Journal de Théorie des Nombres
de Bordeaux **11** (1999), 387–406

The class number one problem for some non-abelian normal CM-fields of degree 24

par F. Lemmermeyer, S. Louboutin, et R. Okazaki

Résumé. Nous déterminons tous les corps de nombres de degré 24, galoisiens mais non-abéliens, à multiplication complexe et tels que les groupes de Galois de leurs sous-corps totalement réels maximaux soient isomorphes à $\mathcal{A}_4$ (le groupe alterné de degré 4 et d'ordre 12) qui sont de nombres de classes d'idéaux égaux à 1. Nous prouvons (*i*) qu'il y a deux tels corps de nombres de groupes de Galois $\mathcal{A}_4 \times \mathcal{C}_2$ (voir Théorème 14), (*ii*) qu'il y a au plus un tel corps de nombres de groupe de Galois $\mathrm{SL}_2(\mathbb{F}_3)$ (voir Théorème 18), et (*iii*) que sous l'hypothèse de Riemann généralisée ce dernier corps candidat est effectivement de nombre de classes d'idéaux égal à 1.

Abstract. We determine all the non-abelian normal CM-fields of degree 24 with class number one, provided that the Galois group of their maximal real subfields is isomorphic to $\mathcal{A}_4$, the alternating group of degree 4 and order 12. There are two such fields with Galois group $\mathcal{A}_4 \times \mathcal{C}_2$ (see Theorem 14) and at most one with Galois group $\mathrm{SL}_2(\mathbb{F}_3)$ (see Theorem 18); if the Generalized Riemann Hypothesis is true, then this last field has class number 1.

1. Introduction

Let us first fix the notation for the groups that will occur: $\mathcal{C}_m$ is the cyclic group of order m, $\mathcal{V} \simeq \mathcal{C}_2 \times \mathcal{C}_2$ is Klein's four group, $\mathcal{Q}_8$ is the quaternion group of order 8, $\mathcal{A}_4$ is the alternating group of order 12 (the normal subgroup of index 2 in the symmetric group $\mathcal{S}_4$), and $\mathrm{SL}_2(\mathbb{F}_3)$ is the group of 2×2-matrices with entries in the finite field of 3 elements and determinant 1.

Let N be a normal CM-field of degree $2n$ and Galois group $G = \mathrm{Gal}(N/\mathbb{Q})$. Since the complex conjugation J is in the centre $Z(G)$ of G, then, its maximal real subfield N^+ is a normal real field of degree n and $G^+ = \mathrm{Gal}(N^+/\mathbb{Q})$ is isomorphic to the quotient group $G/\{\mathrm{Id}, J\}$. As delineated in the introduction in [22], and according to [17], [18], [19] and [23], the completion

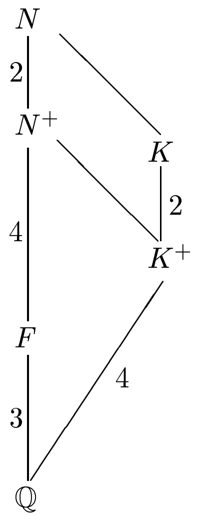
Manuscrit reçu le 27 mai 1998.

of the determination of all the non-abelian normal CM-fields of degree < 32 with class number one is reduced to the determination of all the non-abelian, non-dihedral normal CM-fields of degree 24. Now, there are 15 groups G of order 24, three of them being abelian and one of them having trivial center $Z(G)$ (namely, the symmetric group $\mathcal{S}_4$ of degree 4). Hence, there are 11 possible Galois groups for non-abelian normal CM-fields. Let us also point out that Y. Lefeuvre proved in [16] that there is only one dihedral CM-field of degree 24 with relative class number one, and this dihedral CM-field has class number one. We will focus on two of the remaining ten non-abelian groups: those which have the alternating group $\mathcal{A}_4$ as a quotient. There are only two such groups, namely $\mathrm{SL}_2(\mathbb{F}_3)$ and $\mathcal{A}_4 \times \mathcal{C}_2$ (see e.g. [30]).

Let us also point out that $\mathcal{A}_4 \times \mathcal{C}_2$, $\mathrm{SL}_2(\mathbb{F}_3)$ and $\mathcal{S}_4$ are the only non-abelian groups of order 24 whose 3-Sylow subgroups are not normal. Moreover, if the 3-Sylow subgroup of the Galois group of a non-abelian normal CM-field N of degree 24 is a normal subgroup, then N contains a normal octic CM-subfield M and the relative class number h_M^- of M divides the relative class number h_N^- of N (see [19, Th. 5] or [14, Cor. 1]), which makes the situation easier. For example, quaternion octic CM-fields have even relative class numbers ([17]), hence there are no normal CM-fields with relative class number one with Galois group $\mathcal{C}_3 \times \mathcal{Q}_8$ or $\mathcal{C}_3 \rtimes \mathcal{Q}_8$.

Here is the plan of our investigation: first, we will characterize the normal CM-fields with Galois groups $\mathrm{SL}_2(\mathbb{F}_3)$ or $\mathcal{A}_4 \times \mathcal{C}_2$ and odd relative class number (see Theorems 11 and 10). Then we will reduce the determination of all the normal CM-fields of Galois group $\mathcal{A}_4 \times \mathcal{C}_2$ to the determination of all the imaginary sextic cyclic fields of relative class number 4, and using results from [26] we will complete the determination of all the normal CM-fields with Galois group $\mathcal{A}_4 \times \mathcal{C}_2$ with relative and absolute classs numbers equal to one (see Theorem 14). Next we will give lower bounds on relative class numbers of normal CM-fields of degree 24 with Galois group $\mathrm{SL}_2(\mathbb{F}_3)$ and odd relative class number (see Theorem 16). Since the explicit construction of normal CM-fields of degree 24 with Galois group $\mathrm{SL}_2(\mathbb{F}_3)$ and odd relative class number is far from being easy, we will focus on the class number one problem for these fields and quote a result from [23] (see Proposition 17) which will provide us with a list of 23 cyclic cubic fields such that any normal CM-field of degree 24 with Galois group $\mathrm{SL}_2(\mathbb{F}_3)$ and class number one is the Hilbert 2-class field of a field in that list. Finally, we will complete the determination of all the normal CM-fields of degree 24 with Galois group $\mathrm{SL}_2(\mathbb{F}_3)$ and class number one modulo GRH (see Theorem 18).

1.1. **Diagrams of subfields.** Let N^+ be a normal field of degree 12 with Galois group $\mathcal{A}_4$. This group contains four conjugacy classes: $Cl_1 = \{\mathrm{Id}\}$, DIAGRAM 1



$\mathrm{Gal}(N/\mathbb{Q}) = \mathrm{SL}_2(\mathbb{F}_3)$

$\mathrm{Gal}(N/F) = \mathcal{Q}_8$

$\mathrm{Gal}(N^+/\mathbb{Q}) = \mathcal{A}_4$

$\mathrm{Gal}(F/\mathbb{Q}) = \mathcal{C}_3$

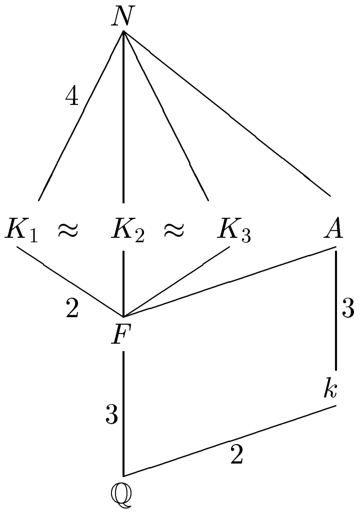
K a CM-field

$Cl_2 = \{(123),\ (142),\ (134),\ (243)\}$, $Cl_3 = \{(132),\ (124),\ (143),\ (234)\}$, and $Cl_4 = \{(12)(34),\ (13)(24),\ (14)(23)\}$. Set $\mathcal{V} = Cl_1 \cup Cl_4$. Then $\mathcal{V}$ is a normal subgroup of $\mathcal{A}_4$, the quotient group $\mathcal{A}_4/\mathcal{V}$ is cyclic of order 3 and $\mathcal{V}$ is isomorphic to Klein's four group. Note that $\mathcal{V}$ is a normal 2-Sylow subgroup of $\mathcal{A}_4$. Let F be the cyclic cubic subfield of N^+ fixed by $\mathcal{V}$. Then N^+/F is biquadratic bicyclic and F is a cyclic cubic field whose conductor we will denote by f.

