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A representation theorem for a class of rigid analytic functions

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Résumé. Soit p un nombre premier, $\mathbb{Q}_p$ le corps des nombres p-adiques et $\mathbb{C}_p$ la complétion d'une clôture algébrique de $\mathbb{Q}_p$. Dans cet article, nous obtenons un théorème de représentation pour les fonctions analytiques rigides sur $\mathbf{P}^1(\mathbb{C}_p)\backslash C(t,\varepsilon)$ qui sont équivariantes par le groupe de Galois $G = Gal_{cont}(\mathbb{C}_p/\mathbb{Q}_p)$, où t désigne un élément Lipschitzien de $\mathbb{C}_p$ et $C(t,\epsilon)$ un $\varepsilon-$voisinage de la $G-$orbite de t.

Abstract. Let p be a prime number, $\mathbb{Q}_p$ the field of p-adic numbers and $\mathbb{C}_p$ the completion of the algebraic closure of $\mathbb{Q}_p$. In this paper we obtain a representation theorem for rigid analytic functions on $\mathbf{P}^1(\mathbb{C}_p)\backslash C(t,\varepsilon)$ which are equivariant with respect to the Galois group $G = Gal_{cont}(\mathbb{C}_p/\mathbb{Q}_p)$, where t is a Lipschitzian element of $\mathbb{C}_p$ and $C(t,\epsilon)$ denotes the $\varepsilon-$neighborhood of the $G-$orbit of t.

1. Introduction

Let p be a prime number, $\mathbb{Q}_p$ the field of p-adic numbers, $\overline{\mathbb{Q}}_p$ a fixed algebraic closure of $\mathbb{Q}_p$ and $\mathbb{C}_p$ the completion of $\overline{\mathbb{Q}}_p$ with respect to the p-adic absolute value. Let $t \in \mathbb{C}_p$ and set $E(t) = \mathbf{P}^1(\mathbb{C}_p)\backslash C(t) = \mathbb{C}_p \cup \{\infty\}\backslash C(t)$ where $C(t)$ denotes the orbit of t with respect to the group G of all continuous automorphisms of $\mathbb{C}_p$ over $\mathbb{Q}_p$. In this paper we are interested in the $G-$equivariant rigid analytic functions on $E(t)$ and their restrictions to affinoids of the form $E(t,\varepsilon) = \mathbb{C}_p \cup \{\infty\}\backslash C(t,\varepsilon)$ where $C(t,\varepsilon)$ stands for the $\varepsilon-$neighborhood of $C(t)$.

These functions are easily described in case t is algebraic over $\mathbb{Q}_p$. For instance, if $t \in \mathbb{Q}_p$ then one can use the equivariant transformation $z \mapsto \frac{1}{z-t}$ to send t to the point at infinity. Then the equivariant rigid analytic functions on $E(t)$ will correspond to the entire functions which are equivariant

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and these are simply power series $f(z) = \sum_{n\geq 0} a_n z^n$ with $a_n \in \mathbb{Q}_p$ for any n and such that $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = 0$.

If t is transcendental over $\mathbb{Q}_p$ it is not obvious that there are any nonconstant equivariant rigid analytic functions on $E(t)$. For certain elements t (called Lipschitzian) such a function $z \mapsto F(t,z)$ is constructed in [APZ2]. In this paper we define for any Lipschitzian element t of $\mathbb{C}_p$ and any natural numbers m, n an equivariant rigid analytic function $F_{m,n}(t,z)$ on $E(t)$, which is related to our basic trace series $F(t,z)$. Then in Theorem 4.2 below we express any equivariant rigid analytic function on an affinoid $E(t,\varepsilon)$ in terms of the above functions $F_{m,n}(t,z)$.

