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The analytic continuation and the order estimate of multiple Dirichlet series

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Résumé. Dans cet article, nous considérons que certaines séries de Dirichlet multiples, dont nous montrons le prolongement analytique en utilisant la formule intégrale de Mellin-Barnes. Des majorations de ces séries sont également obtenues.

Abstract. Multiple Dirichlet series of several complex variables are considered. Using the Mellin-Barnes integral formula, we prove the analytic continuation and an upper bound estimate.

1. Introduction and statement of results

Let $s = \sigma + it$ be a complex variable, and

$$\varphi_k(s) = \sum_{n=1}^{\infty} \frac{a_k(n)}{n^s} \tag{1.1}$$

be a Dirichlet series with complex coefficients $a_k(n)$ $(1 \le k \le r)$. We assume

(I) $\varphi_k(s)$ is convergent absolutely for $\sigma > \alpha_k (> 0)$;
(II) $\varphi_k(s)$ can be continued meromorphically to the whole complex plane $\mathbb{C}$, and holomorphic except for a possible pole at $s = \alpha_k$ of order at most 1, whose residue we denote by R_k;
(III) in any fixed strip $\sigma_1 \le \sigma \le \sigma_2$, the order estimate $\varphi_k(\sigma + it) = O(|t|^A)$ holds as $|t| \to \infty$, where $A = A(\sigma_1, \sigma_2)$ be a non-negative constant.

In the present paper we introduce the multiple Dirichlet series

$$\begin{aligned} &\Phi_r((s_1, \ldots, s_r); (\varphi_1, \ldots, \varphi_r)) \\ &\quad = \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \frac{a_1(m_1)}{m_1^{s_1}} \frac{a_2(m_2)}{(m_1+m_2)^{s_2}} \cdots \frac{a_r(m_r)}{(m_1+\cdots+m_r)^{s_r}} \end{aligned} \tag{1.2}$$

associated with $\varphi_1, \ldots, \varphi_r$, where $s_k = \sigma_k + it_k$ $(1 \le k \le r)$ are complex variables, and prove several basic properties. It is clear that the multiple

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series (1.2) converges absolutely if $\sigma_k > \alpha_k$ $(1 \le k \le r)$. Let $\mathbb{N}, \mathbb{N}_0$ be the sets of positive integers and non-negative integers, respectively. We first prove the following

Theorem 1. *The multiple series $\Phi_r((s_1, \ldots, s_r); (\varphi_1, \ldots, \varphi_r))$ can be continued meromorphically to the whole $\mathbb{C}^r$ space, and its possible singularities are located only on the subsets of $\mathbb{C}^r$ each of which is defined by one of the following equations:*

$$s_j + \cdots + s_r = \alpha_j + \delta_{j+1}\alpha_{j+1} + \cdots + \delta_r \alpha_r - n, \tag{1.3}$$

where $1 \le j \le r$, $n \in \mathbb{N}_0$, and $\delta_k = 0$ or 1 $(2 \le k \le r)$. When $j = r$, (1.3) is to be read as $s_r = \alpha_r - n$. Moreover,

(i) *in the case $j = r \ge 2$, if $\alpha_r \in \mathbb{N}$, then the possible values of n are $0, 1, 2, \ldots, \alpha_r - 1$;*
(ii) *in the case $2 \le j \le r-1$, if $\alpha_j \in \mathbb{N}$ and $\delta_{j+1} = \cdots = \delta_r = 1$, then the possible values of n are $0, 1, 2, \ldots, \alpha_j - 1$;*
(iii) *in the case $j = 1$, if $r = 1$ or if $\delta_2 = \cdots = \delta_r = 1$, then $n = 0$.*

Theorem 2. *If $\varphi_k(s)$ is entire for $1 \le k \le r$, then*

$$\Phi_r((s_1, \ldots, s_r); (\varphi_1, \ldots, \varphi_r))$$

is also entire.

Our $\Phi_r((s_1, \ldots, s_r); (\varphi_1, \ldots, \varphi_r))$ includes various interesting special cases. The function $\varphi_k(s)$ can be the Riemann zeta-function $\zeta(s)$, the Dirichlet L-function $L(s, \chi)$, the automorphic L-function $L(s, f)$ attached to a certain cusp form, etc. When $\varphi_k(s) = \zeta(s)$ $(1 \le k \le r)$, our Φ_r is nothing but the well-known Euler-Zagier sum, whose analytic continuation was recently established by various methods (Arakawa and Kaneko [3], Zhao [13], Akiyama, Egami and Tanigawa [1], Matsumoto [10]; see also Goncharov [6]). Our proof of Theorems 1 and 2 is a generalization of the method given in Matsumoto [10].