Let N be a normal CM-field of degree 24 with Galois group $\mathrm{SL}_2(\mathbb{F}_3)$. There are eight elements of order 3 in $\mathrm{SL}_2(\mathbb{F}_3)$, hence four subgroups of order 3 in $\mathrm{SL}_2(\mathbb{F}_3)$. These four groups are pairwise conjugate. Let K be the fixed field of any of these four subgroups. Then K is a non-normal octic CM-subfield of N whose normal closure is N, and K^+ is a non-normal totally real primitive quartic field whose normal closure is N^+. The same holds for any of the four fields conjugate to K over $\mathbb{Q}$. The (incomplete) lattice of subfields is given in Diagram 1.

Let N be a normal CM-field of degree 24 with Galois group $G = \mathcal{A}_4 \times \mathcal{C}_2$. Note that $Z(G) = \{(1,1),(1,-1)\}$ so that the complex conjugation J of $G = \mathrm{Gal}(N/\mathbb{Q})$ is equal to $(1,-1)$. Let $\tau_1 = (12)(34)$, $\tau_2 = (13)(24)$ and $\tau_3 = (14)(23)$ denote the three elements of order 2 of $\mathcal{A}_4$. Set $G_i = \{(1,1),\ (\tau_i,1),\ (\tau_j,-1),\ (\tau_k,-1)\}$ with $1 \le i \le 3$ and $\{i,j,k\} = \{1,2,3\}$. Then each G_i is a subgroup of $G = \mathcal{A}_4 \times \mathcal{C}_2$ and is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ and $\{\tau_1,\tau_2,\tau_3\} = Cl_4$ being a conjugacy class in $\mathcal{A}_4$ the three groups G_i, $1 \le i \le 3$ are conjugate in G.
Set $G_0 = \{(v,1);\ v \in \mathcal{V}\} = \{(1,1),(\tau_1,1),(\tau_2,1),(\tau_3,1)\}$. For $0 \le i \le 3$

Diagram 2



$\mathrm{Gal}(N/\mathbb{Q}) = \mathcal{A}_4 \times \mathcal{C}_2$

$\mathrm{Gal}(N/F) = \mathcal{C}_2^3$

$\mathrm{Gal}(A/\mathbb{Q}) = \mathcal{C}_6$

$\mathrm{Gal}(F/\mathbb{Q}) = \mathcal{C}_3$

K_i CM-fields

k imaginary

let K_i be the fixed field of G_i, and F be the fixed field of $\mathcal{V} \times \mathcal{C}_2$. The (incomplete) lattice of subfields is given in Diagram 2.

1.2. **Factorizations of Dedekind zeta functions.**

For a CM-field N we let ζ_N denote its Dedekind zeta function, h_N^- its relative class number, $Q_N \in \{1, 2\}$ its Hasse unit index, W_N its group of roots of unity and w_N the order of W_N. Note that $W_N = W_A$ where A denotes the maximal abelian subfield of N. We have

$$h_N^- = \frac{Q_N w_N}{(2\pi)^n} \sqrt{\frac{d_N}{d_{N^+}}} \frac{\mathrm{Res}_{s=1}(\zeta_N)}{\mathrm{Res}_{s=1}(\zeta_{N^+})} \tag{1}$$

Let us make some more general remarks. Using [7] one can easily prove that we have the following factorizations of Dedekind zeta functions into products of Artin's L-functions :

$$(\zeta_N/\zeta_{N^+})(s) = \prod_{\substack{\chi \\ \chi(J) \neq \chi(\mathrm{Id})}} L(s, \chi, N/\mathbb{Q})^{\deg \chi} \tag{2}$$

where χ ranges over all the irreducible characters of $G = \mathrm{Gal}(N/\mathbb{Q})$ which satisfy $\chi(J) \neq \chi(\mathrm{Id})$. In the same way, for any subfield F of N^+, we have

$$(\zeta_N/\zeta_{N^+})(s) = \prod_{\substack{\chi \\ \chi(J) \neq \chi(\mathrm{Id})}} L(s, \chi, N/F)^{\deg \chi} \tag{3}$$

where χ ranges over all the irreducible characters of $G = \mathrm{Gal}(N/F)$ satisfying $\chi(J) \neq \chi(\mathrm{Id})$. We also remind the reader that Artin's L-functions are meromorphic and that Artin's L-functions associated with characters induced by linear characters of subgroups are entire.

1.3. **Prerequisites.**

Here we record some results that will be used later.

Proposition 1.

1. *Let N be a CM-field with maximal real subfield N^+ and let t denote the number of prime ideals of N^+ which ramify in the quadratic extension N/N^+. Then 2^{t-1} divides h_N^-. In particular, if h_N^- is odd then at most one prime ideal of N is ramified in the quadratic extension N/N^+. Moreover, if the narrow class number $h_{N^+}^+$ of N^+ is odd then the 2-rank $r_2(N)$ of the ideal class group of N is equal to $t-1$.*
2. *Let K/L be a finite extension of number fields. If at least one prime ideal of L is totally ramified in the K/L then the norm map from the narrow ideal class group of K to that of L is surjective. If K/L is quadratic, L is totally real and K is totally imaginary then the norm map from the wide ideal class group of K to that of L is surjective.*
3. *Let N be a CM-field with maximal real subfield N^+ and assume h_N^- is odd. Let $H_2(N)$, $H_2(N^+)$ and $H_2^+(N^+)$ denote the 2-class field of N, the 2-class field of N^+ and the narrow 2-class field of N^+, respectively.*
   (a) *The 2-class group of N^+ is cyclic.*
   (b) *$H_2(N) = NH_2(N^+)$ and $H_2(N)$ is a CM-field with maximal real subfield $H_2(N^+)$. Hence, the 2-class group of N is cyclic, for it is isomorphic to the 2-class group of N^+.*
   (c) *If N/N^+ is unramified at all the finite places then $H_2(N) = H_2^+(N^+)$ and $h_{N^+}^+ = 2h_{N^+}$.*
   (d) *If N/N^+ is ramified at some finite place, then $H_2^+(N^+) = H_2(N^+)$ $h_{N^+}^+ = h_{N^+}$ and the narrow 2-class group of N^+ is cyclic.*
4. *Let G denote the wide or the narrow ideal class group of any cyclic cubic field. Then, for any $n \geq 0$ the 2-ranks of the groups $G^{2^n} = \{g^{2^n};\ g \in G\}$ are even. Moreover, the narrow class number of a cyclic cubic field is either equal to its wide class number or equal to four times its wide class number.*

Proof. 1. follows from the ambiguous class number formula ([13, Ch. 13, Lemma 4.1]:

$$\#A_{N/N^+} = 2^{t-1}(V_{N^+} : U_{N^+}^2)h_{N^+}.$$

Here U_{N^+} denotes the unit group of the ring of integers of N^+, and V_{N^+} its subgroup consisting of all units that are norms from N. Now, assume $h_{N^+}^+$ is odd. Since $h_{N^+}^+ = (U_{N^+}^+ : U_{N^+}^2)h_N$ and since this index is a power of 2, we get $h_{N^+} = h_{N^+}^+$, $U_{N^+}^+ = U_{N^+}^2$ and $V_{N^+} = U_{N^+}^2$, which yields $\#A_{N/N^+} = 2^{t-1}h_{N^+}$. It is also well known that the 2-rank of A_{N/N^+} and

Cl_N are the same when h_{N^+} is odd, and we find $2^{t-1} = \# A^{(2)}_{N/N^+} = 2^{r_2(N)}$ which is what we wanted to prove.

2. follows from the proof of Theorem 10.8 in [31].

3.(a): The norm map N from the ideal class group of N to that of N^+ is onto (point 2), hence its kernel has order h_N^-. The kernel of the canonical map j from the ideal class group of N^+ to that of N has order ≤ 2 ([31, Th. 10.3]) and $N \circ j$ sends an ideal class to its square. Therefore, $2^{r_2(N^+)-1}$ divides h_N^-, where $r_2(N^+)$ denotes the 2-rank of the ideal class group of N^+. Therefore, if h_N^- is odd then the 2-class group of N^+ is cyclic.
(b), (c) and (d) follow from the very definitions of these class fields, from the inclusions $N \subseteq NH_2(N^+) \subseteq NH_2^+(N^+) \subseteq H_2(N)$ and from the fact that h_N^- odd implies $NH_2(N^+) = H_2(N)$.

