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# On the distribution of $p^\alpha$ modulo one

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Résumé. Dans cet article, on donne une nouvelle majoration de la discrépance

$$D(N) := \sup_{0\le\gamma\le1} \Big| \sum_{\substack{p\le N \\ \{p^\alpha\}\le\gamma}} 1 - \pi(N)\gamma \Big|$$

de la suite $(p^\alpha)$, lorsque $5/3 \le \alpha < 3$ et $\alpha \ne 2$.

Abstract. In this paper, we give a new upper-bound for the discrepancy

$$D(N) := \sup_{0\le\gamma\le1} \Big| \sum_{\substack{p\le N \\ \{p^\alpha\}\le\gamma}} 1 - \pi(N)\gamma \Big|$$

for the sequence $(p^\alpha)$, when $5/3 \le \alpha < 3$ and $\alpha \ne 2$.

## 1. Introduction.

In 1940 I. M. Vinogradov [14] considered the distribution of the fractional parts of the sequence $(f\sqrt{p})$, where $p$ runs over the set of prime numbers and $f$ a positive constant. This celebrated work motivated the interests of many authors to investigate the distribution of $p^\alpha$ modulo one for fixed $\alpha > 0, \alpha \notin N$(see [1,2,3,4,6,7,9,10]). It is well-known that the sequence $(p^\alpha)$ is uniformly distributed modulo one as $p$ runs over prime values. To make this precise, we consider the discrepancy

$$D(N) := \sup_{0\le\gamma\le1} \left| \sum_{\substack{p\le N \\ \{p^\alpha\}\le\gamma}} 1 - \pi(N)\gamma \right|, \tag{1.1}$$

where $\{t\}$ denotes the fractional part of real $t$.

Leitmann [10] showed that

$$D(N) = O\left(N^{1-\delta}\right) \tag{1.2}$$

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as $N$ tends to infinity, for some $\delta > 0$. R. C. Baker and G. Kolesnik [1] showed that (1.2) holds with

$$\delta = (15000\alpha^2)^{-1}$$

for $\alpha > 1, \alpha \notin N$. This is the sharpest result for large $\alpha$ at present.

We can do much better for small $\alpha$, but one can't find a unified method as $\alpha$ varies, just as Baker and Kolesnik pointed out in their paper [1]: "Of course, one can do much better for small $\alpha$, although the optimal technique varies considerably as $\alpha$ moves over an interval such as $1 < \alpha < 2$". By Vaughan's identity and mean value estimates for Dirichlet polynomial, Balog [2] proved that

$$D(N) = O\left(N^{\frac{1+\alpha}{2}+\varepsilon}\right) \tag{1.3}$$

for $1/2 \leq \alpha < 1$. Throughout the paper, $\varepsilon$ denotes an arbitrarily small positive constant, and $m \sim M$ means that $M < m \leq 2M$. Harman [7] proved that (1.3) holds for $\alpha = 1/2$ independently. Baker and Kolesnik [1] also proves that

$$D(N) = O\left(N^{\frac{157}{168}+\varepsilon}\right) \tag{1.4}$$

for $\alpha = 3/2$ by Van der Corput's method (essentially).

The aim of this paper is to use a new technique to study the case $5/3 \leq \alpha < 3, \alpha \neq 2$. Previous methods only yield weak results in this interval. We shall combine the double large sieve inequality due to Bombieri and Iwaniec [4] and Heath-Brown's method [8] to prove the following

**Theorem 1.** *The estimate*

$$D(N) = O\left(N^{1-\delta+\varepsilon}\right) \tag{1.5}$$

*holds for* $5/3 \leq \alpha < 3, \alpha \neq 2$, *where*

$$\delta = \delta(\alpha) = \begin{cases} 1/26, & \textit{for} \;\; 5/3 \leq \alpha \leq 45/26; \\ (5-2\alpha)/40, & \textit{for} \;\; 45/26 < \alpha \leq 2.1, \alpha \neq 2; \\ 1/50, & \textit{for} \;\; 2.1 < \alpha \leq 317/150; \\ (9-3\alpha)/133, & \textit{for} \;\; 317/150 < \alpha \leq 347/160; \\ (5-\alpha)/151, & \textit{for} \;\; 347/160 < \alpha \leq 129/56; \\ (3-\alpha)/39, & \textit{for} \;\; 129/56 < \alpha < 3. \end{cases}$$

## 2. A special spacing problem

In 1989, Fouvry and Iwaniec [5] first used the double large sieve inequality to estimate exponential sums with monomials. If the exponential factor is large comparable to the variables,then it oscillates too rapidly and can't be controlled by using the double large sieve inequality directly. In order to overcome this problem, Fouvry and Iwaniec appealed to Weyl's method, which reduces the oscillatory behaviour of the exponential factor. Consequently it causes serious problems about the spacing of the resulting points. See Proposition 2 of Fouvry and Iwaniec [5] and Theorem 2 of Liu [11].

Sometimes if the exponential factor is very large, we have to use the Weyl's shift two times. Thus it causes a new spacing problem.

In this section we shall investigate the spacing problem for the points

(2.1)
$$t(m,q;r) := (m+r+q)^\alpha - (m-r+q)^\alpha - (m+r-q)^\alpha + (m-r-q)^\alpha$$

where $\alpha(\alpha-1)(\alpha-2)(\alpha-3) \neq 0, \alpha \in R$.

In this section all constants implied by "$O$", "$\ll$ " depend only on $\alpha$.

