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Stably rational algebraic tori

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Résumé. On montre qu'un tore stablement rationnel avec un corps de décomposition cyclique est rationnel.

abstract. The rationality of a stably rational torus with a cyclic splitting field is proved.

Let X be an irreducible algebraic variety over a field k of characteristic zero. We call X *stably rational* if $X \times_k \mathbf{A}^m$ is rational for some m. In 1984 the first and hitherto only known examples of stably rational but non rational varieties were constructed [1]. Such examples exist over nonclosed fields and over the complex field $\mathbf{C}$. Let now $X = T$ be an algebraic torus over a nonclosed field k. In the category of algebraic tori we have a criterion of stable rationality which enables us to construct stably rational tori. However, among them one has not yet found a nonrational torus.

Conjecture. *Any torus which is stably rational over k is rational over k.*

In this paper we suggest a new approach to this conjecture. In particular, we prove the rationality of any stably rational torus with a cyclic splitting field.

First recall some facts and definitions. Let T be an algebraic torus defined over a field k, L/k the normal finite splitting field of T, $\Pi = Gal(L/k)$, and $\hat{T}$ the Π-module of rational characters of T. An important rôle is played by quasi-split tori. A torus S over k is called quasi-split if the module $\hat{S}$ has a basis permuted by Π. For example, any maximal k-torus of the group $GL_k(n)$ is quasi-split. Dividing the permutation basis into orbits of Π, we get a representation of the Π-module $\hat{S}$ as a direct sum of indecomposable permutation modules. This construction gives a representation of the torus S as a direct product

$$S = R_{F_1/k}(G_m) \times \cdots \times R_{F_t/k}(G_m). \tag{1}$$

The group S is a maximal torus in the general linear group $GL_k(n)$, $n = \dim S$, and

$$S(k) = F_1^* \times \cdots \times F_t^*, \ k \subset F_i \subset L.$$

The following conditions are equivalent [2]:
a) T is stably rational over k ;

b) T can be included in an exact sequence of the form

$$1 \to S_1 \xrightarrow{\alpha} S_2 \xrightarrow{\beta} T \to 1,$$

where S_1 and S_2 are quasi-split k-tori.
Since S_1 is quasi-split, $H^1(M/F, S_1(M)) = 0$ for any Galois extension M/F, $F \supset k$. From here it follows that there exists a rational k-section $\gamma : T \to S_2$ of the morphism β, $\beta \cdot \gamma = Id$. Let $\delta : S_2 \to S_1$ be the rational k-map defined by

$$\delta(g) = \alpha^{-1}(g/\gamma\beta(g)),\ g \in S_2(M),\ M \supset k.$$

Clearly

$$\delta(\alpha(h)g) = h\delta(g),\ h \in S_1(M).$$

The map δ will be called a *covariant* of the representation α. (The map $\gamma : S_2 \to S_1$ with the condition $\gamma(\alpha(h)g) = h\gamma(g)$ is called a covariant of the representation α.)

We have a rational map

$$\varphi = (\beta, \delta) : S_2 \to T \times_k S_1,$$

which is birational over k. Indeed, let $\varphi(g_1) = \varphi(g_2)$. Then $\beta(g_1) = \beta(g_2)$ and we have $g_2 = \alpha(h)g_1$. Hence $\delta(g_1) = \delta(g_2) = h\delta(g_1)$. Since the element $\delta(g_1)$ is invertible, $h = 1$, i.e. $g_1 = g_2$. Since $\dim S_2 = \dim S_1 + \dim T$, all varieties are irreducible and φ is injective on an open subset, we conclude that φ is birational over k. We established the following fact.

Proposition. *Let $\delta : S_2 \to S_1$ be a covariant of an exact representation $\alpha : S_1 \to S_2$ and $W = \delta^{-1}(a)$ the fibre of δ over $a \in S_1(k)$. Then the varieties $T = S_2/\alpha(S_1)$ and W are birationally equivalent over k. All the fibres of δ are stably rational over k.*

This proposition allows us to reformulate the question on stably rational tori in terms of linear representations. Consider one component $R_{F/k}(G_m)$ of S_1, $R_{F/k}(k) = F^*$, where F^* is the multiplicative group of the field F, $(F : k) = r$. Denote by V_r the vector space of dimension r over k. We have the regular exact representation of F^* on V_r, and this action extends to the action of $R_{F/k}(G_m)$ on the family $V_r \otimes_k M$, $M \supset k$, i.e. $R_{F/k}(G_m)$ acts on V_r in the sense of algebraic geometry. Let U be a direct sum of the V_r's corresponding to decomposition (1) of S_1, and let V be the analogous sum corresponding to S_2. The map α defines a linear representation of S_1 on V. The groups S_1 and S_2 act faithfully on U and V. Each S_i has an open orbit in the corresponding space. We shall sometimes identify this orbit and the group itself. We can now extend the covariant $\delta : S_2 \to S_1$ to a rational covariant

$$\delta : V \to U,\ \delta(\alpha(h)v) = h\delta(v),\ h \in S_1(M),\ v \in V(M) = V \otimes_k M.$$

Remark. Since the field k is of characteristic zero, any representation of S_1 can be studied at the level of the group $S_1(k)$, i.e. we can consider usual linear representations of groups of the type F^*.