4. Adapt the statement and proof of [31, Th. 10.1]. $\square$

Proposition 2. *Let $N^+/\mathbb{Q}$ be an $\mathcal{A}_4$-extension, and assume that N/N^+ and M/N^+ are quadratic extensions such that* $\mathrm{Gal}(N/\mathbb{Q}) \simeq \mathrm{Gal}(M/\mathbb{Q}) \simeq \mathrm{SL}_2(\mathbb{F}_3)$*. Then the quadratic subextension M'/N^+ contained in MN/N^+ and different from M and N is normal over $\mathbb{Q}$ with* $\mathrm{Gal}(M'/\mathbb{Q}) \simeq \mathcal{A}_4 \times \mathcal{C}_2$.

Proof. Write $M = N^+(\sqrt{\mu})$ and $N = N^+(\sqrt{\nu})$ for some $\mu, \nu \in N^+$. Since $M/\mathbb{Q}$ and $N/\mathbb{Q}$ are normal, we know that $\mu^{1-\sigma} = \alpha_\sigma^2$ and $\nu^{1-\sigma} = \beta_\sigma^2$ are squares in N^+ for every $\sigma \in \mathrm{Gal}(N^+/\mathbb{Q})$. But then the $(\mu\nu)^{1-\sigma}$ are also squares, and it follows that $M'/\mathbb{Q}$ is normal. Let F denote the cubic subfield of N^+. Then $\mathrm{Gal}(M/F) \simeq \mathrm{Gal}(N/F) \simeq \mathcal{Q}_8$, hence $S(\mu, N^+/F) = S(\nu, N^+/F) = (-1,-1,-1)$ in the notation of Lemma 1 in [15] (the entries in $S(\mu, N^+/F)$ are the elements $\alpha_\sigma^{1+\sigma}$ as σ runs through the automorphisms $\neq 1$ of $\mathrm{Gal}(N^+/F)$). This implies $S(\mu\nu, N^+/F) = (+1,+1,+1)$, hence $\mathrm{Gal}(M'/F) \simeq \mathcal{C}_2 \times \mathcal{C}_2 \times \mathcal{C}_2$, and our claim follows. $\square$

We also recall a result on $\mathcal{A}_4$-extensions of local fields (see [11], Korollar 2.16):

Lemma 3. *Let p be a prime and assume that $K/\mathbb{Q}_p$ is a normal extension of the p-adic field $\mathbb{Q}_p$ with Galois group $\mathcal{A}_4$. Then $p = 2$, and K is the unique $\mathcal{A}_4$-extension of $\mathbb{Q}_2$; in particular, 2 is inert in the cyclic cubic subextension $F = \mathbb{Q}_2(\zeta_7)$ of K and totally ramified in K/F.*

Proof. The inertia subgroup T of $K/\mathbb{Q}_p$ is an invariant subgroup of $\mathcal{A}_4$ and hence must be 1, $\mathcal{V}$ or $\mathcal{A}_4$. Since unramified extensions are cyclic, we have $T \neq 1$. In the case of tame ramification T is cyclic, and since neither $\mathcal{V}$ nor $\mathcal{A}_4$ are cyclic, the ramification must be wild, and we have $p = 2$ or $p = 3$. If we had $p = 3$, then T/V_1 (where V_1 is the first ramification group) must be isomorphic to the 2-Sylow subgroup $\mathcal{V}$ of $\mathcal{A}_4$ contradicting the fact that T/V_1 must be cyclic by Hilbert's theory. The rest of the lemma is a well known result of Weil [32]; for different proofs see [11] and [27]. $\square$

1.4. **Normal CM-fields with Galois group $\mathcal{A}_4 \times \mathcal{C}_2$.**

The lattice of subfields is given in Subsection 1.1, Diagram 2. Let $N_i = F(\sqrt{\alpha_j}\,)$, $i = 1, 2, 3$, denote the three quadratic subextensions of N^+/F; if we write $k = \mathbb{Q}(\sqrt{-m}\,)$ for the complex quadratic subfield of N, then the three non-galois sextic CM-fields inside $N/\mathbb{Q}$ are $K_j = F(\sqrt{-m\alpha_j}\,)$. In the sequel, we fix one of the fields K_1, K_2, K_3 and denote it by K.

There is a simple criterion that allows us to decide whether $F(\sqrt{\alpha})$ is contained in an $\mathcal{A}_4$-extension of $\mathbb{Q}$:

Lemma 4. *Let F/E be a cyclic cubic extension, let σ generate* $\mathrm{Gal}(F/E)$, *and let $\alpha \in F \setminus F^2$. Then $F(\sqrt{\alpha}, \sqrt{\alpha^\sigma}\,)$ is an $\mathcal{A}_4$-extension of E if and only if $N_{F/E}\alpha = \alpha\alpha^\sigma\alpha^{\sigma^2}$ is a square in E.*

Proof. This is an easy exercise in Galois theory. □

Lemma 5. *Let p be an odd prime whose ramification index in the $\mathcal{A}_4$-extension $L/\mathbb{Q}$ is even. Then p splits in the cyclic cubic subfield F of L.*

Proof. Suppose not; then there is exactly on prime ideal $\mathfrak{p}$ in F above p, and since the ramification index of p is even, it must ramify in one of the three quadratic extensions of F. Since these fields are conjugate over $\mathbb{Q}$, the prime ideal $\mathfrak{p}$ ramifies in all three of them. This implies that $\mathfrak{p}$ ramifies completely in the $\mathcal{V}$-extension L/F, but this is impossible for odd p by Hilbert's theory of ramification. □

We next list some useful consequences of the fact that h_N^- is odd:

Lemma 6. *If h_N^- is odd, then so is the narrow class number h_F^+ of F. Moreover, each quadratic subextension of N^+/F is ramified at some prime ideal above each prime p that ramifies in $k/\mathbb{Q}$.*

Proof. Let N_j/F $(j = 1, 2, 3)$ denote the three quadratic subextensions of N^+/F, and let p denote any prime ramified in $k/\mathbb{Q}$. If F_1/F, say, is unramified at p, then so are N_j/F for $j = 2, 3$, since these extensions are conjugate over $\mathbb{Q}$. Thus N^+/F is unramified at p, hence each prime above p in N^+ ramifies in N/N^+. Since h_N^- is odd, p cannot split in $N^+/\mathbb{Q}$ by Proposition 1.1. But then the localization of $N^+/\mathbb{Q}$ at p is an $\mathcal{A}_4$-extension of $\mathbb{Q}_p$, hence $p = 2$ by Lemma 3, and 2 is inert in $F/\mathbb{Q}$ and totally ramified in N^+/F: contradiction. Thus each N_j/F is ramified at some prime dividing p.

Now if h_F^+ is even, then so is h_F by [1]. By Proposition 1.4, the class group of F contains $\mathcal{V} = \mathcal{C}_2 \times \mathcal{C}_2$ as a subgroup. Since, by what we just have proved, N^+/F is disjoint to the Hilbert 2-class field of F, the norm map $N : \mathrm{Cl}(N^+) \longrightarrow \mathrm{Cl}(F)$ is onto, hence $\mathrm{Cl}(N^+)$ has 2-rank at least 2. But now Proposition 1.3(a) shows that h_N^- must be even: contradiction. □

Lemma 7. *If h_N^- is odd, then N/N^+ is unramified outside* 2∞.

Proof. Proposition 1.1 shows that any prime ideal $\mathfrak{p}$ in N^+ that ramifies in N/N^+ lies above a prime p that does not split in $N^+/\mathbb{Q}$; thus the localization $N^+_{\mathfrak{p}}/\mathbb{Q}_p$ has degree 12, hence is an $\mathcal{A}_4$-extension of $\mathbb{Q}_p$. But Lemma 3 then gives $p = 2$. □

Lemma 8. *If h_N^- is odd, then $k/\mathbb{Q}$ ramifies at exactly one finite prime p_0, and this prime splits in $F/\mathbb{Q}$.*

Proof. Let p be an odd prime ramifying in $k/\mathbb{Q}$. We deduce from Lemma 7 that the ramification index e_+ of p in $N^+/\mathbb{Q}$ is even. Lemma 5 then shows that p splits in F.

