})^{-1}.$$

G. Mason considered a natural representation of the Mathieu group M_{24} on the Leech lattice [1,2]. He proved that for any element $g \in M_{24}$ the function $\eta_g(z)$ associated with this representation is an eigenform of all Hecke operators. The functions a_n and $\psi_p(g)$ are virtual characters of the group M_{24}. It was noted that for $p \neq 3$ $\psi_p(g)$ is an effective character.

The problem of determining the nature of the functions $a_n(g)$ and $\psi_p(g)$ for other groups appears naturally.

3. Weyl characters of Lie groups and the characters of modular forms

We shall investigate the restriction of the adjoint representations of simple Lie groups whose Lie algebras are of even rank to finite subgroups in which each element has a rational characteristic polynomial in the adjoint representation. The characters $a_n(g)$ are described in this case by MacDonald's formulas [8]. We shall try to elucidate the nature of the characters $\psi_p(g)$ in this situation.

Let us consider a simple Lie group G_0 whose Lie algebra $Lie(G_0)$ is of even rank. Let g be a finite subgroup of this Lie group such that each element $g \in G$ has in the adjoint representation a rational characteristic polynomial

$$\prod_{k=1}^{s}(x^{a_k} - 1)^{t_k},$$

with which the function

$$\eta_g(z) = \prod_{k=1}^{s} \eta^{t_k}(a_k)$$

is associated.

For any prime p on the group G we define the function

$$\psi_p(g) = p^{k(g)-1}\chi_g(p)$$

where $k(g)$ and $\chi_g(p))$ are the weight and character of the form $\eta_g(z)$. We use

$$ch_{(p-1)\varrho}$$

to denote the Weyl character of an irreducible representation of the Lie group G_0 with leading weight $(p-1)\varrho$, where ϱ is the half-sum of the positive roots of the Lie algebra $Lie(G_0)$.

Theorem 2. *For any element $g \in G$ and odd prime p that is relatively prime to the order of the element g we have*

$$\psi_p(g) = \left(\frac{-1}{p}\right)^{\frac{dimG_0}{2}} p^{\frac{r}{2}-1} ch_{(p-1)\varrho}(g)$$

where r is the rank of the Lie algebra Lie G_0.

This theorem is proved in the article [11].

The Lie algebras of the type A_l where $l \equiv 0, 4, 6, 10, 12, 16, 18, 22 \bmod 24$, of the type B_l where $l \equiv 0, 16 \pmod{24}$, of the type C_l where $l \equiv 0, 16 \pmod{24}$, of the type D_l where $l \equiv 0, 8 \pmod{24}$ have ranks which are multipliers of 24. We can associate cusp forms $\eta_g(z)$ with elements of corresponding Lie groups that have in adjoint representation the characteristic polynomials with rational coefficients. The functions $\psi_p(g)$ in this case are effective characters.

4. Multiplicative η-products and finite subgroups of $SL(5, \mathbb{C})$.

Multiplicative η-products can be associated with elements of finite subgroups in $SL(5, \mathbb{C})$ by means of the adjoint representation. Let us consider this correspondence in detail. It is an interesting problem to find finite groups such that all modular forms associated with elements of these groups by means of some representation are eigenforms of Hecke algebra.

G. Mason has shown that all functions associated with elements of the Mathieu group M_{24} by means of the representation on the Leech Lattice are multiplicative η- products. There are 21 functions of this kind. In the following theorems we shall give other examples of such groups.

Theorem 3. *By means of the adjoint representations all multiplicative η-products whose weight is more than 1 can be associated with finite order elements of the group $SL(5,\mathbb{C})$. The eigenvalues of the element $g \in SL(5,\mathbb{C})$ that corresponds to a given cusp form can be found uniquely,up to a permutation of the values, up to raising eigenvalues to a power coprime with the order of the element g, and up to the multiplication of each eigenvalue by the same fifth root of unity.*

Proof. The adjoint representation Ad of the group $SL(5,\mathbb{C})$ is a subrepresentation of degree 24 of the representation

$$\Phi \otimes \Phi^* : SL(5,\mathbb{C}) \to V \otimes V^* \;\cong\; Hom(V,V),$$

where

$$\Phi : SL(5,\mathbb{C}) \to V$$

is the natural representation of $SL(5,\mathbb{C})$ in 5-dimensional space V and Φ^* is the conjugate representation to Φ.

Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ be the eigenvalues of the operator $\Phi(g)$. The elements

$$\frac{\lambda_l}{\lambda_m},\ 1 \le l, m \le 5$$

are the eigenvalues of the operator $(\Phi \otimes \Phi^*)(g)$. Eliminating one eigenvalue equal to 1 we obtain the set of eigenvalues of the operator $Ad(g)$. Using this method for each multiplicative η- product we find elements $g \in SL(5,\mathbb{C})$ with which they may be associated.

Note that among the eigenvalues of the operator $Ad(g)$ there are no less than 4 units, so for its characteristic polynomial

$$\prod_{k=1}^{s} (x^{a_k} - 1)^{t_k}$$

we have $\sum_{k=1}^{s} t_k \ge 4$. Thus the weight of the modular form $\eta_g(z)$ associated with g is greater than 1. The uniqueness is followed from the concrete calculations.

The results are shown in the following table. We use $\zeta_m$to denote the m-th root of 1. In the right column we write the cusps forms, while in the left we write the eigenvalues of the corresponding element $g \in SL(5,\mathbb{C})$.

eigenvalues	cusp forms
1, 1, 1, 1, 1	$\eta^{24}(z)$
$\zeta_2, \zeta_2, \zeta_2, \zeta_2, 1$	$\eta^8(2z)\eta^8(z)$
$\zeta_2, \zeta_2, 1, 1, 1$	$\eta^{12}(2z)$
$\zeta_3, \zeta_3, \zeta_3, 1, 1$	$\eta^6(3z)\eta^6(z)$
$\zeta_4^3, \zeta_4^2, \zeta_4, \zeta_4, \zeta_4$	$\eta^4(4z)\eta^2(2z)\eta^4(z)$
$\zeta_3^2, \zeta_3^2, \zeta_3, \zeta_3, 1$	$\eta^8(3z)$
$\zeta_4^3, \zeta_4^3, \zeta_4, \zeta_4, 1$	$\eta^4(4z)\eta^4(2z)$
$\zeta_5^3, \zeta_5, \zeta_5, 1, 1$	$\eta^4(5z)\eta^4(z)$
$\zeta_6^4, \zeta_6^3, \zeta_6^3, \zeta_6, \zeta_6$	$\eta^2(6z)\eta^2(3z)\eta^2(2z)\eta^2(z)$
$\zeta_6^5, \zeta_6^4, \zeta_6^2, \zeta_6, 1$	$\eta^4(6z)$
$\zeta_7^4, \zeta_7^2, \zeta_7, 1, 1$	$\eta^3(7z)\eta^3(z)$
$\zeta_6^5, \zeta_6^3, \zeta_6^2, \zeta_6^2, 1$	$\eta^3(6z)\eta^3(2z)$
$\zeta_4^3, \zeta_4^2, \zeta_4^2, \zeta_4, 1$	$\eta^6(4z)$
$\zeta_{14}^{11}, \zeta_{14}^9, \zeta_{14}^7, \zeta_{14}, \zeta_{14}$	$\eta(14z)\eta(7z)\eta(2z)\eta(z)$
$\zeta_8^7, \zeta_8^5, \zeta_8^3, \zeta_8, 1$	$\eta^2(8z)\eta^2(4z)$
$\zeta_9^8, \zeta_9^5, \zeta_9^3, \zeta_9^2, 1$	$\eta^2(9z)\eta^2(3z)$
$\zeta_{12}^9, \zeta_{12}^7, \zeta_{12}^4, \zeta_{12}^3, \zeta_{12}$	$\eta(12z)\eta(6z)\eta(4z)\eta(2z)$
$\zeta_8^5, \zeta_8^5, \zeta_8^3, \zeta_8^2, \zeta_8 1$	$\eta^2(8z)\eta(4z)\eta(2z)\eta^2(z)$
$\zeta_{15}^{12}, \zeta_{15}^{10}, \zeta_{15}^7, \zeta_{15}, 1$	$\eta(15z)\eta(5z)\eta(3z)\eta(z)$
$\zeta_{10}^8, \zeta_{10}^6, \zeta_{10}^5, \zeta_{10}, 1$	$\eta^2(10z)\eta^2(2z)$
$\zeta_{11}^9, \zeta_{11}^5, \zeta_{11}^4, \zeta_{11}^3, \zeta_{11}$	$\eta^2(11z)\eta^2(z)$