Let $M \geq 100, Q \geq 1$, $1 \leq r \leq Q \leq M/10, \Delta > 0$. We set $T = rQM^{\alpha-2}$ and $\mathcal{L} = \log(2MQr)$. Thus for $m \sim M$, $q \sim Q$ and $r$ fixed, we have $t(m,q;r) \sim T$. We use $\mathcal{B}(M,Q,\Delta;r)$ to denote the number of quadruples $(m,\widetilde{m},q,\widetilde{q})$ with $m,\widetilde{m} \sim M$; $q,\widetilde{q} \sim Q$ and $r$ fixed, such that

$$|t(m,q;r) - t(\widetilde{m},\widetilde{q};r)| < \Delta T. \tag{2.2}$$

Our aim is to prove the following:

**Theorem 2.** *Suppose that $Q \ll \varepsilon M^{3/4}$ and $r \leq \varepsilon Q$, then we have*

$$\mathcal{B}(M,Q,\Delta;r) \ll (MQ + \Delta M^2Q^2 + Q^{8/3}) \log^4 2M; \tag{2.3}$$

*If $Q \asymp r$, $Q \ll M^{2/3}$, we have*

$$\mathcal{B}(M,Q,\Delta;r) \ll (MQ + \Delta M^2Q^2 + M^{-2}Q^6) \log^4 2M. \tag{2.4}$$

**Remark 1.** In Theorem 2, the condition $r \leq \varepsilon Q$ can be replaced by $r \leq C(\alpha)Q$, where $C(\alpha)$ is a small constant depending only on $\alpha$. If $\alpha > 1$, then this condiction can be removed.

**Remark 2.** In applications, the contribution of (2.4) is always absorbed by the contribution of (2.3). Since (2.3) is proved only for $r \leq \varepsilon Q$, we state (2.4) for completeness of the Theorem.

*Proof.* Clearly (2.2) implies that

$$\left| t^{\frac{1}{\alpha-2}}(m,q;r) - t^{\frac{1}{\alpha-2}}(\widetilde{m},\widetilde{q};r) \right| < \Delta T^{\frac{1}{\alpha-2}} \tag{2.5}$$

namely,

$$\left| m^{\frac{\alpha}{\alpha-2}} f^{\frac{1}{\alpha-2}}(\frac{r}{m},\frac{q}{m}) - \tilde{m}^{\frac{\alpha}{\alpha-2}} f^{\frac{1}{\alpha-2}}(\frac{r}{\tilde{m}},\frac{\tilde{q}}{\tilde{m}}) \right| \ll \Delta T^{\frac{1}{\alpha-2}}, \tag{2.6}$$

where

$$f(u,v) = (1+u+v)^\alpha - (1+u-v)^\alpha - (1-u+v)^\alpha + (1-u-v)^\alpha.$$

Using Taylor's formula we have for $\beta \neq 0$, $|x| < 1/2$

$$(1+x)^{\beta} = 1 + C_{\beta}^{1}x + C_{\beta}^{2}x^2 + \cdots + C_{\beta}^{11}x^{11} + O(|x|^{12}), \tag{2.7}$$

where $C_{\beta}^{n} = \frac{\beta(\beta-1)\cdots(\beta-n+1)}{n!}$.

For $|u| < 1/4$, $|v| < 1/4$, it follows from (2.7)

$$\begin{aligned} f(u,v) =& \sum_{n=0}^{11} C_{\alpha}^{n}\left((u+v)^n - (-u+v)^n - (u-v)^n + (-u-v)^n\right) \\ &\qquad + O\left(|u|^{12} + |v|^{12}\right) \\ =& 2\sum_{k=1}^{5} C_{\alpha}^{2k}\left((u+v)^{2k} - (u-v)^{2k}\right) \quad + O\left(|u|^{12} + |v|^{12}\right). \end{aligned} \tag{2.8}$$

Moreover,

$$(u+v)^{2k} - (u-v)^{2k} = \sum_{n=0}^{2k} C_{2k}^{n} u^n v^{2k-n} - \sum_{n=0}^{2k} C_{2k}^{n} u^n (-v)^{2k-n} \tag{2.9}$$

$$= 2\sum_{n=1}^{k} C_{2k}^{2n-1} u^{2n-1} v^{2k-2n+1} = 2uv \sum_{n=1}^{k} C_{2k}^{2n-1} u^{2n-2} v^{2k-2n}.$$

Combining (2.6), (2.8) and (2.9) we get

$$\left| m^{\frac{\alpha}{\alpha-2}} (\frac{rq}{m^2})^{\frac{1}{\alpha-2}} g^{\frac{1}{\alpha-2}}(\frac{r}{m}, \frac{q}{m}) - \tilde{m}^{\frac{\alpha}{\alpha-2}} (\frac{r\tilde{q}}{\tilde{m}^2})^{\frac{1}{\alpha-2}} g^{\frac{1}{\alpha-2}}(\frac{r}{\tilde{m}}, \frac{\tilde{q}}{\tilde{m}}) \right| \ll \Delta T^{\frac{1}{\alpha-2}},$$

where

$$g(u,v) = 2C_{\alpha}^{2} + \sum_{k=2}^{5} C_{\alpha}^{2k} \sum_{n=1}^{k} C_{2k}^{2n-1} u^{2(n-1)} v^{2(k-n)} + O\left(\frac{r^{12} + Q^{12}}{M^{10}Qr}\right).$$

By the mean value theorem and the condition $r \leq Q \ll \varepsilon M^{3/4}$, we obtain

$$\begin{aligned} &\left| m h^{\frac{1}{\alpha-2}}(\frac{r}{m}, \frac{q}{m}) - \tilde{m}(\frac{\tilde{q}}{q})^{\frac{1}{\alpha-2}} h^{\frac{1}{\alpha-2}}(\frac{r}{\tilde{m}}, \frac{\tilde{q}}{\tilde{m}}) \right| \\ &\qquad \ll \Delta M + M^{-5}(r^6 + Q^6), \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} h(u,v) = \quad & 1 + d_1(u^2+v^2) + d_2(3u^4 + 10u^2v^2 + 3v^4) \\ & + \sum_{k=4}^{5} \frac{C_{\alpha}^{2k}}{2C_{\alpha}^{2}} \sum_{n=1}^{k} C_{2k}^{2n-1} u^{2(n-1)} v^{2(k-n)} \end{aligned}$$

with $d_1 = \frac{2C_{\alpha}^{4}}{C_{\alpha}^{2}}, d_2 = \frac{C_{\alpha}^{6}}{C_{\alpha}^{2}}$.

