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An almost-sure estimate for the mean of generalized Q-multiplicative functions of modulus 1

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Résumé. Soit $Q = (Q_k)_{k\geq 0}$, $Q_0 = 1$, $Q_{k+1} = q_k Q_k$, $q_k \geq 2$, $k \geq 0$, une échelle de Cantor, Z_Q le groupe compact $\prod_{0\leq j} Z/q_j Z$, et μ sa mesure de Haar normalisée. A un élement x of Z_Q écrit $x = \{a_0, a_1, a_2, ...\}, 0 \leq a_k \leq q_{k+1} - 1, k \geq 0$, on associe la suite $x_k = \sum_{0\leq j\leq k} a_j Q_j$. On montre que si g est une fonction Q-multiplicative unimodulaire, alors

$$\lim_{x_k \to x} \Big(\frac{1}{x_k} \sum_{n \leq x_k - 1} g(n) - \prod_{0\leq j\leq k} \frac{1}{q_j} \sum_{0\leq a\leq q_j - 1} g(aQ_j) \Big) = 0 \quad \mu\text{-p.s.}$$

Abstract Let $Q = (Q_k)_{k\geq 0}$, $Q_0 = 1$, $Q_{k+1} = q_k Q_k$, $q_k \geq 2$, be a Cantor scale, $\mathbf{Z}_Q$ the compact projective limit group of the groups $\mathbf{Z}/Q_k\mathbf{Z}$, identified to $\prod_{0\leq j\leq k-1} \mathbf{Z}/q_j\mathbf{Z}$, and let μ be its normalized Haar measure. To an element $x = \{a_0, a_1, a_2, ...\}$, $0 \leq a_k \leq q_{k+1} - 1$, of $\mathbf{Z}_Q$ we associate the sequence of integral valued random variables $x_k = \sum_{0\leq j\leq k} a_j Q_j$. The main result of this article is that, given a complex Q-multiplicative function g of modulus 1, we have

$$\lim_{x_k \to x} \big(\frac{1}{x_k} \sum_{n \leq x_k - 1} g(n) - \prod_{0\leq j\leq k} \frac{1}{q_j} \sum_{0\leq a< q_j} g(aQ_j) \big) = 0 \quad \mu\text{-a.e.}$$

1. Introduction

Let $\mathbf{N}$ be the set of non-negative integers, and let $Q = (Q_k)_{k\geq 0}$, $Q_0 = 1$, be an increasing sequence of positive integers. Using the greedy algorithm, to every element n of $\mathbf{N}$, one can associate a representation

$$n = \sum_{k=0}^{+\infty} \varepsilon_k(n) Q_k$$

which is unique if for every K,

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$$\sum_{k=0}^{K-1} \varepsilon_k(n) Q_k < Q_K.$$

The simplest examples are the q-adic scale, q integer, $q \geq 2$, and its generalization, the Cantor scale $Q_{k+1} = q_k Q_k$, $Q_0 = 1$, $q_k \geq 2$, $k \geq 0$. In this article, we are concerned with the Cantor scale. For a given integer $n \geq 1$, we denote by $k(n)$ the maximal index k for which $\varepsilon_k(n)$ is different from zero. The integers $\varepsilon_k(n)$ are the digits from n in the basis Q. We recall that if G is an abelian group, a G-valued arithmetical function f such that

$$f(n) = \sum_{k=0}^{k(n)} f(\varepsilon_k(n) Q_k) \quad \text{for} \quad n \geq 1 \quad \text{and} \quad f(0) = 0_G,$$

is called a Q-additive function, an extension of the notion of q-additive function introduced by A. O. Gelfond in the q-adic case [4]. We recall that a real-valued sequence $f(n)$ has an asymptotic distribution if there exists a distribution function F such that for all continuity points x of F, the probability measures defined by $\mu_N(x) = N^{-1}\mathrm{card}\{n \leq N; f(n) \leq x\}$ tends to $F(x)$ as N tends to infinity. In the case of the q-adic scale, necessary and sufficient conditions for the existence of an asymptotic distribution for a real-valued q-additive function have been given by H. Delange in 1972 [3]. J. Coquet [2] considered in 1975 the same kind of problem in cases of Cantor scales and obtained mainly sufficient conditions. In both cases, it appears essential to have information on the difference

