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On inhomogeneous Diophantine approximation with some quasi-periodic expressions, II

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Résumé. On s'intéresse aux valeurs de

$$\mathcal{M}(\theta,\phi)=\liminf_{|q|\to\infty}|q|\|q\theta-\phi\|$$

lorsque θ est un réel ayant un développement en fraction continue quasi-périodique.

Abstract. We consider the values concerning

$$\mathcal{M}(\theta,\phi)=\liminf_{|q|\to\infty}|q|\|q\theta-\phi\|$$

where the continued fraction expansion of θ has a quasi-periodic form. In particular, we treat the cases so that each quasi-periodic form includes no constant. Furthermore, we give some general conditions satisfying $\mathcal{M}(\theta,\phi)=0$.

1. Introduction

Let θ be irrational and ϕ real. We suppose throughout that $q\theta-\phi$ is never integral for any integer q. Define the value of the function

$$\mathcal{M}(\theta,\phi)=\liminf_{|q|\to\infty}|q|\|q\theta-\phi\|\,,$$

which is called *inhomogeneous approximation constant* for the pair θ, ϕ. It is convenient to introduce the functions

$$\mathcal{M}_+(\theta,\phi)=\liminf_{q\to+\infty}q\|q\theta-\phi\|$$

and

$$\mathcal{M}_-(\theta,\phi)=\liminf_{q\to+\infty}q\|q\theta+\phi\|=\liminf_{q\to-\infty}|q|\|q\theta-\phi\|\,.$$

Then $\mathcal{M}(\theta,\phi)=\min\bigl(\mathcal{M}_+(\theta,\phi),\mathcal{M}_-(\theta,\phi)\bigr)$. Several authors have treated $\mathcal{M}(\theta,\phi)$ or $\mathcal{M}_+(\theta,\phi)$ by using their own algorithms (See [1], [2], [4], [5], [11] e.g.), but it has been difficult to find the exact values of $\mathcal{M}(\theta,\phi)$ for

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the concrete pair of θ and ϕ. For example, Cusick, Rockett and Szüsz ([2]) obtain

$$\mathcal{M}\left(\theta, \frac{1}{2}\right) = \frac{1}{4\sqrt{5}} \quad \text{and} \quad \mathcal{M}\left(\theta, \frac{1}{\sqrt{5}}\right) = \frac{1}{5\sqrt{5}}$$

when $\theta = (1+\sqrt{5})/2 = [1; 1, 1, \dots]$. And author ([5]) obtains

$$\begin{aligned} \mathcal{M}\left(\theta, \frac{1}{a}\right) &= \frac{1}{a^2\sqrt{a^2+4}}, \\ \mathcal{M}\left(\theta, \frac{1}{2a}\right) &= \frac{1}{4a^2\sqrt{a^2+4}}, \\ \mathcal{M}\left(\theta, \frac{1}{a^2+4}\right) &= \frac{1}{(a^2+4)\sqrt{a^2+4}} \quad \text{and} \\ \mathcal{M}\left(\theta, \frac{1}{2}\right) &= \frac{1}{4\sqrt{a^2+4}} \quad (a \text{ is odd} \geq 3) \end{aligned}$$

when $\theta = (\sqrt{a^2+4} - a)/2 = [0; a, a, \dots]$. However, it is not easy to apply these methods to find the value $\mathcal{M}(\theta, \phi)$ about the different types of θ.

In [6] author establishes the relationship between $\mathcal{M}(\theta, \phi)$ and the algorithm of Nishioka, Shiokawa and Tamura. If we use this result, we can evaluate the exact value of $\mathcal{M}(\theta, \phi)$ for any pair of θ and ϕ at least when θ is a positive real root of the quadratic equation and $\phi \in \mathbb{Q}(\theta)$. For example,

$$\begin{aligned} \mathcal{M}\left(\theta, \frac{1}{2}\right) &= \begin{cases} \frac{\min(a,b)}{4\sqrt{D}} & \text{if both } a \text{ and } b \text{ are odd,} \\ \frac{a}{4\sqrt{D}} & \text{otherwise,} \end{cases} \\ \mathcal{M}\left(\theta, \frac{1}{\sqrt{D}}\right) &= \frac{a}{D\sqrt{D}} \quad \text{and} \\ \mathcal{M}\left(\theta, \frac{1}{a}\right) &= \frac{1}{a\sqrt{D}} \quad (a \geq 2) \qquad (D = ab(ab+4)) \end{aligned}$$

are given when $\theta = (\sqrt{ab(ab+4)} - ab)/(2a) = [0; a, b, a, b, \dots]$.

Furthermore, in [7] author is so successful applying the Nishioka-Shiokawa-Tamura algorithm that the exact value of $\mathcal{M}(\theta, \phi)$ can be calculated even if θ is a Hurwitzian number, namely its continued fraction expansion has a quasi-periodic form. And it is the first time to find a concrete pair of θ and ϕ so that $\mathcal{M}(\theta, \phi) = 0$. For example, for a positive integer s

$$\mathcal{M}\left(e^{\frac{1}{s}}, \frac{1}{3}\right) = \begin{cases} 0 & \text{if } s \equiv 2 \pmod 3, \\ \frac{1}{18} & \text{otherwise} \end{cases}$$

is given.