Thus the field of rational functions $k(T)$ of a stably rational torus T is the field of invariants of the quasi-split torus S_1 acting faithfully on V by the monomorphism α :

$$k(T) = k(V)^{S_1}.$$

Obviously the converse is also true, i.e. if a quasi-split torus S acts faithfully on a linear space V, the field $k(V)^S$ is stably rational over k. Indeed, the torus S is a subgroup of a maximal k-torus S' of $GL(V)$ and the field $k(V)^S$ is the field of rational functions of the stably rational torus S'/S.

Example. Let L be a finite extension of a field k and F a subfield of $L,\ k \subset F \subset L$. We have an embedding

$$\alpha : S_1 = R_{F/k}(G_m) \to R_{L/k}(G_m) = S_2.$$

Consider the quotient $T = S_2/S_1$. We have an epimorphism of linear spaces over k

$$\delta = Tr_{L/F} : L \to F,\ \delta(ax) = a\delta(x),\ a \in F,\ x \in L,$$

i.e. δ is a covariant of α. The fibre $W = \delta^{-1}(1)$ is an affine space over $k,\ \dim W = \dim T$. The varieties T and W are birationally equivalent, hence the torus T is rational over k.

We now consider a more complicated example and presents a new approach to solution these problem. Let V_m and V_n be vector k-spaces, S_m and S_n maximal k-tori in $GL(V_m)$ and $GL(V_n)$. Then we have a representation of $S_m \times_k S_n$ in the tensor space $V_m \otimes_k V_n = V$. Let N be the image of $S_m \times_k S_n$ in $GL(V)$ under this tensor representation. The group N is contained in a maximal k-torus S of $GL(V)$. There are two exact sequences of k-tori

$$(2)\qquad 1 \to G_{m,k} \to S_m \times_k S_n \to N \to 1,$$

$$(3)\qquad 1 \to N \to S \to T \to 1.$$

We ask whether T is k-rational. Let us write down the exact sequences of Π-modules dual to (2) and (3)

$$(4)\qquad 0 \to \hat{N} \to \hat{S}_m \oplus \hat{S}_n \xrightarrow{\varepsilon} \mathbf{Z} \to 0,$$

$$(5)\qquad 0 \to \hat{T} \to \hat{S} \to \hat{N} \to 0,$$

where $\Pi = Gal(L/k)$, L is the normal finite splitting field of all tori in sequences (2) and (3). We only consider the case $(m, n) = 1$. From (2) we obtain the cohomology exact sequence

$$(\hat{S}_m \oplus \hat{S}_n)^{\Pi} \xrightarrow{\varepsilon} \mathbf{Z} \to H^1(\Pi, \hat{N}) \to 0.$$

The group $Im(\varepsilon)$ contains the integers of the form $am + bn$, this implies that ε is an epimorphism. Hence $H^1(\Pi, \hat{N}) = 0$, it follows that sequence (4) splits, i.e.

$$\hat{N} \oplus \mathbf{Z} \cong \hat{S}_m \oplus \hat{S}_n. \tag{6}$$

This proves that T is stably rational. Note that (6) implies $H^1(k, N) = 0$, i.e. any principal homogeneous space X of N is trivial, $X(k) \cong S_m(k)S_n(k)$ in $S(k)$.

One can view S_m and S_n as subgroups of $GL_k(V)$, $V = V_m \otimes_k V_n$. We denote by R the set of split tensors in $V_m \otimes_k V_n$. The group $S_m \times_k S_n$ acts on R, and R contains an open orbit of $S_m \times_k S_n$, $\dim R = m + n - 1$. Let D_m be a maximal k-diagonal subgroup of $GL_k(V_m)$ calculated with respect to a certain k–basis of V_m, D_n is defined analogously. The set R is stable under $D_m \times_k D_n$, and R contains an open orbit of $D_m \times_k D_n$. Let D be the maximal k-diagonal torus of $GL(V)$ which contains $Im(D_m \times_k D_n) = D_m D_n \subset GL_k(V)$. The factor–group $D/D_m D_n = D_0$ is a split k–torus hence it is k–rational. We have a decomposition $D = D_m D_n \times_k D_0$ as a direct product, $\dim D_0 = (m-1)(n-1)$. Considering the variety D as an open orbit in V and taking into account that orbit of the group $D_m D_n$ is open in R, we obtain a birational decomposition $V \cong R \times_k D_0$. Because R is invariant set with respect to the action of the group $S_m \times S_n$ on V hence the variety $T = S/S_m S_n$ is birationally equivalent to D_0 over k, $S_m S_n = Im(S_m \times_k S_n) \subset GL_k(V)$. The result of our discussion can be summed up as follows.