$$\Big(\frac{1}{x} \sum_{0 \leq n < x} g(n) - \prod_{0 \leq j \leq k(x)} \frac{1}{q_j} \sum_{0 \leq a < q_j} g(aQ_j)\Big),$$

where $g(\cdot)$ is any Q-multiplicative function of modulus 1, and more precisely, to get a characterization of

$$\lim_{x \to +\infty} \Big(\frac{1}{x} \sum_{0 \leq n < x} g(n) - \prod_{0 \leq j \leq k(x)} \frac{1}{q_j} \sum_{0 \leq a < q_j} g(aQ_j)\Big) = 0. \tag{1}$$

In fact, if the sequence $\{q_j\}_{j \geq 0}$ is bounded, the relation 1 is always true. But if $\{q_j\}_{j \geq 0}$ is unbounded, the situation is quite different. In [1], G. Barat constructs a Q-multiplicative function h with values 1 or -1 such that

$$\lim_{x \to +\infty} \prod_{0 \leq j \leq k(x)} \frac{1}{q_j} \sum_{0 \leq a < q_j} h(aQ_j)$$

exists and is a positive number while

$$\liminf_{x \to +\infty} \frac{1}{x} \sum_{n < x} h(n)$$

is less than or equal to zero. This difference is due to the existence of a *first digit phenomenon*, unavoidable for unbounded sequence $\{q_j\}_{0\leq j}$, as remarked by E. Manstavičius in a recent article [6].

Let $\mathbf{Z}_Q$ denote the group of Q-adic integers, considered as the compact projective limit group of $\mathbf{Z}/Q_k\mathbf{Z}$ and identified to $\prod_{0\leq k}\mathbf{Z}/q_k\mathbf{Z}$ (see [5], p. 109). The products

$$\prod_{0\leq j\leq k(n)}\frac{1}{q_j}\sum_{0\leq a<q_j} g(aQ_j)$$

are clearly related to this group in the following way: an element a of $\mathbf{Z}_Q$ can be written $a=(a_0,a_1,\dots), 0\leq a_k\leq q_k-1, 0\leq k$, and we may identify an element of $\mathbf{N}$ with an element of $\mathbf{Z}_Q$ which has only a finite number of digits different from zero. For all $a=(a_0,a_1,\dots)$ belonging to $\mathbf{Z}_Q$, we define on $\mathbf{Z}_Q$ the sequence of $\mathbf{N}$-valued random variables $x_k(\cdot)$ given by $x_k(a)=\sum_{j=0}^k a_jQ_j$, the compact group $\mathbf{Z}_Q$ being endowed with its normalized Haar measure μ, and clearly

$$\prod_{0\leq j\leq k}\frac{1}{q_j}\sum_{0\leq a<q_j} g(aQ_j)=\int_{\mathbf{Z}_Q} g(x_k)d\mu.$$

In this article, we show roughly speaking that although the relation 1 is not always true according to the example of G. Barat (for unbounded sequence $\{q_j\}_{j\geq 0}$), it is almost surely true for a path chosen at random.

2. Results

2.1. Main theorem.

Theorem 1. *Let g be a unimodular Q-multiplicative function and set*

$$m_j(g)=\frac{1}{q_j}\sum_{0\leq a<q_j} g(aQ_j).$$

Then, the relation

$$\lim_{k\to\infty}\Big(\frac{1}{x_k(\cdot)}\sum_{n<x_k(\cdot)} g(n)-\prod_{0\leq j\leq k} m_j(g)\Big)=0$$

holds μ-a.e.

2.2. Consequence of Theorem 1.

Theorem 2. *Let G be a metrizable locally compact abelian group with group law denoted by $+$. Γ denotes the dual group of G endowed with its Haar measure m, and let $f(n)$ be a G-valued Q-additive function. Given a sequence $A(k)$ in G, we denote by F_k^A the distribution of the G-valued* *function defined on* $\mathbf{Z}_Q$ *by* $t \mapsto (f(x_k(t)) - A(k))$*, and by* $\delta_{(a)}$ *the measure consisting in a unit mass at the point* a*.*