Now let t denote the number of odd primes ramifying in $k/\mathbb{Q}$; as we have seen, these split in F, hence there are $3t$ primes ramified in A/F. By Proposition 1.1, we find that $2^{3t-1} \mid h_A^-$; from [24] we get that $h_A^- \mid 4h_N^-$, that is, $2^{3t-3} \mid h_N^-$. This implies $t \leq 1$.

Thus there are two possibilities: $t = 0$ (and then $k/\mathbb{Q}$ is ramified only at 2∞), or $t = 1$; in the last case, there can be no ramification in $k/\mathbb{Q}$ at 2, because otherwise Proposition 1.1 would show that $8 \mid h_A^-$ and $2 \mid h_N^-$, by [24] again.

It remains to show that, in the case $t = 0$, the prime 2 splits in $F/\mathbb{Q}$. To this end, we first show that the three non-normal sextic subfields N_j of N^+ are unramified outside 2. For suppose not; then there is an odd prime p which ramifies in N_j/F, and Lemma 5 shows that p splits in F. Hence K/F is ramified at at least four primes: but then $2 \mid h_N^-$ as above.

Now assume that 2 does not split in $F/\mathbb{Q}$ and write $N_1 = F(\sqrt{\alpha})$. We claim that N_1/F is unramified outside 2. Since N_1/F only ramifies at 2 and since 2 is inert in F (clearly 2 cannot ramify in cyclic cubic extensions), we have $(\alpha) = \mathfrak{a}^2$ or $(\alpha) = (2)\mathfrak{a}^2$ for some ideal $\mathfrak{a}$ of F. Let $\beta \in F$ be such that $(\beta) = \mathfrak{a}^{h_F}$. We get $\alpha^{h_F} = \epsilon\beta^2$ or $\alpha^{h_F} = 2\epsilon\beta^2$ (actually, this last case is impossible in view of Lemma 4) for some $\epsilon \in U_F$ which must be totally positive. This yields $\epsilon = \eta^2 \in U_F^2$ (see the proof of Lemma 6), and since h_F is odd, we get $N_1 = F(\sqrt{\alpha}) = F(\sqrt{\alpha^{h_F}}) = F$ and N_1 would be abelian, a desired contradiction. □

Now $\mathrm{Gal}(N/F) \simeq \mathcal{C}_2 \times \mathcal{C}_2 \times \mathcal{C}_2$, and we get

$$\zeta_N/\zeta_{N^+} = \prod_{i=0}^{3}(\zeta_{K_i}/\zeta_F) = (\zeta_A/\zeta_F)(\zeta_K/\zeta_F)^3$$

(use (3)); from the functional equation we get

$$d_N/d_{N^+} = \prod_{i=0}^{3}(d_{K_i}/d_F) = (d_A/d_F)(d_K/d_F)^3$$

and using (1) we find

$$\frac{h_N^-}{Q_N w_N} = \prod_{i=0}^{3} \frac{h_{K_i}^-}{Q_{K_i} w_{K_i}} = (h_A^-/Q_A w_A)(h_K^-/Q_K w_K)^3.$$

Since $w_K = 2$ (for F is clearly the maximal abelian subfield of K), $w_N = w_A$ and $Q_A = 1$ for $A/\mathbb{Q}$ is cyclic (see [14, point 5 page 352]), we get

$$h_N^- = Q_N h_A^- \left(h_K^-/2Q_K\right)^3.$$

We claim that $Q_K = 1$. In fact, since $w_K = 2$, we have $Q_K = 2$ if and only if we are in the situation (i).2.(a) of Theorem 1 in [14], i.e. if and only if $K = F(\sqrt{\varepsilon}\,)$ for some totally negative unit $\varepsilon \in U_F$. But since F has odd narrow class number (Lemma 6), every totally positive unit is a square, and we get $K = F(\sqrt{-1}\,)$ contradicting the fact that $K/\mathbb{Q}$ is non-abelian.

Therefore, we finally get

Lemma 9. *Let N be a normal CM-field of degree 24 with Galois group $\mathcal{A}_4 \times \mathcal{C}_2$. If h_N^- is odd, then*

$$h_N^- = Q_N h_A^- \left(h_K^-/2\right)^3. \tag{4}$$

Now we are ready to prove

Theorem 10. *Let N be a normal CM-field of degree 24 with Galois group $\mathcal{A}_4 \times \mathcal{C}_2$. If h_N^- is odd then exactly one rational prime p_0 is ramified in the imaginary quadratic field k, $(p_0) = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$ splits in F, $k = \mathbb{Q}(\sqrt{-p_0})$, $h_A^- \equiv 4 \pmod 8$, h_K^- is odd, $Q_N = 2$ and we may choose notation such that $\mathfrak{p}_i$ is the only prime ideal of F which is ramified in K_i/F for $1 \le i \le 3$.*

Proof. By Lemma 8 there is exactly one prime p_0 ramified in k, and p_0 splits in F. Thus at least three finite primes split in A/F, and Proposition 1.1 gives $4 \mid h_A^-$. From [24] we know that h_A^- divides $4h_N^-$, hence we must have $h_A^- \equiv 4 \pmod 8$. Now (4) implies that h_K^- odd and that $Q_N = 2$.

Since h_K^- is odd, there is at most one finite prime ramified in K^-/F by Proposition 1.1, and since F has odd narrow class number by Lemma 6, it must ramify at some finite prime. Since the fields K_i $(i = 1, 2, 3)$ are conjugate over $\mathbb{Q}$, the last claim follows.

It remains to show that $k = \mathbb{Q}(\sqrt{-p_0}\,)$. Since p_0 is the only ramified prime in $k/\mathbb{Q}$, this is clear if p_0 is odd (we can even conclude that $p_0 \equiv 3 \bmod 4$ in this case). If $p_0 = 2$, however, we have to exclude the possibility that $k = \mathbb{Q}(\sqrt{-1})$. This is done as follows: as in the proof of Lemma 8, we write $N_1 = F(\sqrt{\alpha}\,)$ for some totally positive $\alpha \in \mathcal{O}_F$. From Lemma 4 we know that $N\alpha$ is a square; suppose that α is an ideal square. Then $\alpha = \varepsilon\beta^2$ for some unit ε (since F has odd class number), ε is totally positive (since α is), hence ε must be a square (since F has odd narrow class number): contradiction.

Thus there is a prime ideal that divides α to an odd power, and since $N\alpha$ is a square, there are in fact two such prime ideals (necessarily different by what we just have proved), say $\mathfrak{p}$ and $\mathfrak{p}'$. Write $k = \mathbb{Q}(\sqrt{-m}\,)$; then $K_1 = F(\sqrt{-m\alpha}\,)$, and if we had $k = \mathbb{Q}(\sqrt{-1}\,)$, then $K_1 = F(\sqrt{-\alpha}\,)$. But α is divisible by two prime ideals $\mathfrak{p} \neq \mathfrak{p}'$ (to an odd power), hence both $\mathfrak{p}$ and $\mathfrak{p}'$ ramify in K_1/F: contradiction. □

1.5. **Normal CM-fields with Galois group** $\mathrm{SL}_2(\mathbb{F}_3)$.

The lattice of subfields is given in subsection 1.1, Diagram 1. Our first theorem shows that CM-fields $N/\mathbb{Q}$ with Galois group $\mathrm{SL}_2(\mathbb{F}_3)$ and odd relative class number h_N^- arise as narrow Hilbert 2-class fields of their cubic subfields F:

Theorem 11. *Let N be a CM-field with* $\mathrm{Gal}(N/\mathbb{Q}) \simeq \mathrm{SL}_2(\mathbb{F}_3)$*; let N^+ denote its maximal real subfield and F its cyclic cubic subfield. If h_N^- is odd, then $h_F \equiv h_F^+ \equiv 4 \bmod 8$, and h_{N^+} is odd. Moreover, N^+ is the Hilbert 2-class field of F, $h^+(N^+) \equiv 2 \bmod 4$, and N is the narrow Hilbert 2-class field of N^+.*

Conversely, if $F/\mathbb{Q}$ is a cyclic cubic extension with $h_F \equiv h_F^+ \equiv 4 \bmod 8$, let N^+ denote the 2-class field of F. If h_{N^+} is odd, then there exists a normal CM-field N with $\mathrm{Gal}(N/\mathbb{Q}) \simeq \mathrm{SL}_2(\mathbb{F}_3)$ *containing N^+ and with odd class number h_N; in particular, h_N^- is odd.*

Proof. Note that $h_F \equiv h_F^+ \equiv 4 \bmod 8$ if and only if $\mathrm{Cl}_2(F) \simeq \mathrm{Cl}_2^+(F) \simeq \mathcal{C}_2 \times \mathcal{C}_2$ by Proposition 1.4.