In this paper we consider the cases so that each quasi-periodic form includes no constant, and conditions satisfying $\mathcal{M}(\theta, \phi) = 0$.

2. NST algorithm

We first introduce the NST algorithm ([9]). $\theta = [\, a_0;\ a_1,\ a_2,\ \dots]$ denotes the continued fraction expansion of θ, where

$$\theta = a_0 + \theta_0, \qquad a_0 = \lfloor \theta \rfloor\,,$$
$$1/\theta_{n-1} = a_n + \theta_n, \qquad a_n = \lfloor 1/\theta_{n-1} \rfloor \qquad (n = 1,\ 2,\ \dots).$$

The k-th convergent $p_k/q_k = [\ a_0;\ a_1,\ \dots,\ a_k\ ]$ of θ is then given by the recurrence relations

$$p_k = a_n p_{k-1} + p_{k-2} \qquad (k = 0,\ 1,\ \dots), \qquad p_{-2} = 0, \qquad p_{-1} = 1,$$
$$q_k = a_k q_{k-1} + q_{k-2} \qquad (k = 0,\ 1,\ \dots), \qquad q_{-2} = 1, \qquad q_{-1} = 0.$$

Denote $\phi = {}_{\theta}[\ b_0;\ b_1,\ b_2,\ \dots,\ ]$ be the expansion of ϕ in terms of the sequence $\{\theta_0,\ \theta_1,\ \dots\}$, where

$$\phi = b_0 - \phi_0, \qquad b_0 = \lceil \phi \rceil\,,$$
$$\phi_{n-1}/\theta_{n-1} = b_n - \phi_n, \qquad b_n = \lceil \phi_{n-1}/\theta_{n-1} \rceil \quad (n = 1,\ 2,\ \dots\ ).$$

Then, ϕ is represented by

$$\begin{aligned}\phi &= b_0 - b_1\theta_0 + b_2\theta_0\theta_1 - \cdots + (-1)^k b_k\theta_0\theta_1\cdots\theta_{k-1} - (-1)^k\theta_0\theta_1\cdots\theta_{k-1}\phi_k \\ &= b_0 - \sum_{k=0}^{\infty}(-1)^k b_{k+1}\theta_0\theta_1\cdots\theta_k = b_0 - \sum_{k=0}^{\infty} b_{k+1}D_k\,,\end{aligned}$$

where $D_k = q_k\theta - p_k = (-1)^k\theta_0\theta_1\dots\theta_k$. Now, the following theorem is established in [6].

Theorem 1.

$$\mathcal{M}_-(\theta,\phi) = \liminf_{n\to+\infty} \min(B_n\|B_n\theta + \phi\|, B_n^*\|B_n^*\theta + \phi\|)\,,$$

where $B_n = \sum_{k=1}^{n} b_k q_{k-1}$ *and* $B_n^* = B_n - q_{n-1}$.

Remark. It is also known in [6] that $\|B_n\theta + \phi\| = \phi_n|D_{n-1}|$ and $\|B_n^*\theta + \phi\| = (1-\phi_n)|D_{n-1}|$. Together with $\mathcal{M}_+(\theta,\phi) = \mathcal{M}_-(\theta, 1-\phi)$, one can obtain the value $\mathcal{M}(\theta,\phi)$.

3. The case $\mathcal{M}(\theta,\phi) = 0$

Continued fraction expansions of the form

$$[\ c_0;\ c_1,\ \dots,\ c_n,\ \overline{Q_1(k),\ \dots,\ Q_p(k)}\ ]_{k=1}^{\infty}$$

are called *Hurwitzian* if c_0 is an integer, $c_1,\ \dots,\ c_n$ are positive integers, $Q_1(k),\ \dots,\ Q_p(k)$ are polynomials with rational coefficients which takes positive integral values for $k = 1, 2, \dots$ and at least one of the polynomials is not constant. $Q_1(k), \dots, Q_p(k)$ are said to form a *quasi-period*. The expansions

$$e = [\,2;\ \overline{1,\ 2k,\ 1}\,]_{k=1}^{\infty} \qquad \text{and} \qquad e^{1/s} = [\,1;\ \overline{(2k-1)s-1,\ 1,\ 1}\,]_{k=1}^{\infty}$$

where s is a positive integer with $s \ge 2$, are well-known examples (See [3], [8], [10] e.g.). In [7] for a positive integer s we have $\mathcal{M}(e^{1/s}, (e^{1/s}-1)/2) = 0$, $\mathcal{M}(e^{1/s}, 1/2) = 1/8$ and $\mathcal{M}(e^{1/s}, 1/3) = 0$ if $s \equiv 2 \pmod 3$; $1/18$ otherwise.

Then, what is the condition such that $\mathcal{M}(\theta, \phi) = 0$ holds? It seems that a non-constant polynomial in a quasi-periodic part influences whether $\mathcal{M}(\theta, \phi) = 0$ or not. So, we consider the cases each quasi-periodic form includes no constant.

$$\frac{e^{1/s}-1}{e^{1/s}+1} = [\,0;\ \overline{(4k-2)s}\,]_{k=1}^{\infty}\,,$$

where s is a positive integer, or

$$\frac{e^{2/s}-1}{e^{2/s}+1} = [\,0;\ \overline{(2k-1)s}\,]_{k=1}^{\infty}\,,$$

where s is an odd positive integer with $s \ge 3$, is one of the well-known examples (See [10] e.g.).