The following assertions are equivalent:

i) there exists a sequence $A(k)$ *in* G *and a probability measure* ν *on* G *such that the sequence of distributions* F_k^A *converges vaguely to* ν *(i.e.,* $\lim_k \int_G \varphi dF_k^A = \int_G \varphi d\nu$ *for all continuous maps* $\varphi : G \to \mathbf{C}$ *with compact support);*

ii) there exists a sequence $A(k)$ *in* G *and a probability measure* ν *on* G *such that* μ*-a.e., the sequence of random measures* $\frac{1}{x_k(\cdot)} \sum_{n<x_k(\cdot)} \delta_{(f(n)-A(k))}$ *converges vaguely to* ν *as* k *tends to infinity;*

iii) the set X *of characters* g *of* Γ *for which there exists an integer* $N(g)$ *such that* $\prod_{j \geq N(g)} m_j(g) \neq 0$ *is not* m*-negligible.*

Remarks. 1) Assertion iii) is always satisfied if G is a compact metrizable group, for X is not empty (it contains the trivial character), and consequently is not m-negligible.

2) Necessary and sufficient conditions for the continuity of ν can be easily found, since ν appears as a convolution of measures on $\mathbf{Z}_Q$: in fact, the same method as in [7] (p. 84–87), gives that X is a closed and open subgroup. Denoting by H the orthogonal of X and by T_H the canonical projection $G \mapsto G/H$, the measure ν is not continuous if and only if H is finite and

$$\lim_{k\to\infty} \sum_{0\leq j\leq k} \frac{1}{q_j} \sum_{\substack{0\leq a<q_j \\ T_H(f(aQ_j))\neq 0}} 1 < +\infty\,.$$

2.3. **Proof of Theorem 2.** A straightforward adaptation of the argument given in [7] (p 84-87) leads to, *primo* if one of the assumptions i), ii), iii), holds, then, X is a closed and open subgroup of Γ; and *secundo*, there exists a probability measure ν on G and a G-valued sequence $\{A(k)\}_k$ such that for all g in Γ, the sequence

$$\{\overline{g}(A(k)) \prod_{0\leq j\leq k} m_j(g\circ f)\}_k$$

tends to $\hat{\nu}(g)$ where $\hat{\nu}$ is the Fourier transform of ν. This is due to the fact that for g in X there exists an $N(g)$ for which the relation $\prod\limits_{j\geq N(g)} m_j(g) \neq 0$, holds. Hence we get by Theorem 1 that for all g, the sequence

$$\Big\{\frac{1}{x_k(\cdot)} \sum_{n<x_k(\cdot)} g\big(f(n) - A(k)\big)\Big\}_k$$

converges to $\hat{\nu}(g)$ μ-a.e. Next, we use the Fubini theorem on the measured space $(\Gamma \times \mathbf{Z}_Q, m \otimes \mu)$ in an essential way, by saying that since Γ is countable at infinity and $\mathbf{Z}_Q$ is compact, both of the measures m and μ are σ-finite and so, μ-a.e., the sequence $\{\frac{1}{x_k}\sum_{n<x_k(\cdot)} g(f(n)-A(k))\}_k$ converges to $\hat{\nu}(g)$ m-a.e.. In order to prove that μ-a.e., the sequence

$$\{\frac{1}{x_k(\cdot)}\sum_{n<x_k(\cdot)} \delta_{(f(n)-A(k))}\}_k$$

converges vaguely to ν, it suffices to show that for any real-valued continuous function F defined on G whose support is compact, the sequence

$$\{\frac{1}{x_k(\cdot)}\sum_{n<x_k(\cdot)} F(f(n)-A(k))\}_k$$

converges to $\nu(F)$. This can be done as follows. Take any $\varepsilon > 0$; by assumption on F, there exists V, a symmetric neighborhood of the origin in G, such that for all t in G and all u in V, one has $|F(t+u)-F(t)| \le \varepsilon$. Denoting by M the Haar measure on G normalized with respect to m, we have

$$\begin{aligned}
&\big|F(t) - \frac{1}{M(V)}\int_V F(t+u)dM(u)\big| \\
&\qquad = \big|\frac{1}{M(V)}\int_V (F(t+u)-F(t))dM(u)\big| \\
&\qquad \le \frac{1}{M(V)}\int_V \big|(F(t+u)-F(t))\big|dM(u) \\
&\qquad\qquad \le \frac{1}{M(V)}\int_V \varepsilon dM(u) \le \varepsilon.
\end{aligned}$$