Assume that h_N^- is odd. We first claim that N/N^+ is unramified at all finite primes. If at least two finite primes ramify in N/N^+, then h_N^- is even by Proposition 1.1. If exactly one finite prime $\mathcal{P}$ of N is ramified in N/N^+, then any automorphism of $N/\mathbb{Q}$ must leave $\mathcal{P}$ fixed (that is, we have $e_{\mathcal{P}} f_{\mathcal{P}} = 24$). Thus the localization $N_{\mathcal{P}}$ is an $\mathrm{SL}_2(\mathbb{F}_3)$-extension of $\mathbb{Q}_p$; hence it contains an $\mathcal{A}_4$-extension of $\mathbb{Q}_p$, and Lemma 3 shows that $p = 2$. But Weil [32] (see also [11] and [27]) has shown that $\mathbb{Q}_2$ does not admit any $\mathrm{SL}_2(\mathbb{F}_3)$-extension.

Thus N/N^+ is unramified at all finite primes. We claim next that this implies that N^+/F is unramified at all finite primes. In fact, let $\mathcal{P}$ denote a prime in N and let N_T denote its inertia subfield in the extension N/F. Then either $N_T = N$ (and $\mathcal{P}$ is unramified in N/F), or $N_T \subseteq N^+$ (since each subfield $\neq N$ of N/F is contained in N^+). But the second possibility cannot occur since N/N^+ is unramified at all finite primes.

Hence N^+/F is an unramified abelian $\mathcal{V}$-extension, and this implies that the class number h_F is divisible by 4. We claim next that h_{N^+} is odd; this will imply that N^+ is the Hilbert 2-class field of F and hence that $\mathrm{Cl}_2(F) \simeq \mathcal{V}$. Assume therefore that h_{N^+} is even; since $\mathrm{Cl}_2(N^+)$ is cyclic by Proposition 1.3.a), there exists a unique quadratic extension M/N^+ that is unramified everywhere. Since N is ramified at ∞, we conclude that $M \neq N$. Now M is normal over $\mathbb{Q}$, since any $\mathbb{Q}$-automorphism fixes N^+ and maps M to a quadratic extension of N^+ that is unramified everywhere, i.e. to M itself.

Now either $\mathrm{Gal}(M/\mathbb{Q}) \simeq \mathcal{A}_4 \times \mathcal{C}_2$, or $\mathrm{Gal}(M/\mathbb{Q}) \simeq \mathrm{SL}_2(\mathbb{F}_3)$; in the latter case, Proposition 2 shows that the third quadratic extension M'/N^+ contained in MN/N^+ has Galois group $\mathcal{A}_4 \times \mathcal{C}_2$; moreover, since M/N^+ and N/N^+ are unramified at the finite primes, so is M'/N^+. Let $\widetilde{M}$ denote this $\mathcal{A}_4 \times \mathcal{C}_2$-extension, and let p be a prime ramifying in the quadratic subfield k of $\widetilde{M}/\mathbb{Q}$: since N^+/F is unramified, p must ramify in $\widetilde{M}/N^+$, and this yields the desired contradiction.

Finally we know from [2, Thm. 2.5] or [5] that $h^+_{N^+}$ is always even. But the factor group $\mathrm{Cl}^+(N^+)/\mathrm{Cl}(N^+)$ is elementary abelian, and Proposition 3.3.(c) and (d) show that either $h^+_{N^+} = 2h_{N^+}$ or that $\mathrm{Cl}^+_2(N^+)$ is cyclic; in both cases, we conclude that $h^+_{N^+} \equiv 2 \bmod 4$. This implies that N is the narrow Hilbert 2-class field of N^+ as claimed. Moreover, we find that the narrow 2-class number of F divides $2 \cdot (N^+ : F) = 8$, and Proposition 3.4 now shows that $\mathrm{Cl}^+_2(F) \simeq \mathcal{V}$.

For a proof of the converse we notice that $h^+_{N^+}$ is even by the results of [2] and [5]. From [6] or [29] we know that the 2-class group of N^+ is cyclic; in particular, there is a unique quadratic extension N/N^+ that is unramified at the finite primes. By a standard argument, uniqueness implies that $N/\mathbb{Q}$ is normal, and by group theory the Galois group of this extension is $\mathcal{A}_4 \times \mathcal{C}_2$ or $\mathrm{SL}_2(\mathbb{F}_3)$. The first case cannot occur, however, since N/F would be abelian then; thus $\mathrm{Gal}(N/\mathbb{Q}) \simeq \mathrm{SL}_2(\mathbb{F}_3)$. This gives $\mathrm{Gal}(N/F) \simeq \mathcal{Q}$, and now [10, Hilfssatz 2] shows that the class number of N is odd. □

Since N^+/F is unramified we must have $d_{K^+} = d_F = f^2$ (see [12]) and $d_K = d^2_{K^+} = f^4$ (since N/F is unramified at all the finite places, the index of ramification in $N/\mathbb{Q}$ of any prime ideal of N is equal to 1 or 3, hence is odd. Therefore, the quadratic extension K/K^+ is unramified at all the finite places). Finally, we have:

Lemma 12. *Let K be one of the four non-normal octic CM-fields of N. If $h^-_N = 1$ then $h^-_K = 1$.*

Proof. The degree $(N : K)$ is odd, so $h^-_K \mid h^-_N$ (see e.g. [19]). □

This will help us considerably since h^-_K is much easier to compute than h^-_N. Our strategy is first to find an upper bound on f when $h^-_N = 1$ (see Theorem 16), second to determine all the non normal totally real quartic fields K^+ with normal closure of degree 12 and Galois group $\mathcal{A}_4$ and such that $d_{K^+} = d_F = f^2$ and such that f is less than or equal to this bound,

Table 1

D	f	α_F	$P_K(X)$	d_K	h_K
-8	31	$3\alpha^2+4\alpha-5$	$X^6+52X^4+457X^2+2$	$-2^3\cdot 31^4$	**1**
-3	61	$(\alpha^2+4\alpha)/3$	$X^6+15X^4+14X^2+3$	$-61^4\cdot 3$	**1**
-3	67	$3\alpha^2-8\alpha-2$	$X^6+121X^4+3250X^2+3$	$-2^6\cdot 67^4\cdot 3$	22
-7	19	$\alpha^2-\alpha$	$X^6+12X^4+29X^2+7$	$-2^6\cdot 19^4\cdot 7$	4
-7	73	$-\alpha+5$	$X^6+14X^4+41X^2+7$	$-73^4\cdot 7$	**3**
-11	19	α^2-1	$X^6+10X^4+27X^2+11$	$-2^6\cdot 19^4\cdot 11$	6
-11	43	$2\alpha^2+6\alpha+3$	$X^6+73X^4+171X^2+11$	$-43^4\cdot 11$	**3**
-19	9	$-\alpha+3$	$X^6+9X^4+24X^2+19$	$-2^6\cdot 3^8\cdot 19$	2

third to compute h_K^- for each possible K^+, and fourth to compute h_N^- (where $N = KF$) for the few K's for which $h_K^- = 1$.

2. Normal CM-fields with Galois group $\mathcal{A}_4 \times \mathcal{C}_2$ and relative class number one.

The following result is proved in [26]:

Theorem 13. *Let A be an imaginary cyclic sextic field. Then A is the compositum of an imaginary quadratic number field $k = \mathbb{Q}(\sqrt{D}\,)$ and a real cyclic cubic field F whose conductor we denote by f. Assume that exactly one rational prime p_0 is ramified in the quadratic extension $k/\mathbb{Q}$ and that p_0 splits in F. Then, $h_A^- = 4$ if and only if we are in one of the following ten cases:*

$D =$	-4	-4	-8	-3	-3	-7	-7	-11	-11	-19
$f =$	31	43	31	61	67	19	73	19	43	9

In all these cases we have $h_F^+ = 1$.