When $\varphi_k(s) = L(s, \chi_k)$ $(1 \le k \le r)$, where $\chi_1, \ldots, \chi_r$ are Dirichlet characters, the corresponding Φ_r may be called the multiple Dirichlet L-function. The special values at positive integer arguments of these kinds of multiple L-functions have been studied by Goncharov, Arakawa and Kaneko [4] and others. It is also to be noted that a different type of multiple Dirichlet L-functions of the form

$$\sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \frac{\chi_1(m_1)}{m_1^{s_1}} \frac{\chi_2(m_1+m_2)}{(m_1+m_2)^{s_2}} \cdots \frac{\chi_r(m_1+\cdots+m_r)}{(m_1+\cdots+m_r)^{s_r}}$$

has been studied by Goncharov [5], Akiyama and Ishikawa [2], and Ishikawa [8, 9].

By Theorem 2, the multiple Dirichlet L-function

$$\Phi_r((s_1,\dots,s_r);(L(\cdot,\chi_1),\dots,L(\cdot,\chi_r)))$$

is entire if $\chi_1,\dots,\chi_r$ are non-principal. Also, the multiple automorphic L-function

$$\Phi_r((s_1,\dots,s_r);(L(\cdot,f_1),\dots,L(\cdot,f_r)))$$

is entire, where $f_1,\dots,f_r$ are cusp forms. On the other hand, in the present paper we assume that the order of the pole at $s=\alpha_k$ is at most 1, but this is only for simplicity. We may consider the case when $\varphi_k(s)$ has the pole of higher order by the same method.

A natural next problem is to study the order estimate of Φ_r. In this paper we treat the simplest case Φ_2. By the assumption (III) there exists a non-negative constant $\theta_k(\sigma)$ for any σ such that

$$\varphi_k(\sigma+it)=O(|t|^{\theta_k(\sigma)}) \tag{1.4}$$

holds as $|t|\to\infty$. We shall prove

Theorem 3. *Let η be a small positive number. The estimate*

$$\Phi_2((s_1,s_2);(\varphi_1,\varphi_2)) \ll (1+|t_2|)^{-\alpha_2}(1+|t_1+t_2|)^{\theta_1(\sigma_1+\sigma_2-\alpha_2)}R_2$$
$$+(1+|t_2|)^{\theta_2(\eta)+\frac{1}{2}+\max\{0,\frac{1}{2}-\sigma_2+\eta\}}\max\{1+|t_2|,1+|t_1+t_2|\}^{\theta_1(\sigma_1+\sigma_2-\eta)}$$

holds in the region $\{(s_1,s_2)|\ \sigma_1+\sigma_2>\alpha_1+\eta,\sigma_2>\eta\}$, except the points near the set of singularities.

Therefore, if some non-trivial estimate of $\varphi_k(s)$ is known, then Theorem 3 gives a non-trivial estimate of Φ_2. For instance, when $\varphi_k(s)=\zeta(s),L(s,\chi)$, or the automorphic L-function $L(s,f)$ (see Good [7]), sharp values of $\theta_k(\sigma)$ are known.

It is clearly an interesting problem to generalize Theorem 2 to the case of $\Phi_r, r\geq 3$.

2. Proof of Theorems 1 and 2

We prove the theorems by induction. The argument is similar to that developed in the last section of [10], or in [11], [12].

The case $r=1$ is clear from the assumption (II). Consider the case Φ_r, assuming that the theorems are true for Φ_{r-1}. First assume $\sigma_k>\alpha_k$ $(1\leq k\leq r)$. Recall the classical Mellin-Barnes formula

$$\Gamma(s)(1+\lambda)^{-s}=\frac{1}{2\pi i}\int_{(c)}\Gamma(s+z)\Gamma(-z)\lambda^z dz, \tag{2.1}$$

where $\Re s > 0, |\arg\lambda| < \pi, \lambda \neq 0, -\Re s < c < 0$, and the path of integration is the vertical line $\Re z = c$. Put $s = s_r$ and

$$\lambda = \frac{m_r}{m_1 + \cdots + m_{r-1}}$$

in (2.1); we may assume $-\sigma_r < c < -\alpha_r$. Then multiply the both sides by $a_1(m_1)\cdots a_r(m_r)m_1^{-s_1}\cdots(m_1+\cdots+m_{r-2})^{-s_{r-2}}(m_1+\cdots+m_{r-1})^{-s_{r-1}-s_r}$ and sum up with respect to $m_1, \ldots, m_r$. We obtain

$$\begin{aligned}(2.2)\quad \Phi_r((s_1,\ldots,s_r);(\varphi_1,\ldots,\varphi_r)) &= \frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)}\\ &\times \Phi_{r-1}((s_1,\ldots,s_{r-2},s_{r-1}+s_r+z);(\varphi_1,\ldots,\varphi_{r-1}))\varphi_r(-z)dz.\end{aligned}$$