2. Background material

2.1. Let p be a prime number and $\mathbb{Q}_p$ the field of p-adic numbers endowed with the p-adic absolute value $|\ |$, normalized such that $|p| = 1/p$. Let $\overline{\mathbb{Q}}_p$ be a fixed algebraic closure of $\mathbb{Q}_p$ and denote by the same symbol $|\ |$ the unique extension of $|\ |$ to $\overline{\mathbb{Q}}_p$. Further, denote by $(\mathbb{C}_p, |\ |)$ the completion of $(\overline{\mathbb{Q}}_p, |\ |)$ (see [Am], [Ar]). Let $G = Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ endowed with the Krull topology. The group G is canonically isomorphic with the group $Gal_{cont}(\mathbb{C}_p/\mathbb{Q}_p)$ of all continuous automorphisms of $\mathbb{C}_p$ over $\mathbb{Q}_p$. For any $x \in \mathbb{C}_p$ denote $C(x) = \{\sigma(x) | \sigma \in G\}$ the orbit of x and let $\widetilde{\mathbb{Q}_p[x]}$ be the closure of the ring $\mathbb{Q}_p[x]$ in $\mathbb{C}_p$. For any $x \in \overline{\mathbb{Q}}_p$ denote $deg(x) = [\mathbb{Q}_p(x) : \mathbb{Q}_p]$.

2.2. Let $x \in \mathbb{C}_p$. Given a real number $\varepsilon > 0$ let $B(x,\varepsilon) = \{y \in \mathbb{C}_p, |x - y| < \varepsilon\}$ the open ball of radius ε centered at x. If M is a compact subset of $\mathbb{C}_p$ and $\varepsilon > 0$ is a real number, denote by $N(M,\varepsilon)$ the number of all disjoint balls of radius ε which have a non-empty intersection with M. We say that M is *Lipschitzian* if $\lim\limits_{\varepsilon\to 0} \frac{\varepsilon}{|N(M,\varepsilon)|} = 0$. We call an element $x \in \mathbb{C}_p$ *Lipschitzian* if $C(x)$ is Lipschitzian.

According to [APZ2] if x is Lipschitzian then one can integrate Lipschitzian functions (see definition in 2.3 below) with respect to the p-adic Haar measure π_t induced by G on the set $C(x)$.

Let $G_x = \{\sigma \in G : \sigma(x) = x\}$ and P a closed subgroup of G which contains G_x. Then $C_P(x) = \{\sigma(x) : \sigma \in P\}$, the orbit of x with respect to P, is a compact subset of $C(x) = C_G(x)$. If x is Lipschitzian then $C_P(x)$ is a Lipschitzian compact set for any P with $G_x \subset P$. This follows from the fact that for any $\varepsilon > 0$, $N(C_P(x),\varepsilon)$ divides $N(C(x),\varepsilon)$.

Let $x \in \mathbb{C}_p$ and P a closed subgroup of G which contains G_x. For any $\varepsilon > 0$ let $H_P(x,\varepsilon) = \{\sigma \in P : |x - \sigma(x)| < \varepsilon\}$ and $N_P(x,\varepsilon) = N(C_P(x),\varepsilon)$. Then $H_P(x,\varepsilon)$ is an open subgroup of P and $N_P(x,\varepsilon) = [P : H_P(x,\varepsilon)]$. In particular $N(x,\varepsilon) = [G : H(x,\varepsilon)]$, where $H(x,\varepsilon) = H_G(x,\varepsilon)$.

2.3. The notion of rigid analytic function is defined in [FP] (see also [Am]). According to [APZ2], a rigid analytic function defined on a subset D of $\mathbb{C}_p$ is said to be *equivariant* if for any $z \in D$ one has $C(z) \subset D$ and $f(\sigma(z)) = \sigma(f(z))$ for all $\sigma \in G$. A function $f : C(t) \to \mathbb{C}_p$, $t \in \mathbb{C}_p$ is *Lipschitzian* if there exists a real number $c > 0$ such that $|f(x) - f(y)| \leq c|x - y|$ for all $x, y \in C(t)$.