Theorem 1. *Let S_i be a maximal k-torus of $GL_k(V_i)$. Then the quotient space $(V_m \otimes_k V_n)/(S_m \times_k S_n)$ is rational over k if $(m, n) = 1$.* $\triangle$

This theorem was proved by Klyachko [4] by another method. Now consider the problem of rationality of tori with a cyclic splitting field. The idea used in the proof of Theorem 1 allows us to make a step forward in the problem of rationality of tori.

Let L/k be a cyclic extension, $\Pi = Gal(L/k)$ is the cyclic group of order n with generator σ. Let $\mathbf{Z}[\zeta_n]$ be the ring of integers in the cyclotomic field $\mathbf{Q}[\zeta_n]$, where ζ_n is a primitive n-th root of unity. Define a Π-module structure on $\mathbf{Z}[\zeta_n]$ by putting $\sigma(\alpha) = \zeta_n \alpha$. Let T_n be the k-torus with character module $\hat{T}_n = \mathbf{Z}[\zeta_n]$. It is known that all tori T_n are stably rational [2]. Chistov [5] proved that any stably rational k-torus T with a cyclic splitting field is birationally equivalent over k to the product of tori of the form T_n. Moreover, it suffices to check their rationality in the case when n is square-free [3].

Thus, let L_n be a cyclic extension of k of degree n, $\Pi_n = Gal(L_n/k)$, and T_n the L_n/k-torus with character module $\hat{T}_n = \mathbf{Z}[\zeta_n]$, $n = p_1 \cdots p_t$ is square-free, p_i is a prime number. If $n = p$ is a prime number there is an exact sequence of Π_p-modules

$$0 \to \mathbf{Z}[\zeta_p] \to \mathbf{Z}[\Pi_p] \to \mathbf{Z} \to 0, \tag{7}$$

whence by duality one concludes that the torus T_p is isomorphic to the quotient $R_{L_p/k}(G_m)/G_{m,k}$ which, in turn, is an open subset in the projective space $\mathbf{P}^{p-1}$. Thus T_p is k-rational. Now let $t \geq 2$. We have an epimorphism $\mathbf{Z}[\Pi_n] \to \mathbf{Z}[\Pi_n/\Pi_p]$ for every $p|n$, whence the following exact sequence of Π_n-modules

$$0 \to \mathbf{Z}[\zeta_n] \to \mathbf{Z}[\Pi_n] \to \oplus_{p|n} \mathbf{Z}[\Pi_n/\Pi_p]. \tag{8}$$

The dual sequence of k-tori is of the form

$$\prod_{i=1}^{t} R_{F_i/k}(G_m) \to R_{L/k}(G_m) \to T_n \to 1,$$

where F_i is the subfield of $L = L_n$, $(F_i : k) = n/p_i$.

Short sequence (8) is a part of the long exact sequence obtained by tensoring resolutions of the form (7). It is convenient to describe this situation in the language of tensor representations. Let L_p be the subfield of L_n, $(L_p : k) = p$. Then

$$L_n = L_{p_1} \otimes \cdots \otimes L_{p_t},\ F_i = \otimes_{m \neq i} L_{p_m},\ 1 \leq i \leq t.$$

We have embeddings of fields $\psi_i : F_i \to L_n$ which determine natural monomorphisms of groups of linear operators $GL_k(F_i) \to GL_k(L_n)$. The group F_i^* is a maximal k-torus of $GL_k(F_i)$, let D_i be the maximal diagonal subgroup of $GL_k(F_i)$ calculated with respect to a certain basis of extension F_i/k. Choose a point in general position $v = v_1 \otimes \cdots \otimes v_t$, $v_i \in L_{p_i}$, so that for each i the orbits of v under D_i and $S_i = R_{F_i/k}(G_m)$ are open in F_i. Denote by R (resp. R_1) the closure of the orbit of v under $S_1 \times \cdots \times S_t$ (resp. $D_1 \times \cdots \times D_t$), we have $R = R_1$. Let D be the maximal k-diagonal torus of $GL_k(L_n)$ which contains $Im(D_1 \times \cdots \times D_t) = D_1 \cdots D_t$. The factor-group $D/D_1 \cdots D_t = D_0$ is k-rational. The group D is the direct product $D_1 \cdots D_t \times D_0$, $\dim D_0 = (p_1 - 1) \cdots (p_t - 1)$. We have a birational decomposition into direct product

$$D_0 \times R \cong L_n.$$

The group $S_1 \times \cdots \times S_t$ acts on $R \times D_0$ birationally, respecting orbits in R. Let $g \in S_1 \times \cdots \times S_t$. If $g(v, w) = (gv, w')$ and $gv = v$, then $g = 1$, hence $w' = w$. Therefore the set $v \times D_0$ parametrizes the quotient $R_{L/k}(G_m)/(S_1 \times \cdots \times S_t) = T_n$. We have the following statement.

Theorem 2. *Any stably rational torus with a cyclic splitting field is rational over the ground field of characteristic 0.* $\triangle$

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