Next we shall shift the path to $\Re z = M - \eta$, where M is a non-negative integer and η is a small positive number. The function $\varphi_r(-z)$ is of polynomial order with respect to $\Im z$ by the assumption (III), while

$$\Phi_{r-1}((s_1,\ldots,s_{r-2},s_{r-1}+s_r+z);(\varphi_1,\ldots,\varphi_{r-1})) = O(1)$$

in the region $c \leq \Re z \leq M - \eta$ because of the expression (1.2). Hence the integrand on the right-hand side of (2.2) tends to zero when $|\Im z| \to \infty$, so this shifting is possible. Counting the residues of the poles at $s = -\alpha_r, 0, 1, 2, \ldots, M-1$, we obtain

$$\begin{aligned}(2.3)\qquad & \Phi_r((s_1,\ldots,s_r);(\varphi_1,\ldots,\varphi_r))\\ =&\frac{\Gamma(s_r-\alpha_r)\Gamma(\alpha_r)}{\Gamma(s_r)}\Phi_{r-1}((s_1,\ldots,s_{r-2},s_{r-1}+s_r-\alpha_r);(\varphi_1,\ldots,\varphi_{r-1}))R_r\\ &+\sum_{l=0}^{M-1}\binom{-s_r}{l}\Phi_{r-1}((s_1,\ldots,s_{r-2},s_{r-1}+s_r+l);(\varphi_1,\ldots,\varphi_{r-1}))\varphi_r(-l)\\ &+\frac{1}{2\pi i}\int_{(M-\eta)}\frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)}\\ &\qquad\times\Phi_{r-1}((s_1,\ldots,s_{r-2},s_{r-1}+s_r+z);(\varphi_1,\ldots,\varphi_{r-1}))\varphi_r(-z)dz.\end{aligned}$$

The first two terms on the right-hand side are meromorphic in the whole $\mathbb{C}^r$ space by the induction assumption, while we can see that the last integral term is holomorphic in the region

$$\left\{(s_1,\ldots,s_r)\,\middle|\, \begin{array}{c}\sigma_k+\cdots+\sigma_r > \alpha_k+\cdots+\alpha_{r-1}-M+\eta \quad (1\leq k\leq r-1)\\ \sigma_r > -M+\eta\end{array}\right\}$$

by using the induction assumption on the location of singularities of Φ_{r-1}. Since M is arbitrary, this implies the meromorphic continuation of Φ_r to the whole $\mathbb{C}^r$ space.

The location of singularities of Φ_r can also be seen from (2.3). In fact, the factor $\Gamma(s_r - \alpha_r)$ is singular on $s_r = \alpha_r - n$ $(n \in \mathbb{N}_0)$. If $\alpha_r \in \mathbb{N}$, then this singularity is cancelled by the factor $\Gamma(s_r)$ when $n \geq \alpha_r$. By the induction assumption, the factor

$$\Phi_{r-1}((s_1, \ldots, s_{r-2}, s_{r-1} + s_r - \alpha_r); (\varphi_1, \ldots, \varphi_{r-1}))$$

is singular on

$$s_j + \cdots + s_r - \alpha_r = \alpha_j + \delta_{j+1}\alpha_{j+1} + \cdots + \delta_{r-1}\alpha_{r-1} - n,$$

where $1 \leq j \leq r-1$ and $n \in \mathbb{N}_0$, but

(i) if $j \geq 2$, $\alpha_j \in \mathbb{N}$ and $\delta_{j+1} = \cdots = \delta_{r-1} = 1$ then $0 \leq n \leq \alpha_j - 1$, and

(ii) if $j = 1$, and if $r = 2$ or $\delta_2 = \cdots = \delta_{r-1} = 1$ then $n = 0$.

Similarly the factor

$$\Phi_{r-1}((s_1, \ldots, s_{r-2}, s_{r-1} + s_r + l); (\varphi_1, \ldots, \varphi_{r-1})) \quad (l \in \mathbb{N}_0)$$

is singular on

$$s_j + \cdots + s_r + l = \alpha_j + \delta_{j+1}\alpha_{j+1} + \cdots + \delta_{r-1}\alpha_{r-1} - n$$

($1 \leq j \leq r-1, n \in \mathbb{N}_0$), with the same restrictions as above. Collecting the above information we obtain the assertion on the location of singularities of Φ_r.