Using (2.7) again and after a simple calculation, one has

$$|ml(m,q:r)-\tilde{m}(\frac{\tilde{q}}{q})^{\frac{1}{\alpha-2}}l(\tilde{m},\tilde{q}:r)| \ll \Delta M + M^{-5}(r^6+Q^6), \tag{2.11}$$

where

$$l(m,q:r) = 1 + A_1(q,r)m^{-2} + B_1(q,r)m^{-4},$$

with

$$A_1(q,r) = \frac{2C_\alpha^4 C^1_{(\alpha-2)^{-1}}}{C_\alpha^2}(r^2+q^2) := d_3(r^2+q^2) \tag{2.12}$$

(2.13)

$$\begin{aligned} B_1(q,r) &= \frac{C_\alpha^6 C^1_{(\alpha-2)^{-1}}}{C_\alpha^2}(3r^4+10r^2q^2+3q^4) + \frac{4(C_\alpha^4)^2 C^2_{(\alpha-2)^{-1}}}{(C_\alpha^2)^2}(r^2+q^2)^2 \\ &:= d_4(3r^4+10r^2q^2+3q^4) + d_5(r^2+q^2)^2. \end{aligned}$$

From (2.11) we see that

$$m = \left(\frac{\widetilde{q}}{q}\right)^{\frac{1}{\alpha-2}} \widetilde{m} + O\left(\Delta M + M^{-1}(r^2+Q^2)\right). \tag{2.14}$$

By substituting (2.14) into (2.11) we find the first approximation

(2.15)

$$\begin{aligned} m = \left(\frac{\widetilde{q}}{q}\right)^{\frac{1}{\alpha-2}} \widetilde{m} + d_3\left(\left(\frac{\widetilde{q}}{q}\right)^{\frac{1}{\alpha-2}}(r^2+\widetilde{q}^2) - \left(\frac{q}{\widetilde{q}}\right)^{\frac{1}{\alpha-2}}(r^2+q^2)\right)\widetilde{m}^{-1} \\ +O\left(\Delta M + M^{-3}(r^4+Q^4)\right). \end{aligned}$$

Again by substituting (2.15) into (2.11) we get a more precise approximation

$$\begin{aligned} m = A(q,\widetilde{q})\widetilde{m} + B(q,\widetilde{q};r)\widetilde{m}^{-1} + C(q,\widetilde{q};r)\widetilde{m}^{-3} \\ +O\left(\Delta M + M^{-5}(r^6+Q^6)\right), \end{aligned} \tag{2.16}$$

where

$$\begin{aligned} A(q,\widetilde{q}) &= \left(\frac{\widetilde{q}}{q}\right)^{\frac{1}{\alpha-2}}, \\ B(q,\widetilde{q};r) &= d_3\left(\left(\frac{\widetilde{q}}{q}\right)^{\frac{1}{\alpha-2}}(r^2+\widetilde{q}^2) - \left(\frac{q}{\widetilde{q}}\right)^{\frac{1}{\alpha-2}}(r^2+q^2)\right), \end{aligned}$$

$C(q,\widetilde{q};r) =$

$$d_3^2\left(\frac{q}{\widetilde{q}}\right)^{\frac{2}{\alpha-2}}\left(\left(\frac{\widetilde{q}}{q}\right)^{\frac{1}{\alpha-2}}(r^2+\widetilde{q}^2)-\left(\frac{q}{\widetilde{q}}\right)^{\frac{1}{\alpha-2}}(r^2+q^2)\right)(r^2+q^2)$$
$$+\left(d_4(3r^4+10r^2\widetilde{q}^2+3\widetilde{q}^4)+d_5(r^2+\widetilde{q}^2)^2\right)\left(\frac{\widetilde{q}}{q}\right)^{\frac{1}{\alpha-2}}$$
$$-\left(d_4(3r^4+10r^2q^2+3q^4)+d_5(r^2+q^2)^2\right)\left(\frac{q}{\widetilde{q}}\right)^{\frac{3}{\alpha-2}}.$$

Let $\|x\| = \min_{n\in Z}|x-n|$, $e(x) = exp(2\pi ix)$ and $\Delta_1 = \Delta M + M^{-5}(r^6 + Q^6)$. We let $\mathcal{B}_0(M,Q,\Delta_1;r)$ denote the number of triplets $(\widetilde{m},q,\widetilde{q})$ with $\widetilde{m}\sim M$; $q,\widetilde{q}\sim Q$, such that

$$\|A(q,\widetilde{q})\widetilde{m} + B(q,\widetilde{q};r)\widetilde{m}^{-1} + C(q,\widetilde{q};r)\widetilde{m}^{-3}\| \ll \varepsilon^{-1}\Delta_1. \tag{2.17}$$

We assume that $\Delta M < 1/4$, for otherwise Theorem 2 follows from (2.16). If $\Delta M < 1/4$, we claim from (2.16) that

$$\mathcal{B}(M,Q,\Delta;r) \ll \mathcal{B}_0(M,Q,\Delta_1;r) \tag{2.18}$$

Since $\mathcal{B}_0(M,Q,\Delta_1;r)$ is non-decreasing in $\Delta_1$, we have

$$\mathcal{B}_0(M,Q,\Delta_1;r) \le \mathcal{B}_0(M,Q,\Delta_2;r), \tag{2.19}$$

where $\Delta_2 = \Delta M + M^{-1}(r^{2/3} + Q^{2/3})$.