Theorem 4. *The maximal finite subgroups of $SL(5, \mathbb{C})$ whose elements g have characteristic polynomials of the form $\prod_{k=1}^{s}(x^{a_k} - 1)^{t_k}$ in adjoint representation and the corresponding cusp forms $\eta_g(z) = \prod_{k=1}^{s} \eta^{t_k}(a_k z)$ are of the type described in the theorem 1, are the direct products of the group $\mathbb{Z}_5$ (which is generated by the scalar matrix) and one of the following groups: S_4, $A_4 \times \mathbb{Z}_2$, $\mathbb{Q}_8 \times \mathbb{Z}_3$, $D_4 \times \mathbb{Z}_3$, the binary tetrahedral group, the metacyclic group of order 21, D_6, the metacyclic group of order 12: $< S, T : S^3 = T^2 = (ST)^2 >$, all groups of order 16, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_{15}$, $\mathbb{Z}_{14}$, $\mathbb{Z}_{11}$, $\mathbb{Z}_{10}$, $\mathbb{Z}_9$.*

Sketch of the Proof. This proof demands much place to be stated. It was published in the Thesis of the author. We have used various facts of the theory of groups and the theory of group representations. We present the result in the following table where we point which multiplicative η-products are associated with the elements of each group.

order	group	corresponding cusp forms
9	$(\mathbb{Z}_3)^2$	$\eta^{24}(z), \eta^6(3z)\eta^6(z), \eta^8(3z)$
9	$\mathbb{Z}_9$	$\eta^{24}(z), \eta^2(9z)\eta^2(3z), \eta^6(3z)\eta^6(z)$
10	$\mathbb{Z}_{10}$	$\eta^{24}(z), \eta^2(10z)\eta^2(2z),$ $\eta^4(5z)\eta^4(z), \eta^{12}(2z)$
11	$\mathbb{Z}_{11}$	$\eta^{24}(z), \eta^2(11z)\eta^2(z)$
12	$\mathbb{D}_6$	$\eta^{24}(z), \eta^4(6z), \eta^8(3z), \eta^{12}(2z)$
12	$< S^3 = T^2 = (ST)^2 >$	$\eta^{24}(z), \eta^4(6z), \eta^8(3z),$ $\eta^6(4z), \eta^{12}(2z)$
14	$\mathbb{Z}_{14}$	$\eta^{24}(z), \eta^8(2z)\eta^8(z), \eta^3(7z)\eta^3(z),$ $\eta(14z)\eta(7z)\eta(2z)\eta(z)$
15	$\mathbb{Z}_{15}$	$\eta^{24}(z), \eta^6(3z)\eta^6(z),$ $\eta^4(5z)\eta^4(z),$ $\eta(15z)\eta(5z)\eta(3z)\eta(z)$
16	$\mathbb{Z}_4 \times (\mathbb{Z}_2)^2$	$\eta^{24}(z), \eta^8(2z)\eta^8(z),$ $\eta^4(4z)\eta^4(2z),$ $\eta^4(4z)\eta^2(2z)\eta^4(z), \eta^{12}(2z)$
16	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\eta^{24}(z), \eta^8(2z)\eta^8(z),$ $\eta^4(4z)\eta^2(2z)\eta^4(z),$ $\eta^2(8z)\eta(4z)\eta(2z)\eta^2(z), \eta^{12}(2z)$
16	$(\mathbb{Z}_2)^4$	$\eta^{24}(z), \eta^8(2z)\eta^8(z), \eta^{12}(2z)$
16	$\mathbb{D}_8$	$\eta^{24}(z), \eta^8(2z)\eta^8(z), \eta^4(4z)\eta^4(2z),$ $\eta^2(8z)\eta^2(4z), \eta^{12}(2z)$
16	$< T^2 = E, TST = S^3 >$	$\eta^{24}(z), \eta^8(2z)\eta^8(z), \eta^4(4z)\eta^4(2z),$ $\eta^2(8z)\eta^2(4z), \eta^{12}(2z)$
16	$\mathbb{Z}_2 \times \mathbb{D}_4$	$\eta^{24}(z), \eta^8(2z)\eta^8(z),$ $\eta^4(4z)\eta^4(2z), \eta^{12}(2z)$
16	$\mathbb{Z}_2 \times \mathbb{Q}_8$	$\eta^{24}(z), \eta^8(2z)\eta^8(z),$ $\eta^4(4z)\eta^4(2z), \eta^{12}(2z)$
16	$< T^2 = E, TST = S^5 >$	$\eta^{24}(z), \eta^8(2z)\eta^8(z), \eta^4(4z)\eta^4(2z),$ $\eta^2(8z)\eta^2(4z), \eta^{12}(2z)$
16	$< T^2 = S^4 = (ST)^2 >$	$\eta^{24}(z), \eta^8(2z)\eta^8(z), \eta^4(4z)\eta^4(2z),$ $\eta^2(8z)\eta^2(4z)$
16	$< T^4 = S^4 = E, T^{-1}ST = S^{-1} >$	$\eta^{24}(z), \eta^8(2z)\eta^8(z),$ $\eta^4(4z)\eta^4(2z), \eta^6(4z), \eta^{12}(2z)$
16	$< R^2 = S^2 = T^2 = E, TRS = STR = RST >$	$\eta^{24}(z), \eta^8(2z)\eta^8(z),$ $\eta^4(4z)\eta^4(2z), \eta^{12}(2z)$
16	$< R^4 = S^4 = E, (RS)^2 = (R^{-1}S)^2 = E >$	$\eta^{24}(z), \eta^8(2z)\eta^8(z),$ $\eta^6(4z), \eta^{12}(2z)$
21	$< T^3 = E, T^{-1}ST = S^2 >$	$\eta^{24}(z), \eta^3(7z)\eta^3(z), \eta^8(3z)$
24	$\mathbb{Z}_3 \times \mathbb{Q}_8$	$\eta^{24}(z), \eta^6(3z)\eta^6(z),$ $\eta^4(4z)\eta^4(2z), \eta^8(2z)\eta^8(z),$ $\eta(12z)\eta(6z)\eta(4z)\eta(2z),$ $\eta^2(6z)\eta^2(3z)\eta^2(2z)\eta^2(z)$
24	$\mathbb{Z}_2 \times \mathbf{A}_4$	$\eta^{24}(z), \eta^8(2z)\eta^8(z),$ $\eta^{12}(2z), \eta^8(3z), \eta^4(6z)$
24	$< R^3 = S^3 = (RS)^2 >$	$\eta^{24}(z), \eta^8(2z)\eta^8(z),$ $\eta^{12}(2z), \eta^8(3z), \eta^4(6z)$
24	$\mathbb{S}_4$	$\eta^{24}(z), \eta^4(6z), \eta^8(3z), \eta^{12}(2z)$
24	$\mathbb{Z}_3 \times \mathbb{D}_4$	$\eta^{24}(z), \eta^6(3z)\eta^6(z),$ $\eta^4(4z)\eta^4(2z), \eta^8(2z)\eta^8(z),$ $\eta(12z)\eta(6z)\eta(4z)\eta(2z),$ $\eta^2(6z)\eta^2(3z)\eta^2(2z)\eta^2(z),$ $\eta^{12}(2z), \eta^3(6z)\eta^3(2z)$