The function $F_V(t)$ defined by

$$F_V(t) = \frac{1}{M(V)}\int_V F(t+u)dM(u)$$

is the convolution product of F with the characteristic function of V normalized by the constant $M(V)^{-1}$. Therefore, the Fourier transform $\widehat{F_V}$ is integrable and we get

$$\begin{aligned}
\frac{1}{x_k}\sum_{n<x_k} F(f(n)-A(k)) &= \frac{1}{x_k}\sum_{n<x_k}\int_\Gamma \widehat{F}_V(g)\overline{g}(f(n)-A(k))dm(g) \\
&= \int_\Gamma \Big(\frac{1}{x_k}\sum_{n<x_k} \widehat{F}_V(g)\overline{g}(f(n)-A(k))\Big)dm(g).
\end{aligned}$$

By the Lebesgue dominated convergence theorem,

$$\begin{aligned}
&\lim_{k\to+\infty}\frac{1}{x_k}\sum_{n\le x_k-1}F(f(n)-A(k))\\
&\qquad=\lim_{k\to+\infty}\int_\Gamma\Big(\frac{1}{x_k}\sum_{n\le x_k-1}\hat{F}_V(g)\bar{g}(f(n)-A(k))\Big)dm(g)\\
&\qquad=\int_\Gamma\widehat{F}_V(g)\Big(\lim_{k\to+\infty}\frac{1}{x_k}\sum_{n\le x_k-1}\bar{g}\big(f(n)-A(k)\big)\Big)dm(g)\\
&\qquad=\nu(F_V)\ \mu-\text{a.e.}
\end{aligned}$$

Now, since ν is a probability measure and $|F-F_V|\le\varepsilon$, the sequence

$$\{\frac{1}{x_k}\sum_{n\le x_k-1}F(f(n)-A(k))-\nu(F)\}_k$$

is bounded in modulus by 2ε; this implies

$$\lim_{k\to+\infty}\frac{1}{x_k(\cdot)}\sum_{n<x_k(\cdot)}F(f(n)-A(k))=\nu(F)\ \mu-\text{a.e.}$$

Therefore, the sequence $\left\{\frac{1}{x_k(\cdot)}\sum_{n<x_k(\cdot)}\delta_{(f(n)-A(k))}\right\}_k$ converges vaguely μ-a.e. to ν.

3. Proof of Theorem 1

Notation and conventions

Given an arbitrary arithmetical function f, we set

$$S_N(f)=\sum_{0\le n\le N}f(n),\quad M_{N-1}(f)=\sum_{0\le n<Q_N}f(n),\quad \widetilde{M}_N(f)=Q^{-1}M_N(f).$$

Notice that we have the identity $M_{N-1}(f)=S_{Q_N-1}(f)$ and for any Q-multiplicative function f,

$$M_{N-1}(f)=\prod_{0\le k<N}m_k(f).$$

By convention, the result of a summation (resp. a product) on an empty set will be 0 (resp.1).

A - Toolbox.

Proposition 1. *Let g be a Q-multiplicative function of modulus 1 and assume that the sequence $\{\widetilde{M}_k(g)\}_k$ does not tend to 0. Then, there exists a* *sequence* $\{\alpha_k\}_{k\geq 0}$ *of complex numbers of modulus* 1 *such that*

$$\sum_{k=0}^{+\infty} \frac{1}{q_k} \sum_{a=0}^{q_k-1} |1-g(aQ_k)\alpha_k|^2 < +\infty.$$

Proof. By our assumption, all the complex numbers $m_j(g)$ are different from zero. Put $\alpha_j = m_j(\overline{g}(\cdot))|m_j(g(\cdot))|^{-1}$ where $\overline{g}(\cdot)$ is the complex conjugate of $g(\cdot)$. The product $\alpha_j m_j(g)$ is equal to $|m_j(g)|$ and the sequence $\{|\widetilde{M}_{k+1}|\}_k$ is convergent. Therefore,