Gauss has shown how to find a generating polynomial for cyclic cubic fields with prime power conductor (see e.g. [3, Theorem 6.4.6 and Corollary 6.4.12]); the results are $(f, P_F(X)) = (19, X^3-X^2-6X+7)$, $(31, X^3-X^2-10X+8)$, $(43, X^3-X^2-14X-8)$, $(61, X^3-X^2-20X+9)$, $(67, X^3-X^2-22X-5)$, $(73, X^3-X^2-24X+27)$ and $(9, X^3-3X+1)$.

The two cases with $D = -4$ are excluded by Theorem 10; for each of the other eight possibilities for A we now find a totally positive $\alpha_F \in \mathcal{O}_F$ such that $N_{F/\mathbb{Q}}(\alpha_F) = p_0$ (it follows from the proof of Theorem 10 that such an element always exists). We then have $K = F(\sqrt{-\alpha_F}\,)$. According to Table 1, there are only two normal CM-fields of degree 24 with Galois group $\mathcal{A}_4 \times \mathcal{C}_2$ and relative class number one.

Note that we have $h_K^- = h_K$ in all cases. Moreover, it would have been sufficient to compute only the four class numbers written in bold letters since we know that α_K must be primary, i.e., congruent to the square of an element of $\mathcal{O}_F$ (the ring of algebraic integers of F) modulo the ideal $4\mathcal{O}_F$. However, the multiplicative group $(\mathcal{O}_F/4\mathcal{O}_F)^*$ is isomorphic either to $(\mathbb{Z}/2\mathbb{Z})^3)$ or to $(\mathbb{Z}/7\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})^3)$, according as the prime 2 splits in F or is inert in F. In particular, $g \in (\mathcal{O}_F/4\mathcal{O}_F)^*$ is a square in this group if and only if $g^7 = 1$ in this group, i.e., $-\alpha_K$ is primary if and only if $\alpha_K^7 + 1$ is in $4\mathcal{O}_F$. The reader will easily check that α_K is primary if $(D, f) = (-8, 31)$, $(-3, 61)$, $(-7, 73)$ and $(-11, 43)$.

Theorem 14. *There are exactly two normal CM-fields with Galois groups isomorphic to $\mathcal{A}_4 \times \mathcal{C}_2$ and relative class number one. Moreover, both fields have class number one.*

Proof. Set $\gamma_+ = \sqrt{p_0/\alpha_F} + \sqrt{p_0/\sigma(\alpha_F)} + \sqrt{p_0/\sigma^2(\alpha_F)}$. We have $N^+ = F(\gamma_+)$ from which Pari GP [25] readily yields $d_{N^+} = 2^{18} \cdot 31^8$ and $h_{N^+} = 1$ if $D = -8$ and $f = 31$, and $d_{N^+} = 3^6 \cdot 61^8$ and $h_{N^+} = 1$ if $D = -3$ and $f = 61$. $\square$

3. Normal CM-fields with Galois group $\mathrm{SL}_2(\mathbb{F}_3)$ and class number one.

From now on we assume that N is a normal CM-field of degree 24 with Galois group isomorphic to $\mathrm{SL}_2(\mathbb{F}_3)$. We give lower bounds on its relative class number.

3.1. Factorizations of some Dedekind zeta functions.

Let $j = \exp(2\pi i/3) = (-1 + \sqrt{-3})/2$ denote a cube root of unity; with the notation of subsection 1.1, Table 2 gives the irreducible characters of $\mathcal{A}_4$ (see [8, page 181]):

Table 2

	Cl_1	Cl_2	Cl_3	Cl_4
χ_1	1	1	1	1
χ_2	1	j	j^2	1
χ_3	1	j^2	j	1
χ_4	3	0	0	-1

Since $\chi_4 = \phi^*$ is induced by a linear character ϕ of the abelian subgroup $\mathcal{V}$ of $\mathcal{A}_4$, Artin's L function $s \mapsto L(s, \chi_4, N^+/\mathbb{Q})$ is entire and

$$\begin{aligned}\zeta_{N^+}(s) &= L(s,\chi_4,N^+/\mathbb{Q})^3 \prod_{i=1}^{3} L(s,\chi_i,N^+/\mathbb{Q}) \\ &= L(s,\chi_4,N^+/\mathbb{Q})^3 \zeta_F(s) = \zeta(s)L(s,\chi)L(s,\bar{\chi})L(s,\chi_4,N^+/\mathbb{Q})^3\end{aligned}$$

where χ is one of the two conjugate Dirichlet characters associated to the cyclic cubic field F. In particular, $s \in]0,1[$ implies $\zeta_F(s) = \zeta(s)L(s,\chi)L(s,\bar{\chi}) = \zeta(s)|L(s,\chi)|^2 \le 0$. Hence, if $\zeta_{N^+}(s_0) > 0$ for some $s_0 \in]0,1[$ then the entire function $s \mapsto L(s,\chi_4,N^+/\mathbb{Q})$ has a real zero on $]s_0,1[$ and ζ_{N^+} has at least three real zeros on $]s_0,1[$.

The special linear group $\mathrm{SL}_2(\mathbb{F}_3)$ has order 24, contains seven conjugacy classes, two of them (namely $Cl_1 = \{\mathrm{Id}\}$ and $Cl_2 = \{-\mathrm{Id}\}$) consisting of only one element. Next $Cl_1 \cup Cl_2 = \{g_1, g_2\} = Z(\mathrm{SL}_2(\mathbb{F}_3))$, and Table 3 gives the irreducible characters of $\mathrm{SL}_2(\mathbb{F}_3)$ (see [8, pages 403–404, solution to exercise 27.1]).

Table 3

	Cl_1	Cl_2	Cl_3	Cl_4	Cl_5	Cl_6	Cl_7
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	j	j^2	j^2	j
χ_3	1	1	1	j^2	j	j	j^2
χ_4	3	3	-1	0	0	0	0
χ_5	2	-2	0	-1	-1	1	1
χ_6	2	-2	0	$-j$	$-j^2$	j^2	j
χ_7	2	-2	0	$-j^2$	$-j$	j	j^2

According to (2) and Table 3, if N is a normal CM-field of degree 24 and Galois group $G = \mathrm{SL}_2(\mathbb{F}_3)$, then

$$(\zeta_N/\zeta_{N^+})(s) = \left(\prod_{i=5}^{7} L(s,\chi_i,N/\mathbb{Q})\right)^2, \quad s \in]0,1[. \tag{5}$$

3.2. **Lower bounds on residues.**

Lemma 15. 1. *If the absolute value of the discriminant of a number field M other than $\mathbb{Q}$ satisfies*

$$d_M > e^{2(\sqrt{m+1}-1)},$$

then the Dedekind zeta function $s \mapsto \zeta_M(s)$ *of* M *has at most* m *real zeros in the range*

$$s_m = 1 - \frac{2(\sqrt{m+1}-1)^2}{\log d_M} \leq s < 1.$$

In particular, ζ_M *has at most two real zeros in the range* $1-(1/\log d_M) \leq s < 1$.