Now we use the method of Fouvry and Iwaniec [5 ,p319—321] to estimate $\mathcal{B}_0(M,Q,\Delta_2;r)$. For $S>0$ we have the identity

$$\sum_{|s|<S}\left(1-\frac{|s|}{S}\right)e(sx) = \frac{1-\{S\}}{S}\left(\frac{\sin\pi x[S]}{\sin\pi x}\right)^2 + \frac{\{S\}}{S}\left(\frac{\sin\pi x[S+1]}{\sin\pi x}\right)^2.$$

For fixed $(q,\widetilde{q})$, the number of lattice points counted in $\mathcal{B}_0(M,Q,\Delta_2;r)$ is bounded by

$$S^{-1}\sum_{1\le s\le S}\left|\sum_{\widetilde{m}\sim M} e(As\widetilde{m} + Bs\widetilde{m}^{-1} + Cs\widetilde{m}^{-3})\right| + \Delta_2 M, \tag{2.20}$$

where $S = \varepsilon(4\Delta_2)^{-1}$.

Using our assumption and Lemma 4.8 of Titchmarsh [12], the innermost sum in (2.20) equals to

$$\int_M^{2M} e(\pm\|As\|\xi + Bs\xi^{-1} + Cs\xi^{-3})d\xi + O(1) = I + O(1). \tag{2.21}$$

Let $f(t) = t^{2\beta}(r^2+t^2)$ with $\beta = 1/(\alpha-2)$,then

$$f'(t) = 2\beta t^{2\beta-1}(r^2+(\alpha-1)t^2). \tag{2.22}$$

Thus by the mean value theorem we have

$$|B(q,\widetilde{q};r)| = |d_3(q\widetilde{q})^{-\beta}(f(q)-f(\widetilde{q}))| \tag{2.23}$$

$$\begin{aligned} &= |2d_3(q\widetilde{q})^{-\beta}q_0^{\beta-1}(r^2+(\alpha-1)q_0^2)(q-\widetilde{q})| \\ &\asymp Q|q-\widetilde{q}| \end{aligned}$$

for some $q_0 \sim Q$,if $r \leq C(\alpha)Q$ for some small $C(\alpha) > 0$. It should be remarked that if $\alpha > 1$, then this condition can be removed.

We always have $|C(q,\widetilde{q};r)| \ll |B(q,\widetilde{q};r)Q^2$. This is trivial if $|q-\widetilde{q}| \geq Q/20$ since $C(q,\widetilde{q};r) \ll Q^4$ . Now suppose $|q-\widetilde{q}| \leq Q/20$. For fixed $t \neq 0$, we have

$$\tilde{q}^t = q^t + O_t(|\tilde{q}-q|q^{t-1}).$$

Inserting this into the expression of $C(q,\widetilde{q};r)$ we get the desired inequality.

If $\|As\| \geq 3s|B|M^{-2}$, then by Lemma 4.2 of [12] we get

$$I \ll \|As\|^{-1}, \tag{2.24}$$

and if $\|As\| < 3s|B|M^{-2}$, then by Lemma 4.4 of [12] we have

$$I \ll (|B|sM^{-3})^{-1/2}. \tag{2.25}$$

Combining (2.20)—(2.25) we have

$$\mathcal{B}_0(M,Q,\Delta_2;r) \ll \Delta_2 MQ^2 + E_1(M,Q,\Delta_2;r) + E_2(M,Q,\Delta_2;r), \tag{2.26}$$

where

$$E_1(M,Q,\Delta_2;r) = \Delta_2 \sum_{1\leq s\leq S}\sum_{q,\widetilde{q}\sim Q} \min\left(M, \frac{1}{\|As\|}\right),$$

and

$$E_2(M,Q,\Delta_2;r) = \Delta_2 \sum_{(**)} \min\left(M, (|B|sM^{-3})^{-1/2}\right),$$

$$(**): \quad 1 \leq s \leq S; \quad q,\widetilde{q} \sim Q; \quad \|As\| < 3s|B|M^{-2}.$$

$E_j(M,Q,\Delta_2;r)(j=1,2)$ can be estimated in the same way as Fouvry and Iwaniec [5](page 320), and we have

$$E_1(M,Q,\Delta_2;r) \ll MQ\log^3 2M, \tag{2.27}$$

$$E_2(M,Q,\Delta_2;r) \ll (MQ + Q^{8/3})\log^4 2M. \tag{2.28}$$

The estimate (2.3) follows from (2.26)—(2.28). To prove the estimate (2.4), we need only to consider

$$\|A(q,\widetilde{q})\widetilde{m} + B(q,\widetilde{q};r)\widetilde{m}^{-1}\| \ll \varepsilon^{-1}\Delta_3, \tag{2.29}$$

where $\Delta_3 = \Delta M + M^{-3}(r^4+Q^4)$. Since the proof is almost the same as that of (2.3), we omit the details. □

## 3. Some Preliminary Lemmas

In this section we quote some lemmas needed for our proof. We shall use the work of Heath-Brown [8] on the decomposition of sums

$$\sum_{P<n\le P_1} \Lambda(n)f(n) \tag{3.1}$$

with $P_1 \le 2P$. Heath-Brown subdivides the sum (3.1) into expressions of the form

$$\sum_{X<x\le X_1} a(x) \sum_{\substack{Y<y\le Y_1\\ P<xy\le P_1}} b(y)f(xy),$$

with $X < X_1 \le 2X, Y < Y_1 \le 2Y$, $a(x) \ll P^\varepsilon, b(y) \ll P^\varepsilon$ for every fixed $\varepsilon > 0$. The above sum is usually called "Type I" sum if $b(y) = 1$ or $b(y) = \log y$; Otherwise it is called "Type II" sum.