$$\sum_{k=0}^{+\infty} \big(1-\alpha_k m_k(g)\big) < +\infty.$$

From

$$\sum_{k=0}^{+\infty} \big(1-\alpha_k m_k(g)\big) = \sum_{k=0}^{+\infty} \frac{1}{q_k} \sum_{0\leq a<q_k} \big(1-g(aQ_k)\alpha_k\big)$$

we get a fortiori that the series $\sum_{k=0}^{+\infty}\frac{1}{q_k}\sum_{0\leq a<q_k} Re\big(1-g(aQ_k)\cdot\alpha_k\big)$ converges and since $\big|g(aQ_k).\alpha_k\big| = 1$, we deduce

$$\sum_{k=0}^{+\infty} \frac{1}{q_k} \sum_{0\leq a<q_k} \big|1-g(aQ_k)\cdot\alpha_k\big|^2 < +\infty.$$

$\square$

According to Proposition 1, we introduce the sequence of arithmetical functions $g_k^*(n)$ defined by

$$g_k^*(n) = \prod_{0\leq j\leq k} g(a_jQ_j).\alpha_j$$

where n is written in base Q as $n=\sum_{j=0}^k a_jQ_k$. This means that if $k(n)$ is the index of the last digit of n which is different from zero, $g_k^*(n)$ is equal to

$$\Big(\prod_{0\leq j\leq k(n)} g(aQ_j)\alpha_j\Big)\cdot\Big(\prod_{k(n)<j\leq k}\alpha_j\Big).$$

We extend g_k^* by $g_k^*(x) = g_k^*\circ x_k$ which we also denote g_k^*. Moreover, for simplification, we shall use the notation $g^*(aQ_j) = g(aQ_j)\alpha_j$.

Proposition 2. *If the sequence* $\{\widetilde{M}_{k+1}(f)\}_{k\geq 0}$ *does not converge to* 0*, there exists a subset* E_∞ *of* $\mathbf{Z}_Q$ *such that* $\mu(E_\infty) = 1$ *and for every* $a = (a_0, a_1, ...)$ *in* E_∞*, the sequence* $k\mapsto g_k^*(a)$ *converges.*

Proof. The sequence of finite groups $\mathbf{Z}/Q_k\mathbf{Z}$, $k \geq 0$, induces a filtration on the μ-measured space $\mathbf{Z}_Q$, and the complex-valued sequence of adapted functions for this filtration defined by

$$g_k^*(\cdot)\Big(\prod_{0\leq j\leq k}\frac{1}{q_j}\sum_{0\leq a<q_j} g^*(aQ_j)\Big)^{-1}$$

is a martingale. Since we have

$$\prod_{0\leq j\leq k}\frac{1}{q_j}\sum_{0\leq a<q_j} g^*(aQ_j) = \Big|\prod_{0\leq j\leq k}\frac{1}{q_j}\sum_{0\leq a<q_j} g^*(aQ_j)\Big|$$

and

$$\Big|\prod_{0\leq j\leq k}\frac{1}{q_j}\sum_{0\leq a<q_j} g^*(aQ_j)\Big|^{-1}$$

is bounded, this martingale is bounded and so, it converges μ-a.e. But the sequence

$$\Big\{\prod_{0\leq j\leq k}\frac{1}{q_j}\sum_{0\leq a<q_j} g^*(aQ_j)\Big\}_k$$

is convergent. Hence we obtain that the sequence $\{g_k^*()\}$ converges μ - a.e. ☐

Proposition 3. *If the sequence $\{\widetilde{M}_{k+1}(f)\}_k$ does not tend to* 0*, there exists a subset F_∞ of $\mathbf{Z}_Q$ such that $\mu(F_\infty) = 1$ and for every $x = (a_0(x), a_1(x), ...)$ in F_∞, one has*

$$\lim_{\substack{k\to+\infty \\ a_k(x)\neq 0}} \frac{1}{a_k(x)}\sum_{0\leq a<a_k(x)} \big|1 - g^*(aQ_k)\big|^2 = 0.$$

Proof. Assume that the sequence $\{\widetilde{M}_{k+1}(f)\}_k$ does not tend to 0. Using the same notations as in Proposition 2, we have by Proposition 1

$$\sum_{k=0}^{+\infty}\frac{1}{q_k}\sum_{a=0}^{q_k-1} |1 - g(aQ_k)\alpha_k|^2 < +\infty.$$