Let t be a Lipschitzian element of $\mathbb{C}_p$ and $f : C(t) \to \mathbb{C}_p$ a Lipschitzian function. Then the integral

$$\int_{C(t)} f(x) d\pi_t(x)$$

is well defined (see [APZ2]). In particular for any polynomial $P(X) \in \mathbb{C}_p[X]$, any $z \in \mathbb{C}_p \cup \{\infty\} \setminus \mathbb{C}(t)$ and any natural number n the function $z \mapsto f(x,z) = \frac{P(x)}{(z-x)^n}$ is Lipschitzian on $\mathbb{C}(t)$ and we consider the integral $\int_{C(t)} f(x,z) d\pi_t(x)$. Let us denote

$$F_{m,n}(t,z) = \int_{C(t)} \frac{x^m}{(z-x)^n} d\pi_t(x),\ m \geq 0,\ n \geq 0.$$

According to [APZ2] for any $m \geq 0$ one has

$$\int_{C(t)} x^m d\pi_t(x) = Tr(t^m).$$

This shows that $F_{m,0}(t,z) = Tr(t^m) \in \mathbb{Q}_p$, $F_{0,0}(t,z) = 1$ and $1 + F_{1,1}(t, \frac{1}{z}) = F(t,z)$, the trace function associated to t. Also by the equality

$$\frac{1}{(1-u)^m} = (1 + u + u^2 + \ldots + u^n + \ldots)^m = \sum_{s=0}^{\infty} \binom{m+s-1}{s} u^s$$

valid for any positive integer m and any u with $|u| < 1$, it follows that for $|z| > |x|$ one has

$$\frac{x^m}{(z-x)^n} = \sum_{s \geq 0} \binom{n+s-1}{s} \frac{x^{m+s}}{z^{n+s}} .$$

Then one may write:

$$F_{m,n}(t,z) = \sum_{s \geq 0} \binom{m+s-1}{s} \frac{Tr(t^{m+s})}{z^{n+s}}.$$

This formula represents the expansion of $F_{m,n}(t,z)$ in a suitable neighborhood of infinity. As in Theorem 6.1 of [APZ2] one shows that for all $m \geq 0, n \geq 0, F_{m,n}(t,z)$ is an equivariant rigid analytic function defined on $\mathbb{C}_p \cup \{\infty\} \setminus C(t)$.

Remark 2.1. $F'_{m,n}(t,z) = -nF_{m,n+1}(t,z)$ for any $m \geq 0, n \geq 1$. As a consequence one has $F_{m,n+1}(t,z) = \dfrac{(-1)^n}{n!}F^{(n)}_{m,1}(t,z)$, where the derivative is taken with respect to z.

2.4. The above considerations can be generalized as follows: Let $\varepsilon > 0$ be a real number and S a system of right representatives of G with respect to the subgroup $H(t,\varepsilon)$. Assume that the identity element e of G belongs to S and that t is Lipschitzian. For any $\sigma \in S$ the subset $C_\sigma(t,\varepsilon) = \{\tau(t) : \tau \in \sigma H(t,\varepsilon)\}$ is a compact subset of $C(t)$. For $m \geq 0$, $n \geq 0$ denote

$$F^\sigma_{m,n}(t,z) = \int_{C_\sigma(t,\varepsilon)} \frac{x^m}{(z-x)^n}\, d\pi_t(x).$$

It is clear that

$$F_{m,n}(t,z) = \sum_{\sigma \in S} F^\sigma_{m,n}(t,z).$$

In fact this formula represents the Mittag-Leffler decomposition of $F_{m,n}(t,z)$ viewed as a rigid analytic function in the connected affinoid $\mathbb{C}_p \cup \{\infty\} \setminus \cup_{\sigma \in S} B(\sigma(t),\varepsilon) = E(t,\varepsilon)$. In what follows we try to obtain a similar decomposition for any element of the set $A(E(t,\varepsilon))$ of equivariant rigid analytic functions on $E(t,\varepsilon)$.

3. A combinatorial Lemma

Let $\alpha, x, y, \{a_m\}_{m\geq 1}$ be variables. For any $m \geq 1$, let us denote

$$h_m(\alpha) = a_1\alpha^{m-1} + a_2 \binom{m-1}{1} \alpha^{m-2} + \ldots + \binom{m-1}{m-1} a_m \tag{1}$$

where as usually $\binom{m}{k} = \dfrac{m(m-1)\ldots(m-k+1)}{k!}$. For any integer $m \geq 1$ we set

$$A_m(x) = h_m(\alpha) - \frac{1}{1!}h^{(1)}_m(\alpha)x + \ldots + \frac{(-1)^{m-1}}{(m-1)!}h^{(m-1)}_m(\alpha)x^{m-1} \tag{2}$$

where $h^{(k)}_m(\alpha)$ denotes the formal k-th derivative of the polynomial $h_m(\alpha)$ with respect to α.