In any of two expansions of θ above a_k is increasing and $a_k \to \infty$ ($k \to \infty$). So, one may be apt to conjecture that $\mathcal{M}(\theta, \phi) = 0$ for almost all of ϕ. But, there is a case satisfying $\mathcal{M}(\theta, \phi) \ne 0$.

Theorem 2.

$$\mathcal{M}\left(\frac{e^{1/s}-1}{e^{1/s}+1}, \frac{e^{1/s}}{e^{1/s}+1}\right) = \frac{1}{4}\,.$$

Proof. First, note that in the expansion of $\theta = (e^{1/s}-1)/(e^{1/s}+1)$

$$a_n = (4n-2)s \to \infty \quad (n = 1, 2, \dots \to \infty)\,,$$

yielding

$$\theta_{n-1} = \frac{1}{a_n + \theta_n} \to 0 \quad (n = 1, 2, \dots \to \infty)\,.$$

It is convenient to see that

$$q_n|D_{n-1}| = \frac{1}{1 + \theta_n q_{n-1}/q_n} \to 1 \qquad (n = 1, 2, \dots \to \infty)$$

and

$$q_{n-1}|D_{n-1}| = \frac{1}{(4n-2)s + \theta_n + q_{n-2}/q_{n-1}} \to 0 \qquad (n = 1, 2, \dots \to \infty)\,.$$

$\phi = (\theta+1)/2 = e^{1/s}/(e^{1/s}+1)$ is expanded as

$$\phi = {}_{\theta}[1; \overline{(2k-1)s}]_{k=1}^{\infty} = {}_{\theta}[1; \overline{a_k/2}]_{k=1}^{\infty}$$

and

$$\phi_n = \frac{1-\theta_n}{2} \to \frac{1}{2} \qquad (n = 0, 1, 2, \ldots \to \infty)\,.$$

For $n = 1, 2, \ldots$

$$B_n = \sum_{i=1}^{n} \frac{a_i}{2} q_{i-1} = \frac{q_n + q_{n-1} - 1}{2}\,.$$

Hence,

$$\begin{aligned} B_n \|B_n\theta + \phi\| &= B_n \phi_n |D_{n-1}| \\ &= \frac{1}{2}(q_n|D_{n-1}| + q_{n-1}|D_{n-1}| - |D_{n-1}|)\phi_n \\ &\to \frac{1}{2}(1+0-0)\cdot\frac{1}{2} = \frac{1}{4} \qquad (n\to\infty) \end{aligned}$$

and

$$\begin{aligned} B_n^* \|B_n^*\theta + \phi\| &= (B_n - q_{n-1})(1-\phi_n)|D_{n-1}| \\ &= \frac{1}{2}(q_n|D_{n-1}| - q_{n-1}|D_{n-1}| - |D_{n-1}|)(1-\phi_n) \\ &\to \frac{1}{2}(1-0-0)\left(1-\frac{1}{2}\right) = \frac{1}{4} \qquad (n\to\infty)\,, \end{aligned}$$

yielding that $\mathcal{M}_-(\theta,\phi) = 1/4$.

Next, $1-\phi = (1-\theta)/2 = 1/(e^{1/s}+1)$ is expanded as

$$1-\phi = {}_{\theta}[\; 1;\; s+1,\; \overline{(2k-1)s}\;]_{k=2}^{\infty} = {}_{\theta}[\; 1;\; a_1/2+1,\; \overline{a_k/2}\;]_{k=2}^{\infty}$$

and

$$\phi_0 = \frac{1+\theta_0}{2}, \qquad \phi_n = \frac{1-\theta_n}{2} \to \frac{1}{2} \qquad (n = 1, 2, \ldots \to \infty)\,.$$

For $n = 1, 2, \ldots$

$$B_n = 1 + \sum_{i=1}^{n} \frac{a_i}{2} q_{i-1} = \frac{q_n + q_{n-1} + 1}{2}\,.$$

In a similar manner, by

$$B_n\|B_n\theta - \phi\| \to \frac{1}{4} \quad \text{and} \quad B_n^*\|B_n^*\theta - \phi\| \to \frac{1}{4} \quad (n\to\infty)$$

one has $\mathcal{M}_+(\theta,\phi) = 1/4$. Therefore, $\mathcal{M}(\theta,\phi) = \mathcal{M}_\pm(\theta,\phi) = 1/4$. □

Contrary to this result, there is, of course, a case satisfying $\mathcal{M}(\theta,\phi) = 0$.

Theorem 3.

$$\mathcal{M}\left(\frac{e^{1/s}-1}{e^{1/s}+1}, \frac{1}{2}\right) = 0\,.$$

Remark. It is interesting to see that in [7]

$$\mathcal{M}\left(e^{1/s}, \frac{e^{1/s}-1}{2}\right) = 0 \qquad \text{and} \qquad \mathcal{M}\left(e^{1/s}, \frac{1}{2}\right) = \frac{1}{8} \neq 0$$

in comparison with Theorem 2 above and this Theorem.