Let σ_k be defined by $\sigma_k = \frac{1}{q_k}\sum\limits_{a=0}^{q_k-1} |1 - g(aQ_k)\alpha_k|^2$. For x in $\mathbf{Z}_Q$, we write $x = (a_0(x), a_1(x), ...)$, $0 \leq a_k(x) \leq q_k - 1, 0 \leq k$ and we remark that, on the sequence of the $a_k(x)$ different from 0, one has

$$\begin{aligned}\frac{1}{a_k(x)}\sum_{0\leq a<a_k(x)} \big|1 - g^*(aQ_k)\big|^2 &\leq \frac{1}{a_k(x)}\sum_{0\leq a<q_k} \big|1 - g^*(aQ_k)\big|^2) \\ &\leq \frac{q_k}{a_k(x)}\big(\frac{1}{q_k}\sum_{0\leq a<q_k} \big|1 - g^*(aQ_k)\big|^2\big)\end{aligned}$$

$$\leq \quad \frac{q_k}{a_k(x)}\sigma_k.$$

Since $\sum_k \sigma_k < +\infty$, it is known that there exists an increasing positive function h tending to infinity when k tends to infinity such that $\sum_k \sigma_k h(k) < +\infty$ and $\prod_{k=0}^{+\infty}(1-\sigma_k h(k)) > 0$. We consider the set $F(h)$ of points x in $\mathbf{Z}_Q$ such that for all k, the inequality

$$[q_k\sigma_k h(k)] \leq a_k(x) \leq q_k - 1$$

holds, where $[\cdot]$ denotes the integral part function. This set $F(h)$ is closed, and its measure $\mu(F(h))$ is equal to

$$\prod_{k=0}^{+\infty}\frac{1}{q_k}(q_k - [q_k\sigma_k h(k)]),$$

and we have

$$\mu F(h) \geq \prod_{k=0}^{+\infty}\frac{1}{q_k}(q_k - q_k\sigma_k h(k)).$$

Now, we remark that this last product can be written $\prod_{k=0}^{+\infty}(1-\sigma_k h(k))$ and so, $\mu F(h) \neq 0$. For an x in $F(h)$, we consider the condition $[q_k\sigma_k h(k)] \leq a_k(x) \leq q_k - 1$, when $a_k(x) \neq 0$. If $[q_k\sigma_k h(k)]$ is not 0, then we have

$$\begin{aligned}\frac{q_k}{a_k(x)}\sigma_k \leq \frac{q_k}{[q_k\sigma_k h(k)]}\sigma_k \quad &\leq \tfrac{q_k\sigma_k h(k)}{[q_k\sigma_k h(k)]}\cdot\tfrac{q_k}{q_k\sigma_k h(k)}\sigma_k \\ &\leq \tfrac{q_k\sigma_k h(k)}{[q_k\sigma_k h(k)]}\tfrac{1}{h(k)} \leq \tfrac{2}{h(k)}\end{aligned}$$

and in this case, we get $\lim_{k\to+\infty}\frac{q_k}{a_k(x)}\sigma_k = 0$. The case where $[q_k\sigma_k h(k)] = 0$ remains. We have $0 \leq q_k\sigma_k h(k) < 1$, i.e. $q_k\sigma_k < 1/h(k)$. Hence

$$\frac{q_k}{a_k(x)}\sigma_k \leq \frac{q_k}{1}\sigma_k \leq q_k\sigma_k \leq \frac{1}{h(k)} = o(1), \quad k \to +\infty.$$

To obtain our result, we remark that the sequence of functions h_r indexed by positive integers r and defined by $h_r(n) = h(n)$ if $n > r$ and $h(n)r^{-1}$ otherwise, satisfies the same requirements as h. Now, the sequence of closed sets $F(h_r)$ is increasing with r and $\lim_{r\to+\infty}\mu(F(h_r)) = 1$. This gives immediately that F_∞, the union of the $F(h_r)$, is a measurable set of measure 1. Now, if x belongs to F_∞, it belongs to some $F(h_r)$ and as a consequence, along the sequence k such that $a_k(x) \neq 0$, we have

$$\begin{aligned}\frac{1}{a_k(x)}\sum_{0\leq a<a_k(x)}\left|1-g^*(aQ_k)\right|^2 \quad &\leq \quad \frac{q_k}{a_k(x)}\sigma_k \leq q_k\sigma_k \\ &\leq \quad \frac{2}{h_r(k)} = o(1), \quad k \to +\infty.\end{aligned}$$