5. Multiplicative η-products and regular representations of groups of order 24

Let us continue to study relationships between multiplicative η-products and representations of finite groups.

Theorem 5. *Let G be any group of order 24, let Φ be its regular representation,and let*

$$P_g(x) = \prod_{k=1}^{s} (x^{a_k} - 1)^{t_k}$$

be the characteristic polynomial of the operator $\Phi(g)$ for an element $g \in G$. Then the function

$$\eta_g(z) = \prod_{k=1}^{s} \eta^{t_k}(a_k)$$

is multiplicative η-product.

Proof. In the monograph [9] the generating elements and defining relations are given for all non-abelian groups of order 24. Applying these data, we can write out the conjugacy classes,the subgroups and the factorgroups for the groups under consideration. Furthermore, by using the well-known representations of abelian groups and of the groups D_n, S_4 and Q_8 and by applying the orthogonality relations for the characters of representations, we can construct the tables of irreducible representations and find their eigenvalues. Since the regular representation is a direct sum in which any irreducible representation enters with multiplicity equal to its dimension, it follows that these tables can be used to write out the eigenvalues of the regular representations. The calculations are technical and too cumbersome, and therefore we present only the result, namely, the lists of cusp forms that correspond to the elements of the groups of order 24. It turns out that in all these groups the same functions correspond to all elements of the same order and among 30 multiplicative η-products, only eight of them appear and two of them, namely, $\eta(24z)$ and $\eta^3(8z)$ have half-integer weights.