Proof. $\phi = 1/2$ is expanded as

$$\begin{aligned} 1/2 &= {}_\theta[\ 1;\ s+1,\ \overline{(8k-2)s,\ (4k+1)s}\ ]_{k=1}^{\infty} \\ &= {}_\theta[\ 1;\ a_1/2+1,\ \overline{a_{2k},\ a_{2k+1}/2}\ ]_{k=1}^{\infty} \end{aligned}$$

and $\phi_0 = 1/2$, for $n = 1, 2, \dots$

$$\phi_{2n-1} = 1 - \frac{1}{2}\theta_{2n-1} \to 1, \quad \phi_{2n} = \frac{1}{2} - \theta_{2n} \to \frac{1}{2} \qquad (n \to \infty)\,.$$

Since for $n = 1, 2, \dots$

$$B_{2n-1} = \frac{a_1}{2} + 1 + \sum_{i=1}^{n-1}\left(a_{2i}q_{2i-1} + \frac{1}{2}a_{2i+1}q_{2i}\right) = \frac{1}{2}q_{2n-1} + q_{2n-2}\,,$$

one finds that

$$\begin{aligned} B_{2n-1}\|B_{2n-1}\theta + \phi\| &= B_{2n-1}\phi_{2n-1}|D_{2n-2}| \\ &= \left(\frac{1}{2}q_{2n-1}|D_{2n-2}| + q_{2n-2}|D_{2n-2}|\right)\phi_{2n-1} \\ &\to \left(\frac{1}{2}\cdot 1 + 0\right)\cdot 1 = \frac{1}{2}\,, \end{aligned}$$

$$\begin{aligned} B_{2n-1}^*\|B_{2n-1}^*\theta + \phi\| &= (B_{2n-1} - q_{2n-2})(1 - \phi_{2n-1})|D_{2n-2}| \\ &= \frac{1}{2}q_{2n-1}|D_{2n-2}|(1-\phi_{2n-1}) \to \frac{1}{2}\cdot 1 \cdot (1-1) = 0\,, \end{aligned}$$

$$\begin{aligned} B_{2n}\|B_{2n}\theta + \phi\| &= (B_{2n-1} + b_{2n}q_{2n-1})\phi_{2n}|D_{2n-1}| \\ &= \left(q_{2n}|D_{2n-1} + \frac{1}{2}q_{2n-1}|D_{2n-1}|\right)\phi_{2n} \\ &\to \left(1 + \frac{1}{2}\cdot 0\right)\cdot\frac{1}{2} = \frac{1}{2}\,, \end{aligned}$$

$$\begin{aligned} B_{2n}^*\|B_{2n}^*\theta + \phi\| &= (B_{2n} - q_{2n-1})(1-\phi_{2n})|D_{2n-1}| \\ &= \left(q_{2n}|D_{2n-1}| - \frac{1}{2}q_{2n-1}|D_{2n-1}|\right)(1-\phi_{2n}) \\ &\to \left(1 - \frac{1}{2}\cdot 0\right)\left(1 - \frac{1}{2}\right) = \frac{1}{2} \end{aligned}$$

as n tends to infinity. Therefore, we have $\mathcal{M}(\theta, 1/2) = \mathcal{M}_{\pm}(\theta, 1/2) = 0$. $\square$

We shall show one more case satisfying $\mathcal{M}(\theta, \phi) = 0$.

Theorem 4.

$$\mathcal{M}\left(\frac{e^{1/s}-1}{e^{1/s}+1}, \frac{1}{3}\right) = 0\,.$$

Proof. When $s \equiv 0 \pmod 3$, $\phi = 1/3$ is expanded as

$$\frac{1}{3} = {}_{\theta}[\ 1;\ \frac{2}{3}a_1 + 1,\ \overline{a_{2k},\ \frac{2}{3}a_{2k+1}}\ ]_{k=1}^{\infty}$$

and $\phi_0 = 2/3$, for $n = 1, 2, \dots$

$$\phi_{2n-1} = 1 - \frac{2}{3}\theta_{2n-1} \to 1, \quad \phi_{2n} = \frac{2}{3} - \theta_{2n} \to \frac{2}{3} \qquad (n \to \infty)\,.$$

Since for $n = 1, 2, \dots$

$$B_{2n-1} = \frac{2}{3}a_1 + 1 + \sum_{i=1}^{n-1}\left(a_{2i}q_{2i-1} + \frac{2}{3}a_{2i+1}q_{2i}\right) = \frac{2}{3}q_{2n-1} + q_{2n-2}\,,$$

one finds that

$$\begin{aligned} B^*_{2n-1}\|B^*_{2n-1}\theta + \phi\| &= (B_{2n-1} - q_{2n-2})(1 - \phi_{2n-1})|D_{2n-2}| \\ &= \frac{2}{3}q_{2n-1}|D_{2n-2}|(1 - \phi_{2n-1}) \to \frac{2}{3}\cdot 1 \cdot (1-1) = 0\,, \end{aligned}$$

as n tends to infinity. Hence, we have $\mathcal{M}_-(\theta, 1/3) = 0$.