□

Proposition 4. *If the sequence* $\{\widetilde{M}_{k+1}(f)\}_{k\geq 0}$ *converges to zero, then*

$$\lim_{N\to+\infty}\frac{1}{N}\sum_{0\leq n<N} g(n)=0.$$

Proof. This Proposition is due to J.Coquet [2]. □

B- End of the proof

1- First case: *the sequence* $\{\widetilde{M}_{k+1}(f)\}_k$ *tends to zero.*

From Proposition 4, $\lim\limits_{N\to+\infty}\sum\limits_{0\leq n\leq N-1} f(n)=0$, and $(x_k)_k$ tends to infinity μ-a.e. due to the fact that $x_k(a)$ is bounded if and only if a has only a finite number of nonzero digits. This means exactly that a is an integer; but $\mu(\mathbf{N})=0$.

2- Second case: *the sequence* $\{\widetilde{M}_{k+1}(f)\}_k$ *does not tend to zero.*

We consider the intersection of the sets E_∞ and F_∞ given in Proposition 2 and Proposition 3 respectively. Notice that $\mu(E_\infty\cap F_\infty)=1$. Our aim is to prove that for every ξ in $E_\infty\cap F_\infty$

$$\lim_{k\to+\infty}\Big(\frac{1}{x_k(\xi)}\sum_{n<x_k(\xi)} g(n)-\widetilde{M}_{k+1}(g)\Big)=0.$$

The sequence of functions $k\mapsto g_k^*(n)$ and the constants α_j are defined as in Proposition 2. Let ξ be an element of $E_\infty\cap F_\infty$ and denote $x_k(\xi)$ by x_k for short. We have:

$$\begin{aligned}
S_{x_k}(g_k^*) &= \big(\sum_{0\leq a<a_k} g(aQ_k)\alpha_k\big)M_{k-1}(g_{k-1}^*)+(g(a_kQ_k)\alpha_k)S_{x_{k-1}}(g_{k-1}^*)\\
&= \big(\sum_{0\leq a<a_k} g(aQ_k)\alpha_k\big)M_{k-1}(g_{k-1}^*)+(g_k^*(\xi))(\overline{g}_{k-1}^*(\xi))S_{x_{k-1}}(g_{k-1}^*),
\end{aligned}$$

and by iteration

$$\begin{aligned}
S_{x_k}(g_k^*) &= \sum_{j=0}^{k} g_k^*(\xi)\overline{g_j^*(\xi)}\big(\sum_{0\leq a<a_j(\xi)} g(aQ_j)\alpha_j\big)\ \big(\prod_{r=0}^{j-1}\sum_{a=0}^{q_r-1} g_{j-1}^*(aQ_r)\big)\\
&= \sum_{j=0}^{k} g_k^*(\xi)\overline{g_j^*(\xi)}\big(\sum_{0\leq a<a_j(\xi)} g(aQ_j)\alpha_j\big)\big(M_{j-1}(g_{j-1}^*)\big)
\end{aligned}$$

If $a_j(\xi)\neq 0$, the choice of ξ in F_∞ implies

$$\sum_{0\leq a<a_j(\xi)} g^*(aQ_j)=a_j(\xi)(1+\varepsilon_j),$$

with $\varepsilon_j = o(1)$ when j tends to infinity. Since g is of modulus 1 and $Q_j^{-1}M_{j-1}(g_{j-1}^*)$ is bounded by 1,

$$\begin{aligned}\Big|S_{x_k}(g_k^*) - \sum_{j=0}^{k}(g_k^*(\xi))\overline{g_j^*(\xi)}\big(a_j(\xi)\big)\big(M_{j-1}(g_{j-1}^*)\big)\Big| \\ = \Big|S_{x_k}(g_k^*) - \sum_{j=0}^{k}(g_k^*(\xi))\overline{g_j^*(\xi)}\cdot\big(a_j(\xi)\big)\cdot\big(\big(Q_j^{-1}M_{j-1}(g_{j-1}^*)\big)Q_j\big)\Big| \\ \leq \sum_{j=0}^{k}\varepsilon_j a_j(\xi)Q_j.\end{aligned}$$