2. *Let* M *be a normal field of degree* 12 *with Galois group* $\mathcal{A}_4$. *Then,* $s \in]0,1[$ *and* $\zeta_M(s) > 0$ *implies that* ζ_M *has at least three real zeros in the range* $]s,1[$. *In particular,* $1-(1/\log d_M) \leq s < 1$ *implies* $\zeta_M(s) \leq 0$.
3. *Let* N *be a normal CM-field of degree* 24 *with Galois group isomorphic to* $\mathrm{SL}_2(\mathbb{F}_3)$. *Set*

$$\epsilon_N = 1 - \frac{24\pi e^{1/12}}{d_N^{1/24}} \geq 1 - \frac{24\pi e^{1/12}}{f^{2/3}} = \epsilon_f. \tag{6}$$

Then, we have $\zeta_N(1-(2/\log d_N)) \leq 0$ *and*

$$\mathrm{Res}_{s=1}(\zeta_N) \geq \epsilon_N \frac{2}{e \log d_N}.$$

Proof. The first part of point 2 was proved in the previous subsection. The second part of point 2 follows from point 1. The first part of Point 3 follows from point 2 and (5), and the second part of point 2 follows from [20]. Let us now prove point 1. Assume ζ_M has at least $m+1$ real zeros in the range $[s_m, 1[$. According to the proof of [28, Lemma 3] for any $s > 1$ we have

$$f(s) = \frac{m+1}{s-s_m} - \frac{1}{s-1} \leq \left(\sum_{\rho \in]0,1[} \frac{1}{s-\rho}\right) - \frac{1}{s-1} < \frac{1}{2}\log d_M + h(s)$$

where

$$h(s) = \left(\frac{1}{s} - \frac{n}{2}\log \pi\right) + \frac{r_1}{2}\frac{\Gamma'}{\Gamma}(s/2) + r_2\left(\frac{\Gamma'}{\Gamma}(s) - \log 2\right)$$

and where ρ ranges over all the real zeros in $]0,1[$ of ζ_M. Setting

$$t_1 = \frac{\sqrt{m+1}-s_m}{\sqrt{m+1}-1} = 1 + \frac{1-s_m}{\sqrt{m+1}-1} = 1 + \frac{2(\sqrt{m+1}-1)}{\log d_M} < 2,$$

we get

$$f(t_1) = \frac{\sqrt{m+1}-1}{t_1-1} = \frac{(\sqrt{m+1}-1)^2}{1-s_m} = \frac{1}{2}\log d_M < \frac{1}{2}\log d_M + h(t_1),$$

and since $h(t_1) < h(2) < 0$, we have a contradiction. Indeed, let $\gamma = 0.577\cdots$ denote Euler's constant. Since $h'(s) > 0$ for $s > 0$ we do have

$$\begin{aligned} h(t_1) < h(2) &= \frac{1-n(\gamma+\log\pi)}{2} + r_2(1-\log 2) \\ &\leq \frac{1-n(\gamma+\log\pi-1+\log 2)}{2} \;<\; 0. \end{aligned}$$

□

3.3. **Lower bounds on h_N^-.**.

From now on we assume that $h_N = 1$. By Theorem 11, we see that N/F is unramified at all the finite places, therefore $d_N = d_{N^+}^2 = f^{16}$, $\epsilon_N = \epsilon_f$, and according to Lemma 15.3 we have

$$\mathrm{Res}_{s=1}(\zeta_N) \geq \epsilon_N \frac{2}{e\log d_N} \geq \epsilon_N \frac{1}{8e\log f}$$

and

$$h_N^- = \frac{Q_N w_N}{(2\pi)^{12}}\sqrt{\frac{d_N}{d_{N^+}}}\frac{\mathrm{Res}_{s=1}(\zeta_N)}{\mathrm{Res}_{s=1}(\zeta_{N^+})} \geq \epsilon_f \frac{f^4/\log f}{2e(2\pi)^{12}\mathrm{Res}_{s=1}(\zeta_{N^+})}. \tag{7}$$

The field N^+ is totally real of degree 12 and unramified over its cyclic cubic subfield F of conductor f; therefore we have $d_{N^+} = d_F^4 = f^8$. Since N^+/F is an unramified abelian quartic extension, we have (see [21] or [23]):

$$\mathrm{Res}_{s=1}(\zeta_{N^+}) \leq \frac{1}{2^{11}}(\log f + 0.05)^{11}. \tag{8}$$

Combining (7) and (8) gives the following Theorem:

Theorem 16. *Let N be a normal CM-field of degree 24 with $\mathrm{Gal}(N/\mathbb{Q})$ isomorphic to $\mathrm{SL}_2(\mathbb{F}_3)$. If h_N^- is odd then*

$$h_N^- \geq \epsilon_N \frac{f^4}{4e\left(\pi(\log f + 0.05)\right)^{12}}. \tag{9}$$

In particular, $h_N^- = 1$ implies $f \leq 83000$.

We remark that the bound of f obtained through (8) is about ten times smaller than the bound $f < 10^6$ obtained from general upper bounds on residues of Dedekind zeta functions.

According to numerical computations of bounds on $\mathrm{Res}_{s=1}(\zeta_{N^+})$ for all the F's of prime conductors $f \leq 83000$ and according to (7) we proved in [23]:

Proposition 17. *(See [23]). Let N be a normal CM-field of degree 24 with Galois group isomorphic to $\mathrm{SL}_2(\mathbb{F}_3)$. Assume that the class number of N is 1. Then,*

1. *The class number h_F and narrow class number h_F^+ of F are equal to* 4, *which implies that the conductor f of F is a prime congruent to* $1 \bmod 6$.
2. N^+ *is the narrow Hilbert* 2*-class field of F, $h_{N^+}^+$ is* 2, *and N is the second narrow Hilbert* 2*-class field of F.*
3. *Finally, f is one of the following* 23 *prime values:* $f = 163$, 277, 349, 397, 547, 607, 709, 853, 937, 1399, 1789, 2131, 2689, 2803, 3271, 4567, 5197, 6079, 8011, 10957, 11149, 14407 *or* 18307.

3.4. **The determination.**

Our first job is the construction of the 2-class fields of these 23 cubic fields. To this end, we first construct a quadratic unramified extension of F and then use the formulas in [12] to compute the quartic polynomial $P_{K^+}(X)$ with integer coefficients whose roots generate N^+.

The construction of the quadratic unramified extension of F is straightforward from the theoretical point of view; we did our computations using PARI, and since the latest versions of KANT and PARI (which we did not yet have at our disposal) allow the direct computation of Hilbert class fields, we may skip the details here. Table 4 gives the results of our computations, including the class number of a quartic field K^+ generated by a root of $P_{K^+}(X)$.

We can check that the fields with $h_{K^+} = 2$ (these class numbers were computed using PARI) in this table really have even class number by explicitly constructing the unramified quadratic extension. In each case where the class number of K^+ in the wide sense is odd we now compute the quadratic extension K/K^+ which is unramified at all finite primes (since N/N^+ is unramified outside ∞ and N^+/K^+ is cyclic of degree 3, K/K^+ must also be unramified at all finite primes) and give an octic polynomial $P_K(x)$ such that K is generated by a (suitable) root of $P_K(x)$. Note that the fact that this construction works implies that the class number of K^+ is odd – again by [2]. Finally we compute the class number of K; the results are collected in Table 5.

This leaves us with the problem of computing the relative class numbers of the fields N for $f = 163$, 349, 397, 853 and 937. Florian Hess (Berlin) first checked that $h_K = 1$ for these fields using KASH (see [9]) and then computed $\mathrm{Cl}(N)$:

f	163	349	397	853	937
Cl(N)	1	$(3,3)$	$(3,3)$	$(11,11)$	$(19,19)$

The computation for $f = 163$ holds under the assumption of the validity of GRH, and for the other conductors the groups given in this table are subgroups of Cl(N) without any assumption. It is, however, reasonable to assume that these subgroups are actually the whole class group.

TABLE 4

f	$P_{K^+}(X)$	h_{K^+}
163	$x^4 - x^3 - 7x^2 + 2x + 9$	1
277	$x^4 - x^3 - 11x^2 + 4x + 12$	2
349	$x^4 - x^3 - 10x^2 + 3x + 20$	1
397	$x^4 - 13x^2 - 2x + 19$	1
547	$x^4 - 2x^3 - 19x^2 + 11x + 10$	1
607	$x^4 - 2x^3 - 13x^2 + 7x + 33$	2
709	$x^4 - 17x^2 - 13x + 35$	1
853	$x^4 - x^3 - 28x^2 + 31x - 2$	1
937	$x^4 - x^3 - 16x^2 + 11x + 54$	1
1399	$x^4 - x^3 - 23x^2 + 18x + 68$	2
1789	$x^4 - 31x^2 - 44x + 69$	2
2131	$x^4 - x^3 - 25x^2 + 10x + 123$	1
2689	$x^4 - x^3 - 65x^2 + 256x - 192$	2
2803	$x^4 - x^3 - 52x^2 + 17x + 4$	1
3271	$x^4 - x^3 - 51x^2 + 36x + 592$	2
4567	$x^4 - x^3 - 76x^2 + 273x - 169$	1
5197	$x^4 - x^3 - 46x^2 + 55x + 243$	2
6079	$x^4 - 2x^3 - 81x^2 - 38x + 992$	1
8011	$x^4 - x^3 - 127x^2 + 148x + 3292$	1
10957	$x^4 - x^3 - 148x^2 + 13x + 4563$	1
11149	$x^4 - 2x^3 - 121x^2 - 326x - 128$	1
14407	$x^4 - x^3 - 141x^2 - 220x + 2000$	2
18307	$x^4 - x^3 - 174x^2 - 373x + 2147$	1

Theorem 18. *There is at most one CM-field N with* $\mathrm{Gal}(N/\mathbb{Q}) \simeq \mathrm{SL}_2(\mathbb{F}_3)$ *and class number* 1, *namely the maximal solvable extension of the cubic field of conductor* 163 *that is unramified at all finite primes. It is the splitting field of the polynomial* $x^8 + 9x^6 + 23x^4 + 14x^2 + 1$. *If GRH holds then N has class number* 1.