Thus, we have the following table:

group	corresponding cusp forms
$\mathbb{Z}_{24}$	$\eta(24z), \eta^2(12z), \eta^3(8z), \eta^4(6z), \eta^6(4z), \eta^8(3z), \eta^{12}(2z), \eta^{24}(z)$
$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$	$\eta^2(12z), \eta^4(6z), \eta^6(4z), \eta^8(3z), \eta^{12}(2z), \eta^{24}(z)$
$(\mathbb{Z}_2)^3 \times \mathbb{Z}_3$	$\eta^4(6z), \eta^8(3z), \eta^{12}(2z), \eta^{24}(z)$
$\mathbb{S}_4$	$\eta^6(4z), \eta^8(3z), \eta^{12}(2z), \eta^{24}(z)$
$(\mathbb{Z}_3) \times \mathbb{D}_4$	$\eta^2(12z), \eta^4(6z), \eta^6(4z), \eta^8(3z), \eta^{12}(2z), \eta^{24}(z)$
$(\mathbb{Z}_3) \times \mathbb{Q}_8$	$\eta^2(12z), \eta^4(6z), \eta^6(4z), \eta^8(3z), \eta^{12}(2z), \eta^{24}(z)$
$(\mathbb{Z}_2) \times \mathbb{D}_6$	$\eta^4(6z), \eta^8(3z), \eta^{12}(2z), \eta^{24}(z)$
$(\mathbb{Z}_2) \times \mathbf{A}_4$	$\eta^4(6z), \eta^8(3z), \eta^{12}(2z), \eta^{24}(z)$
$\mathbb{D}_{12}$	$\eta^2(12z), \eta^4(6z), \eta^6(4z), \eta^8(3z), \eta^{12}(2z), \eta^{24}(z)$
$< R, S : R^4 = S^6 = (RS)^2 = (R^{-1}S)^2 >$	$\eta^4(6z), \eta^6(4z), \eta^8(3z), \eta^{12}(2z), \eta^{24}(z)$
$< R, S : R^3 = S^3 = (RS)^2 >$	$\eta^4(6z), \eta^6(4z), \eta^8(3z), \eta^{12}(2z), \eta^{24}(z)$
$< R, S : RSR^{-1} = S^2, R^8 = E, S^3 = E >$	$\eta^2(12z), \eta^4(6z), \eta^6(4z), \eta^3(8z), \eta^8(3z), \eta^{12}(2z), \eta^{24}(z)$
$< R, S : R^2 = S^6 = (RS)^2 >$	$\eta^2(12z), \eta^4(6z), \eta^6(4z), \eta^8(3z), \eta^{12}(2z), \eta^{24}(z)$
$\mathbb{Z}_2\times < R, S : R^4 = S^6 = E, R^{-1}SR = S^{-1} >$	$\eta^2(12z), \eta^4(6z), \eta^6(4z), \eta^8(3z), \eta^{12}(2z), \eta^{24}(z)$
$(\mathbb{Z}_4) \times \mathbb{S}_3$	$\eta^2(12z), \eta^6(4z), \eta^4(6z), \eta^8(3z), \eta^{12}(2z), \eta^{24}(z)$

6. Multiplicative η-products and representations of dihedral groups

Theorem 6. *For the dihedral groups D_n, where $3 \leq n \leq 23, n \neq 13, 17, 19$ there is such exact representation Φ that $\forall g \in D_n$ the operater $\Phi(g)$ has such characteristic polynomial*

$$\prod_{k=1}^{s} (x^{a_k} - 1)^{t_k}$$

that the function

$$\eta_g(z) = \prod_{k=1}^{s} \eta^{t_k}(a_k)$$

is multiplicative η-product. For other dihedral groups there is no such representation.

We note that all multiplicative η- products can be associated with elements of dihedral groups (for different n).

This result has been published in [15].

7. The arithmetic interpretation of the Fourier coefficients of multiplicative η-products

At the end of the article we shall consider the arithmetic interpretation of the Fourier coefficients of multiplicative η-products and their Mellin transformation. From this point of view the multiplicative η- products of the weight one have been studied by Japanese mathematicians M. Koike, T. Kondo, T. Tasaka and others [4,5,6]. The multiplicative η-products of the weight 2 have been studied by French mathematician Ligozat [7]. Dummit, Kisilevsky and McKay have found for 16 of 28 multiplicative η-products of the integer weight L- functions with grossen-characters of imaginary quadratic fields which are equal to the Mellin transformations of this forms. They have proved that for other 12 multiplicative η-products of the integer weight this correspondence is impossible.

In the following theorem we present the analogous formulas where instead of the ring of integers of an imaginary quadratic field we consider orders in the algebra of quaternions and the Cayley algebra.