Lastly, if $\varphi_r(s)$ is entire then $R_r = 0$, hence the first term on the right-hand side of (2.3) vanishes. Therefore inductively we can show that Φ_r is entire. All the assertions of Theorems 1 and 2 are now established.

3. Estimation of certain integrals

In this section, as a preparation for the proof of Theorem 3, we consider the integral of the form

$$\text{(3.1)} \quad J(u,v) = \int_{-\infty}^{\infty} (1 + |u+y|)^p \, (1 + |v+y|)^q \, (1 + |y|)^r \times \exp\left(A|u+y| + B|y|\right) dy,$$

where u, v, p, q, r, A, B are real numbers with $A + B < 0$. To evaluate this integral, we use the estimate

$$\text{(3.2)} \quad \int_{-\infty}^{\infty} (1 + |y|)^\rho \exp(A|u+y| + B|y|) dy = O\left((1 + |u|)^{\rho+\delta} e^{B|u|} + e^{A|u|}\right),$$

where u, ρ, A, B are real numbers with $A + B < 0, \delta = 1$ or 0 according as $A = B$ or $A \neq B$, and the implied constant depends only on ρ, A and B. This estimate is a special case of Lemma 3 in Matsumoto [11].

Temporarily we assume $|u| \geq |v|$. Divide the integral (3.1) as

$$\int_{|y|>2|u|} + \int_{2|u|\geq|y|>2|v|} + \int_{|y|\leq 2|v|},$$

and estimate the factor $(1+|u+y|)^p\,(1+|v+y|)^q\,(1+|y|)^r$ in the integrand as

$$\begin{cases} O\big((1+|y|)^{p+q+r}\big) & \text{if } |y|>2|u|, \\ O\big(\max\{1,U^p\}(1+|y|)^{q+r}\big) & \text{if } 2|u|\geq|y|>2|v|, \\ O\big(\max\{1,U^p\}\max\{1,V^q\}(1+|y|)^r\big) & \text{if } |y|\leq 2|v|, \end{cases}$$

where $U=1+|u|$ and $V=1+|v|$. Then replace the intervals of all integrals by $(-\infty,\infty)$, and apply (3.2). The result is that

$$\begin{aligned} J(u,v) \ll & U^{p+q+r+\delta}e^{B|u|}+e^{A|u|} \\ & + \max\{1,U^p\}\{U^{q+r+\delta}e^{B|u|}+e^{A|u|}\} \\ & + \max\{1,U^p\}\max\{1,V^q\}\{U^{r+\delta}e^{B|u|}+e^{A|u|}\}, \end{aligned}$$

hence

Lemma 1. *Using the notations as above, we have*

$$J(u,v) \ll (1+U^p)(1+U^q+V^q)U^{r+\delta}e^{B|u|} + (1+U^p)(1+V^q)e^{A|u|}. \tag{3.3}$$

This is proved under the assumption $|u|\geq|v|$, but the case $|u|<|v|$ can be treated similarly, and the same conclusion (3.3) holds.

In particular, if $A=B=-\frac{\pi}{2}$ then $\delta=1$, then (3.3) implies the following

Lemma 2. *We have*

$$\begin{aligned} &\int_{-\infty}^{\infty}(1+|u+y|)^p(1+|v+y|)^q(1+|y|)^r\exp\Big(-\frac{\pi}{2}|u+y|-\frac{\pi}{2}|y|\Big)\,dy \\ &\quad = O\left((1+U^p)(1+U^q+V^q)(1+U^{r+1})\exp\Big(-\frac{\pi}{2}|u|\Big)\right), \end{aligned}$$

with the implied constant depending only on p,q and r.

4. Proof of Theorem 3

Putting $r=2$ and $M=0$ in (2.3), we have

$$\begin{aligned} \Phi_2((s_1,s_2);(\varphi_1,\varphi_2)) = & \frac{\Gamma(s_2-\alpha_2)\Gamma(\alpha_2)}{\Gamma(s_2)}\varphi_1(s_1+s_2-\alpha_2)R_2 \\ & + \frac{1}{2\pi i}\int_{(-\eta)}\frac{\Gamma(s_2+z)\Gamma(-z)}{\Gamma(s_2)}\varphi_1(s_1+s_2+z)\varphi_2(-z)dz, \end{aligned}$$

and the last integral is holomorphic in the region

$$\{(s_1, s_2) \mid \sigma_1 + \sigma_2 > \alpha_1 + \eta, \quad \sigma_2 > \eta\}.$$