**Lemma 1.** *Let $f(n)$ $(n = 1, 2, \cdots)$ be a complex valued function. Let $P \ge 2, P_1 \le 2P$. Let $u, v, z$ be positive numbers satisfying*

$$2 \le u < v \le z \le P, \tag{3.2}$$

$$u^2 \le z, 128uz^2 \le P_1, 2^{18}P_1 \le v^3. \tag{3.3}$$

*Then the sum (3.1) may be decomposed into $(\log P)^6$ sums, each of which is either of Type I with $Y \ge z$ or of Type II with $u \le Y \le v$.*

**Lemma 2.** *Let $x_1, x_2, \cdots, x_J$ be real and $H \ge 1$. Then the discrepancy of sequence $(x_j)$,which is defined by*

$$D := \sup_{0\le\gamma\le1} |\sum_{\substack{j\le J\\ \{x_j\}\le\gamma}} 1 - J\gamma|,$$

*satisfies*

$$D \le 6JH^{-1} + 4\sum_{h=1}^{H} h^{-1} \left|\sum_{j=1}^{J} e(hx_j)\right|. \tag{3.4}$$

**Lemma 3.** *Let $L > K$, $Q > 0$, and let $z_k$ be complex numbers. We then have*

$$\left|\sum_{K<k\le L} z_k\right|^2 \le \left(2 + \frac{L-K}{Q}\right) \sum_{|q|<Q} (1 - \frac{|q|}{Q}) \sum_{K<k\pm q\le L} z_{k+q}\overline{z_{k-q}}.$$

**Lemma 4.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be two finite sets of real numbers, $\mathcal{X} \subset [-X, X]$, $\mathcal{Y} \subset [-Y, Y]$. Then for any complex functions $u(x)$ and $v(y)$ we have*

$$|\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} u(x)v(y)|^2 \leq 20(1+XY) \sum_{\substack{x,x'\in\mathcal{X}\\ |x-x'|\leq Y^{-1}}} |u(x)u(x')| \sum_{\substack{y,y'\in\mathcal{Y}\\ |y-y'|\leq X^{-1}}} |v(y)v(y')|.$$

**Lemma 5.** *Let $Q \geq 1, m \sim M, q \sim Q$. Let $\alpha(\neq 0,1)$ be a real number, $t(m,q) = (m+q)^\alpha - (m-q)^\alpha, T = M^{\alpha-1}Q$, and let $\mathcal{A}(M,Q,\Delta)$ be the number of lattice points $(m, m_1, q, q_1)$ such that $m, m_1 \sim M, q, q_1 \sim Q$, and*

$$|t(m,q) - t(m_1,q_1)| < \Delta T.$$

*Suppose $\eta$ is a sufficiently small positive constant. If $Q < \eta M^{3/4}$,then*

$$\mathcal{A}(M,Q,\Delta) \ll (MQ + \Delta M^2Q^2 + Q^{8/3}) \log^4 4MQ,$$

*the implied constant relying at most on $\eta$ and $\alpha$.*

Lemma 1 and 2 are Lemma 1 and 2 of Baker and Kolesnik [1] respectively. Lemma 3 is Lemma 2 of Fouvry and Iwaniec [5]. Lemma 4 is the double large sieve inequality, see Proposition 1 of Fouvry and Iwaniec [5]. Lemma 5 is Theorem 2 of Liu [11].

## 4. Estimations of Exponential Sums

We first estimate the bilinear forms of type

$$S(M,N) = \sum_{m\sim M}\sum_{n\sim N} a_m b_n e(hm^\alpha n^\alpha), \tag{4.1}$$

with $MN \sim X$ and $a_m \ll 1$, $b_n \ll 1$.

**Proposition 1.** *Suppose that $\frac{223}{139} < \alpha < 3, \alpha \neq 2$, $0 < \theta < \frac{3}{68}$, $\frac{6-3\alpha}{37} \leq \theta \leq \frac{9-3\alpha}{97}$, $h \leq X^\theta$. Then we have*

$$S(M,N) \ll X^{1-\theta+\varepsilon} \tag{4.2}$$

*if*

$$X^{\max(\alpha+19\theta-2,12\theta)} \ll N \ll X^{\min(\alpha-1-2\theta,1-32\theta/3)}. \tag{4.3}$$

*Proof.* Take $Q_1 = [X^{2\theta}]$. By Cauchy's inequality and Lemma 3 we get

$$|S(M,N)|^2 \ll \frac{M^2N^2}{Q_1} + \frac{MN}{Q_1}$$

$$\sum_{q_1=1}^{Q_1} \sum_{m\sim M} \left| \sum_{N<n\leq 2N-q_1} \overline{b_{n+q_1}} b_{n-q_1} e\left(hm^\alpha t(n,q_1)\right) \right| \tag{4.4}$$

$$\ll \frac{M^2N^2}{Q_1} + \frac{MN}{Q_1} \log X S_1(Q_1^*)$$

for some $1 \ll Q_1^* \ll Q_1$, where

$$S_1(Q_1^*) = \sum_{q_1\sim Q_1^*} \sum_{m\sim M} \left| \sum_{N<n\leq 2N-q_1} \overline{b_{n+q_1}} b_{n-q_1} e\left(hm^\alpha t(n,q_1)\right) \right| \tag{4.5}$$

with

$$t(n,q_1) = (n+q_1)^\alpha - (n-q_1)^\alpha.$$

It reduces to show that

$$S_1(Q_1^*) \ll MN \log X. \tag{4.6}$$

Take $Q_2 = [\frac{Q_1^*}{\log X}]^2 + 10$. Using Lemma 3 again gives

$$|S_1(Q_1^*)|^2 \ll \frac{(Q_1^*MN)^2}{Q_2} + \frac{Q_1^*MN\log X}{Q_2} S_2(Q_1^*, Q_2^*) \tag{4.7}$$

for some $1 \ll Q_2^* \ll Q_2$, where

$$S_2(Q_1^*, Q_2^*) = \sum_{q_1\sim Q_1^*} \sum_{q_2\sim Q_2^*} \sum_{n\sim N} \left| \sum_{m\sim M} e\left(hm^\alpha t(n,q_2;q_1)\right) \right| \tag{4.8}$$

and $t(n,q_2;q_1)$ is defined by (3.1).