Theorem 7. *Let $\mathbb{H}$ be the algebra of quaternions over $\mathbb{Q}$ and Γ_4 is the lattice of the Hurwitz quaternions :*

$$\alpha = \frac{a+bi+cj+dk}{2}, a \equiv b \equiv c \equiv d \pmod 2, a,b,c,d \in \mathbb{Z}.$$

Then

$$\frac{1}{12} \sum_{\alpha \in \Gamma_4 \subset \mathbb{H}} \alpha^6 e^{2\pi i z N(\alpha)} = \eta^8(z)\eta^8(2z).$$

Furthermore

$$\frac{1}{8} \sum_{\substack{\alpha \in \mathbb{H}, \\ a+b+c+d \equiv 1 \pmod 2}} \alpha^4 e^{2\pi i z N(\alpha)} = \eta^{12}(2z),$$

where the summation is taken over such quaternions $a+bi+cj+dk$,that $a+b+c+d \equiv 1 \pmod 2$, $a,b,c,d \in \mathbb{Z}$.

Theorem 8. *Let Ca be the Cayley algebra. Then we can construct in Ca the order on which the bilinear form*

$$< \alpha, \beta >= \alpha\bar{\beta} + \beta\bar{\alpha}$$

defines the structure of the even unimodular lattice of the type Γ_8 where the root system E_8 is closed under the multiplication in Cayley algebra.

Then the sum

$$\frac{1}{12} \sum_{\alpha \in \Gamma_8 \subset Ca} \alpha^8 e^{2\pi i z N\alpha}$$

over all elements of this order is equal to the cusp form $\eta^{24}(z)$.

These theorems have been published in [16].

References

[1] G. Mason, M_{24} *and certain automorphic forms.* Contemp. Math. **45** (1985), 223–244.

[2] G. Mason, *Finite groups and Hecke operators.* Math.Ann. **283** (1989), 381–409.

[3] D. Dummit, H. Kisilevsky, J. McKay, *Multiplicative products of η-functions.* Contemp. Math. **45** (1985), 89–98.

[4] T. Hiramatsu, *Theory of automorphic forms of weight 1.* Adv. Stud. Pure Math. **13** (1988), 503–584.

[5] M. Koike, *Higher reciprocity law , modular forms of weight 1 and elliptic curves.* Nagoya Math.J. **98** (1985), 109–115.

[6] M. Koike, *On McKay's conjecture.* Nagoya Math.J. **95** (1984), 85–89.

[7] G. Ligozat, *Courbes modulaires de gendre 1.* Bull. Soc. Math. France **43** (1975), 80 pp.

[8] I.G. MacDonald, *Affine systems of roots and the Dedekind η-function.* Sb. Perev. Mat. **16** (1972), 3–49.

[9] H.S.M. Coxeter, W.O.J. Mozer, *Generators and relations for discrete groups.* Second edition, Band 14 Springer-Verlag, Berlin-Göttingen-New York 1965 ix+161 pp.

[10] G. Shimura, *An introduction to the arithmetic theory of automorphic functions.* Kanô Memorial Lectures, No. 1. Publications of the Mathematical Society of Japan, No. 11. Iwanami Shoten, Publishers, Tokyo; Princeton University Press, Princeton, N.J., 1971. xiv+267 pp.

[11] T. Kondo, *Examples of multiplicative η- products.* Sci. Pap. Coll. Arts and Sci. Univ. Tokyo. **35** (1986), 133–149.

[12] G.V. Voskresenskaya, *Modular forms and group representations.* Matem. Zametki **52** (1992), 25–31.

[13] G.V. Voskresenskaya, *Cusp forms and finite subgroups in $SL(5, \mathbb{C})$.* Fun. anal. and appl. **29** (1995), 71–73.

[14] G.V. Voskresenskaya, *Modular forms and regular representations of groups of order 24.* Matem. Zametki **60** (1996), 292–294.

[15] G.V. Voskresenskaya, *Modular forms and the representations of dihedral groups.* Matem. Zametki **63** (1998), 130–133.

[16] G.V. Voskresenskaya, *Hypercomplex numbers, root systems and modular forms, "Arithmetic and geometry of varieties".* Samara, (1992), 48–59.

Galina Voskresenskaya
Deptartment of Mathematics,
Samara State University
Acad. Pavlov str.1
Samara, 443011, Russia
E-mail : vosk@info.ssu.samara.ru