Consequently

$$\Big|S_{x_k}(g_k^*) - \sum_{j=0}^{k} g_k^*(\xi)\overline{g_j^*(\xi)}a_j(\xi)M_{j-1}(g_{j-1}^*)\Big| = o(x_k), \quad (k \to +\infty).$$

Since ξ belongs to E_∞, $\{g_k^*(\xi)\}_k$ converges, and as a consequence, the sequence $\eta_{j,k} = |g_k^*(\xi).\overline{g_j^*(\xi)} - 1|$ tends to 0 when k and j, $j \leq k$, tend to infinity independently.

This implies

$$\Big|\sum_{j=0}^{k} g_k^*(\xi)\overline{g_j^*(\xi)}a_j(\xi)M_{j-1}(g_{j-1}^*) - \sum_{j=0}^{k} a_j(\xi)M_{j-1}(g_{j-1}^*)\Big| \leq \sum_{j=0}^{k}\eta_{j,k}a_j(\xi)Q_j,$$

and so, when $k \to +\infty$,

$$\Big|\sum_{j=0}^{k} g_k^*(\xi)\overline{g_j^*(\xi)}a_j(\xi)M_{j-1}(g_{j-1}^*) - \sum_{j=0}^{k} a_j(\xi)M_{j-1}(g_{j-1}^*)\Big| = o(x_k).$$

Moreover, $Q_j^{-1}M_{j-1}(g_{j-1}^*)$ tends to a limit, say $\widetilde{M}_\infty(g_\infty^*)$. Hence

$$\begin{aligned}\Big|\sum_{j=0}^{k} a_j(\xi)\cdot M_{j-1}(g_{j-1}^*) - \widetilde{M}_\infty(g_\infty^*)\sum_{j=0}^{k} a_j(\xi)Q_j\cdot\Big| \\ \leq \sum_{j=0}^{k}\big|Q_j^{-1}M_{j-1}(g_{j-1}^*) - \widetilde{M}_\infty(g_\infty^*)\big|\cdot a_j(\xi)Q_j = o(x_k),\ (k \to +\infty).\end{aligned}$$

Finally

$$\begin{aligned}
&\Big|S_{x_k}(g_k^*) - \widetilde{M}_\infty(g_\infty^*)\sum_{j=0}^{k} a_j(\xi)Q_j\Big| \\
&\qquad \le \Big|S_{x_k}(g_k^*) - \sum_{j=0}^{k} g_k^*(\xi)\overline{g_j^*(\xi)}\cdot a_j(\xi)\big)\cdot M_{j-1}(g_{j-1}^*)\Big| \\
&\qquad + \Big|\sum_{j=0}^{k} g_k^*(\xi)\overline{g_j^*(\xi)}\cdot a_j(\xi)\cdot M_{j-1}(g_{j-1}^*) - \sum_{j=0}^{k} a_j(\xi)\cdot M_{j-1}(g_{j-1}^*)\Big| \\
&\qquad + \Big|\sum_{j=0}^{k} a_j(\xi)\cdot M_{j-1}(g_{j-1}^*) - \widetilde{M}_\infty(g_\infty^*)\sum_{j=0}^{k} a_j(\xi)Q_j\Big|.
\end{aligned}$$

It then follows that

$$S_{x_k}(g_k^*) = \widetilde{M}_\infty(g_\infty^*)\cdot x_k + o(x_k), (k \to +\infty).$$

To obtain the result, it is enough to notice that from $Q_{k+1}^{-1}M_k(g_k^*) - \widetilde{M}_\infty(g_\infty^*) = o(1)$ we obtain $S_{x_k}(g_k^*) = Q_{k+1}^{-1}M_k(g_k^*)\cdot x_k + o(x_k)$, and replacing g_k^* by its value, we get

$$S_{x_k}(g_k^*) = S_{x_k}(g)\prod_{j=0}^{k}\alpha_j, \quad M_k(g_k^*) = M_k(g)\prod_{j=0}^{k}\alpha_j.$$

and this leads to $S_{x_k}(g) - \big(M_k(g)Q_{k+1}^{-1}\big)\cdot x_k = o(x_k)$.

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