No loss of generality, we suppose that $Q_1^* \ll Q_2^*$. It suffices for us to show for fixed $q_1$ that

$$S_2(q_1) = \sum_{q_2\sim Q_2^*} \sum_{n\sim N} \left| \sum_{m\sim M} e\left(hm^\alpha t(n,q_2;q_1)\right) \right| \ll MN\log^3 X. \tag{4.9}$$

If $Q_1^* \gg Q_2^*$, we shall estimate the sum over $q_1$.

Take $Q_3 = [\frac{Q_2^*}{\log X}]^2 + 10$. Using Lemma 3 again, we get that

$$|S_2(q_1)|^2 \ll \frac{(Q_2^*MN)^2}{Q_3} + \frac{Q_2^*MN\log X}{Q_3} S_3(Q_2^*, Q_3^*) \tag{4.10}$$

for some $1 \ll Q_3^* \ll Q_3$, where

$$S_3(Q_2^*, Q_3^*) = \sum_{q_3 \sim Q_3^*} \sum_{m \sim M} \left| \sum_{n \sim N} \sum_{q_2 \sim Q_2^*} e\left(ht(m, q_3)t(n, q_2; q_1)\right) \right|. \tag{4.11}$$

It is easy to show that $Q_2 = o(N^{3/4})$ and $Q_3 = o(M^{3/4})$. Let $F = hM^\alpha N^\alpha$ and $G = \frac{FQ_1^* Q_2^* Q_3^*}{MN^2}$. It is easy to check that $G \gg 1$. By Lemma 4 we obtain that

$$|S_3(Q_2^*, Q_3^*)|^2 \ll G\mathcal{A}\mathcal{B}, \tag{4.12}$$

where $\mathcal{A}$ is the number of quadruples $(m, \widetilde{m}, q_3, \widetilde{q_3})$ such that

$$|t(m, q_3) - t(\widetilde{m}, \widetilde{q_3})| \ll \left(ht(N, Q_2^*; Q_1^*)\right)^{-1},$$

for $m, \widetilde{m} \sim M; q_3, \widetilde{q_3} \sim Q_3^*$, and where $\mathcal{B}$ is the number of quadruples $(n, \widetilde{n}, q_2, \widetilde{q_2})$ such that

$$|t(n, q_2; q_1) - t(\widetilde{n}, \widetilde{q_2}; q_1)| \ll \left(ht(M, Q_3^*)\right)^{-1},$$

for $n, \widetilde{n} \sim N; q_2, \widetilde{q_2} \sim Q_2^*$. Then $\mathcal{A}, \mathcal{B}$ can be estimated by Lemma 5 and Theorem 2, respectively; And we get that

$$\mathcal{A} \ll \left(MQ_3^* + \frac{M^2 Q_3^{*2}}{G} + Q_3^{*8/3}\right) \log^4 X \tag{4.13}$$

and

$$\mathcal{B} \ll \left(NQ_2^* + \frac{N^2 Q_2^{*2}}{G} + Q_2^{*8/3}\right) \log^4 X. \tag{4.14}$$

Combining (4.12)—(4.14) we get that

(4.15)

$$\begin{aligned} S_3(Q_2^*, Q_3^*)^2 \log^{-8} X \ll\ & FN^{-1}Q_1^*(Q_2^* Q_3^*)^2 + M^2 N^2 Q_2^{*2} \\ & + FN^{-2} Q_1^* Q_2^{*11/3} Q_3^{*2} + M^2 N Q_2^* Q_3^{*2} + M^3 N^4 Q_2^* Q_3^* F^{-1} Q_1^{*-1} \\ & + M^2 Q_2^{*8/3} Q_3^{*2} + F(MN)^{-1} Q_1^* Q_2^{*2} Q_3^{*11/3} \\ & + F^{-1} M^{-1} N^{-2} Q_1^* (Q_2^* Q_3^*)^{11/3}. \end{aligned}$$

Now it is easy to check that under the conditions of Proposition 1 we have

$$S_3(Q_2^*, Q_3^*) \ll MNQ_2^* \log^4 X. \tag{4.16}$$

Thus Proposition 1 follows from inserting (4.16) into (4.10). □

**Proposition 2.** *Suppose that* $2 < \alpha < 3$, $0 < \theta \le \frac{3-\alpha}{39}$, $0 < \theta \le \frac{3}{136}$, $0 < \theta \le \frac{12-3\alpha}{269}$, *and* $0 < \theta \le \frac{\alpha}{2} - 1$, $h \le X^{\theta}$. *Then (4.2) holds for*

$$X^{24\theta} \ll N \ll X^{\min\left(\frac{12-3\alpha-197\theta}{3}, 1-\frac{64}{3}\theta\right)}. \tag{4.17}$$

*Proof.* Take $Q_1 = [X^{2\theta}]$ and Lemma 3 gives us

$$|S(M,N)|^2 \ll \frac{M^2N^2}{Q_1} + \frac{MN\log X}{Q_1} \sum_{q_1 \sim Q_1^*} \sum_{n \sim N} \left| \sum_{m \sim M} e\left(hm^{\alpha} t(n,q_1)\right) \right| \tag{4.18}$$

for some $1 \ll Q_1^* \ll Q_1$.

Let $S_1$ denote the triple exponential sum in the right side of (4.18). Take $Q_2 = [\frac{Q_1^*}{\log X}]^2 + 10$ and Lemma 3 gives

$$\begin{aligned} |S_1|^2 \ll & \frac{(Q_1^* MN)^2}{Q_2} \\ & + \frac{Q_1^* MN \log X}{Q_2} \sum_{q_2 \sim Q_2^*} \sum_{m \sim M} \left| \sum_{n \sim N} \sum_{q_1 \sim Q_1^*} e\left(ht(m,q_2)t(n,q_1)\right) \right| \end{aligned} \tag{4.19}$$

for some $1 \ll Q_2^* \ll Q_2$.