Lemma 3.1. *For any x, y and any $m \geq 1$ one has:*

$$A_m(x) = \sum_{r=1}^{m} a_r \binom{m-1}{r-1} (\alpha - x)^{m-r} \tag{3}$$

and

$$A_m(y) = \sum_{r=1}^{m} \binom{m-1}{r-1} A_r(x)(x-y)^{m-r}.$$

Proof. Equality (3) states that $A_m(x) = h_m(\alpha - x)$, which follows directly from the Taylor expansion (2). As for the second equality, by applying (3) and using the identity

$$\binom{m-1}{r-1}\binom{r-1}{j-1} = \binom{m-1}{j-1}\binom{m-j}{r-j}$$

one contains

$$\begin{aligned}
&\sum_{r=1}^{m}\binom{m-1}{r-1}A_r(x)(x-y)^{m-r}\\
&=\sum_{r=1}^{m}\binom{m-1}{r-1}(x-y)^{m-r}\sum_{j=1}^{r}a_j\binom{r-1}{j-1}(\alpha-x)^{r-j}\\
&=\sum_{j=1}^{m}a_j\binom{m-1}{j-1}\sum_{r=j}^{m}\binom{m-j}{r-j}(x-y)^{m-r}(\alpha-x)^{r-j}\\
&=\sum_{j=1}^{m}a_j\binom{m-1}{j-1}(\alpha-y)^{m-j}.
\end{aligned}$$

This equals $A_m(y)$ by (3) and so the lemma is proved. □

4. Equivariant rigid analytic functions on $E(t,\varepsilon)$

4.1. Let t be an element of $\mathbb{C}_p$, let $\varepsilon > 0$ be a real number and denote by $B(C(t),\varepsilon)$ the union of all disjoint open balls $B(x,\varepsilon)$ which have a nonempty intersection with $C(t)$. Choose $\alpha \in \overline{\mathbb{Q}}_p$ such that $|t-\alpha| < \varepsilon$. Then one has $H(t,\varepsilon) = H(\alpha,\varepsilon)$. Let S be a system of right representatives of G with respect to $H(t,\varepsilon)$ and assume $e \in S$. One has $B(C(t),\varepsilon) = \bigcup_{\sigma\in S} B(\sigma(\alpha),\varepsilon)$. Consider the affinoid $E(t,\varepsilon) = \mathbb{C}_p \cup \{\infty\} \backslash B(C(t),\varepsilon)$ and let $A(E(t,\varepsilon))$ be the set of equivariant rigid analytic functions on $E(t,\varepsilon)$. If t is Lipschitzian then the functions $F_{m,n}(t,z)$ defined in Section 2 are elements of $A(E(t,\varepsilon))$. In this section we shall prove that all the elements of $A(E(t,\varepsilon))$ can be expressed in terms of the functions $F_{m,n}(t,z)$, $m,n \geq 0$.

4.2. We have the following proposition.

Proposition 4.1. *Let t be an element of $\mathbb{C}_p$. Denote $K_t = \widetilde{\mathbb{Q}_p[t]} \bigcap \overline{\mathbb{Q}}_p$ and let $\varepsilon > 0$ and $\alpha \in K_t$ such that $|\alpha - t| < \varepsilon$. There exists a sequence $\{\alpha_n\}_{n\geq 1}$ of elements of K_t and a sequence $\{\varepsilon_n\}_{n\geq 1}$ of positive real numbers such that:*

(i) $\varepsilon_1 = \varepsilon$, $\alpha_1 = \alpha$,

(ii) For any $n \geq 1$ one has $\varepsilon_{n+1} \leq \inf\{\varepsilon_n/2, |t-\alpha_n|\}$,

(iii) $|t-\alpha_n| < \varepsilon_n$, $n \geq 1$, and deg α_n is smallest with this property.