$1 - \phi = 2/3$ is expanded as

$$\frac{2}{3} = {}_{\theta}[\ 1;\ \frac{1}{3}a_1 + 1,\ \overline{a_{2k},\ \frac{1}{3}a_{2k+1}}\ ]_{k=1}^{\infty}$$

and $\phi_0 = 1/3$, for $n = 1, 2, \dots$

$$\phi_{2n-1} = 1 - \frac{1}{3}\theta_{2n-1} \to 1, \quad \phi_{2n} = \frac{1}{3} - \theta_{2n} \to \frac{1}{3} \qquad (n \to \infty)\,.$$

Since for $n = 1, 2, \dots$

$$B_{2n-1} = \frac{1}{3}a_1 + 1 + \sum_{i=1}^{n-1}\left(a_{2i}q_{2i-1} + \frac{1}{3}a_{2i+1}q_{2i}\right) = \frac{1}{3}q_{2n-1} + q_{2n-2}\,,$$

one finds that

$$\begin{aligned} B^*_{2n-1}\|B^*_{2n-1}\theta - \phi\| &= (B_{2n-1} - q_{2n-2})(1 - \phi_{2n-1})|D_{2n-2}| \\ &= \frac{1}{3}q_{2n-1}|D_{2n-2}|(1 - \phi_{2n-1}) \to \frac{1}{3}\cdot 1 \cdot (1-1) = 0\,, \end{aligned}$$

as n tends to infinity. Hence, we have $\mathcal{M}_+(\theta, 1/3) = 0$.
Therefore, $\mathcal{M}(\theta, 1/3) = 0$.

When $s \equiv 1 \pmod 3$, $\phi = 1/3$ is expanded as

$$\frac{1}{3} = {}_{\theta}[\ 1;\ \frac{2}{3}(a_1+1),\ \overline{\frac{2}{3}a_{6k-4},\ \frac{2a_{6k-3}+1}{3},\ a_{6k-2},\ \frac{2}{3}a_{6k-1},\ a_{6k},\ \frac{2a_{6k+1}-1}{3}}\ ]_{k=1}^{\infty}$$

and $\phi_0 = 2/3$, for $n = 1, 2, \dots$

$$\phi_{6n-5} = \frac{2}{3}(1-\theta_{6n-5}) \to \frac{2}{3}, \qquad \phi_{6n-4} = \frac{2}{3}(1-\theta_{6n-4}) \to \frac{2}{3},$$
$$\phi_{6n-3} = 1 - \frac{2}{3}\theta_{6n-3} \to 1, \qquad \phi_{6n-2} = \frac{2}{3} - \theta_{6n-2} \to \frac{2}{3},$$
$$\phi_{6n-1} = 1 - \frac{2}{3}\theta_{6n-1} \to 1, \qquad \phi_{6n} = \frac{2}{3} - \theta_{6n} \to \frac{2}{3}$$

as n tends to infinity. Since for $n = 1, 2, \dots$

$$\begin{aligned} B_{6n-5} &= \frac{2}{3}(a_1+1) + \sum_{i=1}^{n-1}\left(\frac{2}{3}a_{6i-4}q_{6i-5} + \frac{2a_{6i-3}+1}{3}q_{6i-4}\right. \\ &\qquad \left. + a_{6i-2}q_{6i-3} + \frac{2}{3}a_{6i-1}q_{6i-2} + a_{6i}q_{6i-1} + \frac{2a_{6i+1}-1}{3}q_{6i}\right) \\ &= \frac{2}{3}(q_{6n-5} + q_{6n-6}), \end{aligned}$$

one finds that

$$\begin{aligned} B^*_{6n-3}\|B^*_{6n-3}\theta + \phi\| &= (B_{6n-3} - q_{6n-4})(1-\phi_{6n-3})|D_{6n-6}| \\ &= \frac{2}{3}q_{6n-3}|D_{6n-4}|(1-\phi_{6n-3}) \to \frac{2}{3}\cdot 1\cdot(1-1) = 0, \\ B^*_{6n-1}\|B^*_{6n-1}\theta + \phi\| &= \frac{2}{3}q_{6n-1}|D_{6n-2}|(1-\phi_{6n-1}) \to \frac{2}{3}\cdot 1\cdot(1-1) = 0, \end{aligned}$$

as n tends to infinity. Hence, we have $\mathcal{M}_-(\theta, 1/3) = 0$.

$1 - \phi = 2/3$ is expanded as

$$\frac{2}{3} = {}_{\theta}[1; \frac{1}{3}(a_1+1), \overline{\frac{1}{3}a_{6k-4}, \frac{a_{6k-3}+2}{3}, a_{6k-2},\ \frac{1}{3}a_{6k-1}, a_{6k}, \frac{a_{6k+1}-2}{3}}\ ]_{k=1}^{\infty}$$

and $\phi_0 = 1/3$, for $n = 1, 2, \dots$

$$\phi_{6n-5} = \frac{1}{3}(1-\theta_{6n-5}) \to \frac{1}{3}, \qquad \phi_{6n-4} = \frac{1}{3}(1-\theta_{6n-4}) \to \frac{1}{3},$$
$$\phi_{6n-3} = 1 - \frac{1}{3}\theta_{6n-3} \to 1, \qquad \phi_{6n-2} = \frac{1}{3} - \theta_{6n-2} \to \frac{1}{3},$$
$$\phi_{6n-1} = 1 - \frac{1}{3}\theta_{6n-1} \to 1, \qquad \phi_{6n} = \frac{1}{3} - \theta_{6n} \to \frac{1}{3}$$

as n tends to infinity. Since for $n=1,2,\dots$

$$B_{6n-5}=\frac{1}{3}(q_{6n-5}+q_{6n-6})\,,$$

one finds that

$$B_{6n-3}^*\|B_{6n-3}^*\theta-\phi\|=\frac{1}{3}q_{6n-3}|D_{6n-4}|(1-\phi_{6n-3})\to\frac{1}{3}\cdot 1\cdot(1-1)=0\,,$$
$$B_{6n-1}^*\|B_{6n-1}^*\theta-\phi\|=\frac{1}{3}q_{6n-1}|D_{6n-2}|(1-\phi_{6n-1})\to\frac{1}{3}\cdot 1\cdot(1-1)=0\,,$$

as n tends to infinity. Hence, we have $\mathcal{M}_+(\theta,1/3)=0$.
Therefore, $\mathcal{M}(\theta,1/3)=0$.