Hence in this region, by using Stirling's formula and (1.4), we have

$$\begin{aligned}\Phi_2((s_1, s_2); (\varphi_1, \varphi_2)) &\ll (1 + |t_2|)^{-\alpha_2} (1 + |t_1 + t_2|)^{\theta_1(\sigma_1+\sigma_2-\alpha_2)} R_2 \\ + (1 + |t_2|)^{-\sigma_2+\frac{1}{2}} e^{\frac{\pi}{2}|t_2|} &\int_{-\infty}^{\infty} (1 + |t_2 + y|)^{\sigma_2-\eta-\frac{1}{2}} (1 + |t_1 + t_2 + y|)^{\theta_1(\sigma_1+\sigma_2-\eta)} \\ &\times (1 + |y|)^{\theta_2(\eta)+\eta-\frac{1}{2}} \exp\left(-\frac{\pi}{2}|t_2 + y| - \frac{\pi}{2}|y|\right) dy.\end{aligned}$$

We apply Lemma 2 with $u = t_2, v = t_1+t_2, p = \sigma_2-\eta-\frac{1}{2}, q = \theta_1(\sigma_1+\sigma_2-\eta)$, and $r = \theta_2(\eta) + \eta - \frac{1}{2}$. Since $q \geq 0$ and $r + 1 \geq 0$, the estimate of Lemma 2 can be written as

$$O\left((1 + U^p)U^{r+1} \max\{U, V\}^q \exp\left(-\frac{\pi}{2}|u|\right)\right).$$

Hence

$$\begin{aligned}&\Phi_2((s_1, s_2); (\varphi_1, \varphi_2)) \\ &\ll (1 + |t_2|)^{-\alpha_2} (1 + |t_1 + t_2|)^{\theta_1(\sigma_1+\sigma_2-\alpha_2)} R_2 + (1 + |t_2|)^{\theta_2(\eta)-\sigma_2+\eta+1} \\ &\quad \times \left\{1 + (1 + |t_2|)^{\sigma_2-\eta-\frac{1}{2}}\right\} \max\left\{1 + |t_2|, 1 + |t_1 + t_2|\right\}^{\theta_1(\sigma_1+\sigma_2-\eta)},\end{aligned}$$

which implies the assertion of Theorem 3.

References

[1] S. Akiyama, S. Egami, Y. Tanigawa, *An analytic continuation of multiple zeta functions and their values at non-positive integers.* Acta Arith. **98** (2001), 107–116.

[2] S. Akiyama, H. Ishikawa, *On analytic continuation of multiple L-functions and related zeta-functions.* In: Analytic number theory (Beijing/Kyoto, 1999), 1–16, Dev. Math., 6, Kluwer Acad. Publ., Dordrecht, 2002

[3] T. Arakawa, M. Kaneko, *Multiple zeta values, poly-Bernoulli numbers, and related zeta functions.* Nagoya Math. J. **153** (1999), 189–209.

[4] T. Arakawa, M. Kaneko, *On multiple L-values*, in preparation.

[5] A. B. Goncharov, *Multiple polylogarithms, cyclotomy and modular complexes.* Math. Res. Letters **5** (1998), 497–516.

[6] A. B. Goncharov, *Multiple polylogarithms and mixed Tate motives*, preprint.

[7] A. Good, *The square mean of Dirichlet series associated with cusp forms.* Mathematika **29** (1982), 278–295.

[8] H. Ishikawa, *On analytic properties of a multiple L-function.* In: Analytic extension formulas and their applications (Fukuoka, 1999/Kyoto, 2000), 105–122, Int. Soc. Anal. Appl. Comput., 9, Kluwer Acad. Publ., Dordrecht, 2001.

[9] H. Ishikawa, *A multiple character sum and a multiple L-function.* Arch. Math. **79** (2002), 439–448.

[10] K. Matsumoto, *Asymptotic expansions of double zeta-functions of Barnes, of Shintani, and Eisenstein series.* Nagoya Math. J., to appear.

[11] K. Matsumoto, *The analytic continuation and the asymptotic behaviour of certain multiple zeta-functions I.* J. Number Theory, to appear.

[12] K. MATSUMOTO, *The analytic continuation and the asymptotic behaviour of certain multiple zeta-functions II.* In: Analytic and Probabilistic Methods in Number Theory, Proc. 3rd Intern. Conf. in Honour of J. Kubilius (Palanga, Lithuania, Sept 2001), 188–194, A. Dubickas et al. (eds.), TEV, Vilnius, 2002.

[13] J. ZHAO, *Analytic continuation of multiple zeta functions.* Proc. Amer. Math. Soc. **128** (2000), 1275–1283.

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