Let $S_2$ denote the quadruple sum in the right side of (4.19). Take $Q_3 = [\frac{Q_2^*}{\log X}]^2 + 10$ and Lemma 3 gives

$$|S_2|^2 \ll \frac{(Q_1^* Q_2^* MN)^2}{Q_3} + \frac{Q_1^* Q_2^* MN \log X}{Q_3} S_3 \tag{4.20}$$

for some $1 \ll Q_3^* \ll Q_3$, where

$$S_3 = \sum_{q_1 \sim Q_1^*} \sum_{q_3 \sim Q_3^*} \sum_{n \sim N} \left| \sum_{m \sim M} \sum_{q_2 \sim Q_2^*} e\left(ht(m,q_2)t(n,q_3;q_1)\right) \right|.$$

No loss of generality we suppose that $Q_1^* \ll Q_3^*$; Otherwise we estimate the sum over $q_1$. Thus we have

$$S_3 \ll \sum_{q_1 \sim Q_1^*} S_4(q_1), \tag{4.21}$$

where

$$S_4(q_1) = \sum_{q_3 \sim Q_3^*} \sum_{n \sim N} \left| \sum_{m \sim M} \sum_{q_2 \sim Q_2^*} e\left(ht(m,q_2)t(n,q_3;q_1)\right) \right|.$$

Take $Q_4 = [\frac{Q_3^*}{\log X}]^2 + 10$ and Lemma 3 gives

$$|S_4(q_1)|^2 \ll \frac{(Q_2^* Q_3^* MN)^2}{Q_4} + \frac{Q_2^* Q_3^* MN \log X}{Q_4} S_5 \tag{4.22}$$

for some $1 \ll Q_4^* \ll Q_4$, where

$$S_5 = \sum_{q_2 \sim Q_2^*} \sum_{q_4 \sim Q_4^*} \sum_{m \sim M} \left| \sum_{n \sim N} \sum_{q_3 \sim Q_3^*} e\left(ht(m, q_4; q_2) t(n, q_3; q_1)\right) \right|.$$

Again we suppose that $Q_2^* \ll Q_4^*$. Let

$$S_6(q_1, q_2) = \sum_{q_4 \sim Q_4^*} \sum_{m \sim M} \left| \sum_{n \sim N} \sum_{q_3 \sim Q_3^*} e\left(ht(m, q_4; q_2) t(n, q_3; q_1)\right) \right|,$$

and $G = \frac{F Q_1^* Q_2^* Q_3^* Q_4^*}{M^2 N^2}$.

By Lemma 4 we get that

$$|S_6(q_1, q_2)|^2 \ll G \mathcal{B}_1 \mathcal{B}_2, \tag{4.23}$$

where $\mathcal{B}_1$ denotes the number of quadruples $(m, \widetilde{m}, q_4, \widetilde{q_4})$ such that

$$|t(m, q_4; q_2) - t(\widetilde{m}, \widetilde{q_4}; q_2)| \ll \left(ht(N, Q_3^*; Q_1^*)\right)^{-1},$$

for $m, \widetilde{m} \sim M; q_4, \widetilde{q_4} \sim Q_4^*$, and where $\mathcal{B}_2$ is the number of quadruples $(n, \widetilde{n}, q_3, \widetilde{q_3})$ such that

$$|t(n, q_3; q_1) - t(\widetilde{n}, \widetilde{q_3}; q_1)| \ll \left(ht(M, Q_4^*; Q_2^*)\right)^{-1},$$

for $n, \widetilde{n} \sim N; q_3, \widetilde{q_3} \sim Q_3^*$. By Theorem 2, we have

$$\mathcal{B}_1 \ll \left( M Q_4^* + \frac{M^2 Q_4^{*2}}{G} + Q_4^{*8/3} \right) \log^4 X \tag{4.24}$$

and

$$\mathcal{B}_2 \ll \left( N Q_3^* + \frac{N^2 Q_3^{*2}}{G} + Q_3^{*8/3} \right) \log^4 X. \tag{4.25}$$

Combining (4.23)—(4.25) we obtain

$$S_6(q_1, q_2) \ll MN Q_3^* \log^4 X \tag{4.26}$$

under the conditions of Proposition 2. Now Proposition 2 follows from (4.18)—(4.22) and (4.26). □

Next we estimate the triple sum

$$S(H, M, N) = \sum_{h \sim H} \sum_{m \sim M} \sum_{n \sim N} a_m b_n c_h e(h m^{\alpha} n^{\alpha}), \tag{4.27}$$

with $MN \sim X$ and $a_m \ll 1$, $b_n \ll 1$, $c_h \ll 1$.

**Proposition 3.** *Suppose that* $3/2 < \alpha < 3$, $0 < \theta \le \frac{3-\alpha}{30}$. *Then for* $X^{2\theta} \ll N \ll X^{1/3}$, *we have*

$$S(H,M,N) \ll HX^{1-\theta+\varepsilon}. \tag{4.28}$$

*Proof.* We apply Heath-Brown's method [8]. Suppose $1 \le Q \le HN$ is a parameter to be specified later. For a fixed $q(1 \le q \le Q)$, define

$$T_q = \{(n,h) : n \sim N, h \sim H, \frac{16N^\alpha H(q-1)}{Q} < hn^\alpha \le \frac{16N^\alpha Hq}{Q}\}.$$

Then we have

$$S(H,M,N) = \sum_{q=1}^{Q} \sum_{m\sim M} a_m \sum_{(n,h)\in T_q} b_n c_h e(hm^\alpha n^\alpha).$$

Cauchy's inequality gives

$$\begin{aligned} |S(H,M,N)|^2 &\ll QM \sum_q \sum_{m\sim M} \left| \sum_{(n,h)\in T_q} b_n c_h e(hm^\alpha n^\alpha) \right|^2 \\ &\ll QM \sum_{(*)} \left| \sum_{m\sim M} e(m^\alpha \lambda) \right|, \end{aligned} \tag{4.29}$$

where $\lambda = h_1 n_1^\alpha - h_2 n_2^\alpha$, and $(*)$ denotes a sum over $h_1, h_2 \sim H$, $n_1, n_2 \sim N$ such that

$$|\lambda| \ll HN^\alpha Q^{-1}.$$

Applying the exponent pair $(\frac{1}{14}, \frac{11}{14})$ we have

$$\sum_{m\sim M} e(m^\alpha \lambda) \ll \min\left(M, \frac{M}{M^\alpha |\lambda|}\right) + M^{\frac{\alpha+10}{14}} |\lambda|^{\frac{1}{14}}. \tag{4.30}$$