When $s\equiv 2 \pmod 3$, $\phi=1/3$ is expanded as

$$\frac{1}{3}={}_\theta[\,1;\ \frac{2a_1+1}{3},\ \overline{\frac{1}{3}a_{6k-4},\ \frac{2}{3}(a_{6k-3}+1),\ a_{6k-2},\ \frac{2}{3}a_{6k-1},\ a_{6k},\ \frac{2}{3}(a_{6k+1}-1)}\ ]_{k=1}^\infty$$

and $\phi_0=2/3$, for $n=1,2,\dots$

$$\phi_{6n-5}=\frac{1}{3}(1-2\theta_{6n-5})\to\frac{1}{3},\qquad \phi_{6n-4}=\frac{1}{3}(2-\theta_{6n-4})\to\frac{2}{3},$$
$$\phi_{6n-3}=1-\frac{2}{3}\theta_{6n-3}\to 1,\qquad \phi_{6n-2}=\frac{2}{3}-\theta_{6n-2}\to\frac{2}{3},$$
$$\phi_{6n-1}=1-\frac{2}{3}\theta_{6n-1}\to 1,\qquad \phi_{6n}=\frac{2}{3}-\theta_{6n}\to\frac{2}{3}$$

as n tends to infinity. Since for $n=1,2,\dots$

$$\begin{aligned}B_{6n-5}&=\frac{2a_1+1}{3}+\sum_{i=1}^{n-1}\Bigg(\frac{1}{3}a_{6i-4}q_{6i-5}+\frac{2}{3}(a_{6i-3}+1)q_{6i-4}\\&\qquad+a_{6i-2}q_{6i-3}+\frac{2}{3}a_{6i-1}q_{6i-2}+a_{6i}q_{6i-1}+\frac{2}{3}(a_{6i+1}-1)q_{6i}\Bigg)\\&=\frac{1}{3}(2q_{6n-5}+q_{6n-6})\,,\end{aligned}$$

one finds that

$$B_{6n-3}^*\|B_{6n-3}^*\theta+\phi\|=\frac{2}{3}q_{6n-3}|D_{6n-4}|(1-\phi_{6n-3})\to\frac{2}{3}\cdot 1\cdot(1-1)=0\,,$$
$$B_{6n-1}^*\|B_{6n-1}^*\theta+\phi\|=\frac{2}{3}q_{6n-1}|D_{6n-2}|(1-\phi_{6n-1})\to\frac{2}{3}\cdot 1\cdot(1-1)=0\,,$$

as n tends to infinity. Hence, we have $\mathcal{M}_-(\theta,1/3)=0$.

$1-\phi=2/3$ is expanded as

$$\frac{2}{3}={}_\theta[\,1;\ \frac{a_1+2}{3},\ \overline{\frac{2}{3}a_{6k-4},\ \frac{a_{6k-3}+1}{3},\ a_{6k-2},\ \frac{1}{3}a_{6k-1},\ a_{6k},\ \frac{a_{6k+1}-1}{3}}\ ]_{k=1}^\infty$$

and $\phi_0 = 1/3$, for $n = 1, 2, \dots$

$$\phi_{6n-5} = \frac{1}{3}(2 - \theta_{6n-5}) \to \frac{2}{3}, \qquad \phi_{6n-4} = \frac{1}{3}(1 - 2\theta_{6n-4}) \to \frac{1}{3},$$
$$\phi_{6n-3} = 1 - \frac{1}{3}\theta_{6n-3} \to 1, \qquad \phi_{6n-2} = \frac{1}{3} - \theta_{6n-2} \to \frac{1}{3},$$
$$\phi_{6n-1} = 1 - \frac{1}{3}\theta_{6n-1} \to 1, \qquad \phi_{6n} = \frac{1}{3} - \theta_{6n} \to \frac{1}{3}$$

as n tends to infinity. Since for $n = 1, 2, \dots$

$$B_{6n-5} = \frac{1}{3}(q_{6n-5} + 2q_{6n-6}),$$

one finds that

$$B^*_{6n-3}\|B^*_{6n-3}\theta - \phi\| = \frac{1}{3}q_{6n-3}|D_{6n-4}|(1 - \phi_{6n-3}) \to \frac{1}{3}\cdot 1 \cdot (1-1) = 0,$$
$$B^*_{6n-1}\|B^*_{6n-1}\theta - \phi\| = \frac{1}{3}q_{6n-1}|D_{6n-2}|(1 - \phi_{6n-1}) \to \frac{1}{3}\cdot 1 \cdot (1-1) = 0,$$

as n tends to infinity. Hence, we have $\mathcal{M}_+(\theta, 1/3) = 0$.
Therefore, $\mathcal{M}(\theta, 1/3) = 0$. □