Jürgen Klüners has called our attention to the fact that the field in Theorem 18 is the one with minimal discriminant and Galois group $\mathrm{SL}_2(\mathbb{F}_3)$ (see [4]).

ACKNOWLEDGEMENT

The authors thank Florian Hess for computing the class groups of the fields of degree 24 and the referee for his comments.

TABLE 5

f	$P_K(x)$	h_K
163	$x^8+9x^6+23x^4+14x^2+1$	1
349	$x^8+18x^6+75x^4+85x^2+1$	1
397	$x^8+150x^6+135x^4+22x^2+1$	1
547	$x^8+1057x^6+1739x^4+554x^2+1$	3
709	$x^8+125x^6+215x^4+42x^2+1$	7
853	$x^8+93x^6+1755x^4+3546x^2+1$	1
937	$x^8+1486x^6+341591x^4+13077x^2+1$	1
2131	$x^8+557x^6+31119x^4+4262x^2+25$	13
2803	$x^8+2026x^6+28103x^4+7001x^2+1$	3
4567	$x^8+7971x^6+4253156x^4+44743x^2+1$	3
6079	$x^8+69x^6+1155x^4+807x^2+16$	$13\cdot 19$
8011	$x^8+140x^6+4070x^4+5260x^2+1$	13
10957	$x^8+1786x^6+795291x^4+15178x^2+1$	7
11149	$x^8+2505801756x^6+140624201977542x^4$	
	$+474240540x^2+1$	7
18307	$x^8+534x^6+15015x^4+6582x^2+1$	$3\cdot 7$

Added in proof. Meanwhile, the second author succeeded in removing the GRH condition from the result of Theorem 18. See his articles [*Computation of relative class numbers of CM-fields by using Hecke L-functions*, Math. Comp. **69** (2000), 371–393] and [*Computation of $L(0,X)$ and of relative class numbers of CM-fields*, preprint Univ. Caen (1998)].

REFERENCES

[1] J.V. Armitage, A. Fröhlich, Classnumbers and unit signatures. *Mathematika* **14** (1967), 94–98.

[2] C. Bachoc, S.-H. Kwon, Sur les extensions de groupe de Galois $\tilde{\mathcal{A}}_4$. *Acta Arith.* **62** (1992), 1–10.

[3] H. Cohen *A course in computational algebraic number theory.* Grad. Texts Math. **138**, Springer-Verlag, Berlin, Heidelberg, New York (1993).

[4] H. Cohen, F. Diaz y Diaz, M. Olivier, Tables of octic fields with a quartic subfield. Math. Comp., à paraître.

[5] A. Derhem, Capitulation dans les extensions quadratiques non ramifiées de corps de nombres cubiques cycliques. thèse, Université Laval, Québec (1988).

[6] Ph. Furtwängler, Über das Verhalten der Ideale des Grundkörpers im Klassenkörper. *Monatsh. Math. Phys.* **27** (1916), 1–15.

[7] H. Heilbronn, Zeta-functions and L-functions. *Algebraic Number Theory, Chapter VIII, J.W.S. Cassels and A. Fröhlich, Academic Press,* (1967).

[8] G. James, M. Liebeck, *Representations and Characters of groups. Cambridge University Press,* (1993).

[9] KANT V4, by M. Daberkow, C. Fieker, J. Klüners, M. Pohst, K. Roegner, M. Schörnig, and K. Wildanger, *J. Symbolic Computation* **24** (1997), 267–283.

[10] H. Koch, Über den 2-Klassenkörperturm eines quadratischen Zahlkörpers. I, J. Reine Angew. Math. **214/215** (1963), 201–206.

[11] P. Kolvenbach, Zur arithmetischen Theorie der $SL(2,3)$-Erweiterungen. Diss. Köln, 1982.
[12] S.-H. Kwon, Corps de nombres de degré 4 de type alterné. *C. R. Acad. Sci. Paris* **299** (1984), 41–43.
[13] S. Lang, *Cyclotomic Fields II.* Springer-Verlag 1980
[14] F. Lemmermeyer, Ideal class groups of cyclotomic number fields I. *Acta Arith.* **72** (1995), 347–359.
[15] F. Lemmermeyer, Unramified quaternion extensions of quadratic number fields. *J. Théor. N. Bordeaux* **9** (1997), 51–68.
[16] Y. Lefeuvre, Corps diédraux à multiplication complexe principaux. *Ann. Inst. Fourier*, à paraître.
[17] S. Louboutin, R. Okazaki, Determination of all non-normal quartic CM-fields and of all non-abelian normal octic CM-fields with class number one. *Acta Arith.* **67** (1994), 47–62.
[18] S. Louboutin, R. Okazaki, The class number one problem for some non-abelian normal CM-fields of 2-power degrees. *Proc. London Math. Soc.* **76**, No.3 (1998), 523–548.
[19] S. Louboutin, R. Okazaki, M. Olivier, The class number one problem for some non-abelian normal CM-fields. *Trans. Amer. Math. Soc.* **349** (1997), 3657–3678.
[20] S. Louboutin, Lower bounds for relative class numbers of CM-fields. *Proc. Amer. Math. Soc.* **120** (1994), 425–434.
[21] S. Louboutin, Majoration du résidu au point 1 des fonctions zêta de certains corps de nombres. *J. Math. Soc. Japan* **50** (1998), 57–69.
[22] S. Louboutin, The class number one problem for the non-abelian normal CM-fields of degree 16. *Acta Arith.* **82** (1997), 173–196.
[23] S. Louboutin, Upper bounds on $|L(1,\chi)|$ and applications. *Canad. J. Math.*, 50 (4), (1998), 794–815.
[24] R. Okazaki, Inclusion of CM-fields and divisibility of class numbers. *Acta Arith.*, à paraître.
[25] User's Guide to PARI-GP, by C. Batut, K. Belabas, D. Bernardi, H. Cohen and M. Olivier, last updated Nov. 14, 1997; Laboratoire A2X, Univ. Bordeaux I.
[26] Y.-H. Park, S.-H. Kwon, Determination of all imaginary abelian sextic number fields with class number ≤ 11. *Acta Arith.* **82** (1997), 27–43 .
[27] A. Rio, Dyadic exercises for octahedral extensions. preprint 1997;
URL: `http://www-ma2.upc.es/~rio/papers.html`
[28] H.M. Stark, Some effective cases of the Brauer-Siegel theorem. *Invent. Math.* **23** (1974), 135–152.
[29] O. Taussky, A remark on the class field tower. J. London Math. Soc. **12** (1937), 82–85.
[30] A. D. Thomas, G. V. Wood, *Group Tables*, Shiva Publishing Ltd, Kent, UK 1980.
[31] L.C. Washington, *Introduction to Cyclotomic Fields.* Grad. Texts Math. **83**, Springer-Verlag 1982; 2nd edition 1997.
[32] A. Weil, Exercices dyadiques. *Invent. Math.* **27** (1974), 1–22.

Franz LEMMERMEYER
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn / Germany
E-mail : `lemmerm@mpim-bonn.mpg.de`

Stéphane LOUBOUTIN
Université de Caen
Dept de Mathematique et Mecanique
BP 5186
14032 Caen Cedex, France
E-mail : `loubouti@math.unicaen.fr`

Ryotaro OKAZAKI
Department of Mathematics
Faculty of Engineering
Doshisha University
Kyotanabe, Kyoto, 610-03 Japan
E-mail : `rokazaki@dd.iij4u.or.jp`