By the same argument as in Heath-Brown [8], we get that

$$|S(H,M,N)|^2 X^{-\varepsilon} \ll M^2NHQ + H^{\frac{29}{14}} M^{\frac{\alpha+24}{14}} N^{\frac{\alpha+28}{14}} Q^{\frac{-1}{14}}. \tag{4.31}$$

if $1 \ll Q \ll HN$.

Finally take $Q = [HNX^{-2\theta}]$ and Proposition 3 follows. □

## 5. Proof of Theorem 1

Let $D_P$ denote the discrepancy of the sequence $\{p^\alpha : p \sim P\}$. By a simple splitting argument, there exists a $P$ such that $1 \le P \le N$ and

$$D(N) \ll D_P \log N. \tag{5.1}$$

Suppose $P \gg N^{1/2}$. Let $H_0 = [P^\delta]$, where $\delta$ is given in Theorem 1. Applying Lemma 2, we obtain

$$\begin{aligned} D_P &\ll P^{1-\delta} + \sum_{h \le H_0} h^{-1} \left| \sum_{p \sim P} e(hp^\alpha) \right| \\ &\ll P^{1-\delta} + P^{1/2} \log P + \sum_{h \le H_0} h^{-1} \left| \sum_{P < n \le P_1} \Lambda(n) e(hn^\alpha) \right| \qquad (5.2) \\ &\ll \log P \left( P^{1-\delta} + H^{-1} \sum_{h \sim H} \left| \sum_{P < n \le P_1} \Lambda(n) e(hn^\alpha) \right| \right) \end{aligned}$$

for some $1 \ll H \ll H_0$ and $P < P_1 \le 2P$.

Applying Lemma 1 with $f(n) = e(hn^\alpha)$, we find

$$D_P \log^{-1} P \ll P^{1-\delta} + H^{-1} \sum_{h \sim H} |S(h)|, \tag{5.3}$$

where $S(h)$ is either a Type I sum with

$$Y \ge 8^{-1} P^{1/2-\delta}$$

or a Type II sum with

$$8^{-1} P^{2\delta} \le Y \le 2^7 P^{1/3}$$

If $S(h)$ is a Type II sum, then by Proposition 3 (take $\theta = \delta$) we have

$$\sum_{h \sim H} |S(h)| \ll H P^{1-\delta+\varepsilon}. \tag{5.4}$$

It suffices for us to show that for any $S(h)$ of Type I we have

$$S(h) \ll P^{1-\delta+\varepsilon}. \tag{5.5}$$

We prove (5.5) by Propositions 1 and 2.

**Case 1.** $\frac{5}{3} \le \alpha \le \frac{45}{26}$.

In this interval, we have $\delta = \frac{1}{26}$. We take $\theta = \frac{1}{26}$ in Proposition 1 and we find that (5.5) holds for $P^{6/13} \ll Y \ll P^{23/39}$. For $Y \gg P^{23/39}$, using the exponent pair $(\frac{1}{14}, \frac{11}{14})$ on the sum over $y$ and estimating the sum over $x$ trivially, we get

$$S(h) P^{-\varepsilon} \ll X \left( hX^\alpha Y^{\alpha-1} \right)^{\frac{1}{14}} Y^{\frac{11}{14}} \ll P^{\frac{911+78\alpha}{14 \cdot 78}} \ll P^{\frac{25}{26}}. \tag{5.6}$$

**Case 2.** $\frac{45}{26} < \alpha \le \frac{21}{10}$.

In this interval, we have $\delta = \frac{5-2\alpha}{40}$. By Proposition 1 we have (5.5) holds for $P^{\frac{1}{2}-\delta} \ll Y \ll P^{\frac{8\alpha-5}{15}}$. We use the exponent pair $(\frac{1}{14}, \frac{11}{14})$ again for $Y \gg P^{\frac{8\alpha-5}{15}}$.

**Case 3.** $\frac{21}{10} < \alpha \leq \frac{317}{150}$.

In this interval, we have $\delta = \frac{1}{50}$. By Proposition 2 we see that (5.5) holds for $P^{\frac{1}{2}-\delta} \ll Y \ll P^{\frac{43}{75}}$. For $Y \gg P^{\frac{43}{75}}$ we use the exponent pair $(\frac{1}{30}, \frac{26}{30})$ and (5.5) still holds.

**Case 4.** $\frac{317}{150} < \alpha \leq \frac{347}{160}$.

In this interval, we have $\delta = \frac{9-3\alpha}{133}$ and (5.5) can be proved as the same in Case 3.

**Case 5.** $\frac{347}{160} < \alpha \leq \frac{129}{56}$.

In this case, we have $\delta = \frac{5-\alpha}{151}$. By Proposition 2 we know that (5.5) holds for $P^{\frac{1}{2}-\delta} \ll Y \ll P^{\frac{1}{2}}$. For $P^{\frac{1}{2}} \ll Y \ll P^{1-24\delta}$, (5.5) also follows from Proposition 2 with $X$ and $Y$ interchanged. For $Y \gg P^{1-24\delta}$ , we use the exponent pair $(\frac{1}{30}, \frac{26}{30})$ again.

**Case 6.** $\frac{129}{56} < \alpha < 3$.

In this case, we have $\delta = \frac{3-\alpha}{39}$ and (5.5) could be proved as the same in Case 5.

Combining all the above, we see that (5.5) holds in any cases. This completes the proof of Theorem 1.

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