4. The cases $\mathcal{M}((e^{2/s}-1)/(e^{2/s}+1), \phi) = 0$

Let us calculate $\mathcal{M}(\theta, \phi)$ when

$$\theta = \frac{e^{2/s}-1}{e^{2/s}+1} = [\,0;\ \overline{(2k-1)s}\,]_{k=1}^{\infty},$$

where s is an odd positive integer with $s \geq 3$. The situations are a little bit different from the previous results. Notice that $a_n = (2n-1)s \to \infty$, so $\theta_{n-1} = 1/(a_n + \theta_n) \to 0$ ($n = 1, 2, \dots \to \infty$). $\lim_{n\to\infty} q_n|D_{n-1}| = 1$ and $\lim_{n\to\infty} q_{n-1}|D_{n-1}| = 0$ hold for this θ too. The first result is quite different from Theorem 2.

Theorem 5.

$$\mathcal{M}\left(\frac{e^{2/s}-1}{e^{2/s}+1}, \frac{e^{2/s}}{e^{2/s}+1}\right) = 0.$$

Proof. $\phi = (\theta+1)/2 = e^{2/s}/(e^{2/s}+1)$ is expanded as

$$\begin{aligned}\phi &= {}_\theta[\,1;\ \overline{\frac{(6k-5)s+1}{2},\ (6k-3)s,\ \frac{(6k-1)s-1}{2}}\,]_{k=1}^{\infty}\\ &= {}_\theta[\,1;\ \overline{\frac{a_{3k-2}+1}{2},\ a_{3k-1},\ \frac{a_{3k}-1}{2}}\,]_{k=1}^{\infty}\end{aligned}$$

and

$$\phi_{3n} = \frac{1-\theta_{3n}}{2} \to \frac{1}{2} \qquad (n = 0, 1, 2, \ldots \to \infty)\,,$$

$$\phi_{3n-2} = 1 - \frac{1}{2}\theta_{3n-2} \to 1, \quad \phi_{3n-1} = \frac{1}{2} - \theta_{3n} \to \frac{1}{2} \qquad (n = 1, 2, \ldots \to \infty)\,.$$

Since for $n = 1, 2, \ldots$

$$\begin{aligned} B_{3n} &= \sum_{i=1}^{n} \left(\frac{a_{3i-2}+1}{2} q_{3i-3} + a_{3i-1} q_{3i-2} + \frac{a_{3i}-1}{2} q_{3i-1} \right) \\ &= \frac{1}{2}(q_{3n} + q_{3n-1} - 1)\,. \end{aligned}$$

one finds that

$$\begin{aligned} B^*_{3n+1} \| B^*_{3n+1}\theta + \phi \| &= (B_{3n+1} - q_{3n})(1 - \phi_{3n+1})|D_{3n}| \\ &= \frac{1}{2}(q_{3n+1}|D_{3n}| - |D_{3n}|)(1 - \phi_{3n+1}) \\ &\to \frac{1}{2}(1-0)(1-1) = 0 \quad (n \to \infty)\,, \end{aligned}$$

yielding $\mathcal{M}_-(\theta, \phi) = 0$.

Next, $1 - \phi = (1-\theta)/2 = 1/(e^{2/s}+1)$ is expanded as

$$1 - \phi = {}_{\theta}[\, 1;\ \frac{a_1+3}{2},\ \overline{a_{3k-1},\ \frac{a_{3k}-1}{2},\ \frac{a_{3k+1}+1}{2}}\ ]_{k=2}^{\infty}$$

and

$$\phi_0 = \frac{1+\theta_0}{2}, \qquad \phi_{3n-2} = 1 - \frac{1}{2}\theta_{3n-2} \to 1,$$

$$\phi_{3n-1} = \frac{1}{2} - \theta_{3n-1}, \quad \phi_{3n} = \frac{1}{2}(1 - \theta_{3n}) \to \frac{1}{2} \quad (n = 1, 2, \ldots \to \infty)\,.$$

Since for $n = 1, 2, \ldots$

$$B_{3n-2} = \frac{1}{2} q_{3n-2} + q_{3n-3} + \frac{1}{2}\,.$$

one finds that

$$\begin{aligned} B^*_{3n-2} \| B^*_{3n-2}\theta - \phi \| &= \frac{1}{2}(q_{3n-2}|D_{3n-3}| + |D_{3n-3}|)(1 - \phi_{3n-2}) \\ &\to \frac{1}{2}(1+0)(1-1) = 0 \quad (n \to \infty)\,, \end{aligned}$$

yielding $\mathcal{M}_+(\theta, \phi) = 0$. Therefore, $\mathcal{M}(\theta, \phi) = \mathcal{M}_\pm(\theta, \phi) = 0$. □

Theorem 6.

$$\mathcal{M}\left(\frac{e^{2/s}-1}{e^{2/s}+1}, \frac{1}{2} \right) = 0\,.$$

Proof. $\phi = 1/2$ is expanded as

$$\frac{1}{2} = {}_\theta[\, 1;\ \frac{a_1+1}{2},\ \overline{\frac{a_{3k-1}+1}{2},\ a_{3k},\ \frac{a_{3k+1}-1}{2}}\ ]_{k=1}^{\infty}$$

and

$$\phi_0 = \frac{1}{2}, \qquad \phi_{3n-2} = \frac{1}{2}(1-\theta_{3n-2}) \to \frac{1}{2},$$

$$\phi_{3n-1} = 1 - \frac{1}{2}\theta_{3n-1} \to 1, \quad \phi_{3n} = \frac{1}{2} - \theta_{3n} \to \frac{1}{2} \quad (n = 1, 2, \ldots \to \infty)\,.$$

Since for $n = 1, 2, \ldots$

$$\begin{aligned} B_{3n-2} &= \frac{a_1+1}{2} + \sum_{i=1}^{n-1}\left(\frac{a_{3i-1}+1}{2}q_{3i-2} + a_{3i}q_{3i-1} + \frac{a_{3i+1}-1}{2}q_{3i}\right) \\ &= \frac{1}{2}(q_{3n-2} + q_{3n-3})\,, \end{aligned}$$

one finds that

$$\begin{aligned} B_{3n-1}^{*}\|B_{3n-1}^{*}\theta + \phi\| &= (B_{3n-1} - q_{3n-2})(1-\phi_{3n-1})|D_{3n-2}| \\ &= \frac{1}{2}q_{3n-1}|D_{3n-2}|(1-\phi_{3n-1}) \to \frac{1}{2}\cdot 1 \cdot (1-1) = 0 \end{aligned}$$

as n tends to infinity. Therefore, we have $\mathcal{M}(\theta, 1/2) = \mathcal{M}_{\pm}(\theta, 1/2) = 0$. □

Theorem 7.

$$\mathcal{M}\left(\frac{e^{2/s}-1}{e^{2/s}+1}, \frac{1}{3}\right) = 0\,.$$

Proof. When $s \equiv 3$, $s \equiv 5$, $s \equiv 1 \pmod 6$, the situation is completely the same as the case of

$$\theta = \frac{e^{1/s}-1}{e^{1/s}+1}$$

with $s \equiv 0$, $s \equiv 1$, $s \equiv 2 \pmod 3$, respectively. □

5. Some conditions satisfying $\mathcal{M}(\theta, \phi) = 0$

We have already seen several examples so that $\mathcal{M}(\theta, \phi) = 0$ holds. Then, what is the condition of $\mathcal{M}(\theta, \phi) = 0$? Of course, the following is clear.

Theorem 8. *If $\phi_n \to 0$ or $\phi_n \to 1$ $(n \to \infty)$ for infinitely many positive integers n, then $\mathcal{M}(\theta, \phi) = 0$.*

Proof. First, we shall show that $\theta_{n-1} < B_n|D_{n-1}| < 4$ for any positive integer n. Since

$$B_n = \sum_{i=1}^{n} b_i q_{i-1} \le \sum_{i=1}^{n}(a_i+1)q_{i-1} = q_n + 2q_{n-1} + (q_{n-2} + \cdots + q_1) < 4q_n\,,$$

we obtain

$$B_n|D_{n-1}| < \frac{4q_n}{q_n + \theta_n q_{n-1}} < 4\,.$$

On the other hand,

$$B_n|D_{n-1}| \geq \frac{\sum_{i=1}^{n} q_{i-1}}{q_n + \theta_n q_{n-1}} > \frac{1}{a_n + \theta_n} = \theta_{n-1}\,.$$

If $\phi_n \to 0$ $(n \to \infty)$, then

$$B_n\|B_n\theta + \phi\| = B_n|D_{n-1}|\phi_n \to 0 \quad (n \to \infty)\,.$$

If $\phi_n \to 1$ $(n \to \infty)$, then

$$B_n^*\|B_n^*\theta + \phi\| = B_n^*|D_{n-1}|(1 - \phi_n) \to 0 \quad (n \to \infty)\,.$$

□

Corollary. *When $b_n = 1$, $\phi_{n-1} \to 0$ if and only if $\phi_n \to 1$ $(n \to \infty)$.*

This is very generous. So, we state the following.

Theorem 9. *If $|a_n - b_n| \leq c$ and $a_n \to \infty$ $(n \to \infty)$ for infinitely many positive integers n, then $\mathcal{M}(\theta, \phi) = 0$. Here, c is a constant not depending upon n.*

Remark. In fact, $a_n = b_n \to \infty$ $(n \to \infty)$ holds in all previous theorems above implying $\mathcal{M}(\theta, \phi) = 0$.

Proof. If $|a_n - b_n| \leq c$, then $\dfrac{1}{\theta_{n-1}} - \dfrac{\phi_{n-1}}{\theta_{n-1}} < c + 2$ or $0 < 1 - \phi_{n-1} < (c+2)\theta_{n-1}$. And if $\lim_{n\to\infty} a_n = \infty$, then

$$\theta_{n-1} = \frac{1}{a_n + \theta_n} \to 0 \quad (n \to \infty)\,.$$

Thus, $1 - \phi_{n-1} \to 0$ $(n \to \infty)$ entails that

$$B_{n-1}^*\|B_{n-1}^*\theta + \phi\| = B_{n-1}^*|D_{n-2}|(1 - \phi_{n-1}) \to 0 \quad (n \to \infty)\,.$$

□

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