The proof easily follows by induction on n since any ball $B(t,\varepsilon)$ contains elements of K_t (see [APZ1]).

In what follows we work with sequences $\{\alpha_n\}_n$ and $\{\varepsilon_n\}_n$ as in Proposition 4.1. It is clear that $\lim_n \varepsilon_n = 0$, and $t = \lim_n \alpha_n$. Note also that the ball $B(\alpha_{n+1}, \varepsilon_{n+1})$ is contained in $B(\alpha_n, \varepsilon_n)$ for all $n \geq 1$. Let us consider the subgroup $H(t,\varepsilon_n) = H(\alpha_n, \varepsilon_n)$ defined in Section 2. Denote $d_n = [G : H(T, \varepsilon_n)] = N(t, \varepsilon_n)$, and let S_n be a fixed system of representatives of right cosets of G with respect to $H(t,\varepsilon_n)$. We shall assume that the identity element e of G belongs to each S_n. We remark that $S_1 = S$ and d_n divides d_{n+1} for all $n \geq 1$.

4.3. Let $f \in A(E(t,\varepsilon))$. Then (see [FP], Ch I) f admits a Mittag-Leffler decomposition: $f(z) = \sum_{\sigma\in S} f_\sigma(z) + f(\infty)$ where $f(\infty)$ is the value of f at infinity and

$$f_\sigma(z) = \sum_{m\geq 1} \frac{a_{\sigma,m}}{(z-\sigma(\alpha))^m},\ \lim_m \frac{|a_{\sigma,m}|}{\varepsilon^m} = 0,\ \sigma \in S. \tag{4}$$

Since f is equivariant then for any $z \in E(t,\varepsilon)$ and any $\tau \in G$ one has $\sum_{\sigma\in S} \tau(f_\sigma(z)) = \sum_{\sigma \in S} f_\sigma(\tau(z))$ and $\tau(f(\infty)) = f(\infty)$. Hence $f(\infty) \in \mathbb{Q}_p$ and for any $\sigma \in S$ one can write:

$$f_\sigma(\sigma(z)) = \sigma(f_e(z)),\ a_{\sigma,m} = \sigma(a_{e,m}),\ m \geq 1. \tag{5}$$

Next we remark that for any $\tau \in H(t,\varepsilon)$ and any $\sigma \in S$ the element $\sigma(\tau(\alpha))$ belongs to $B(\sigma(\alpha),\varepsilon)$, and so the function $f_\sigma(z) = \sum_{m\geq 1} \frac{a_{\sigma,m}}{(z-\sigma(\alpha))^m}$ can also be written as

$$f_\sigma(z) = \sum \frac{a_{\sigma\tau,m}}{(z-\sigma\tau(\alpha))^m} = f_{\sigma\tau}(z)$$

where

$$a_{\sigma\tau,m} = \sum_{i=1}^{m} \binom{m-1}{i-1} a_{\sigma,i}(\sigma(\alpha) - \sigma\tau(\alpha))^{m-i}.$$

In what follows we shall assume that $f(\infty) = 0$.

4.4. At this point we derive another convenient expression for $f(z)$, using the above elements α_n. Fix $n \geq 1$. Then $d = d_1$ divides $d_n = [G : H(t,\varepsilon_n)]$. Denote $q_n = d_n/d$ and let $B(\alpha_n^{(j)}, \varepsilon_n)$, $1 \leq j \leq q_n$ be all the balls of radius ε_n centered at suitable conjugates of α_n and such that these balls cover $C_{H(t,\varepsilon)}(T) = C_e(t,\varepsilon)$. Then

$$f_e(z) = \frac{d}{d_n} \sum_{1\leq j\leq q_n} \sum_{m\geq 1} \frac{A_{e,m}^{(j)}}{(z-\alpha_n^{(j)})^m} \tag{6}$$

where

$$A_{e,m}^{(j)} = \sum_{i=1}^{m} \binom{m-1}{i-1} a_{e,i}(\alpha - \alpha_n^{(j)})^{m-i}\,. \tag{7}$$

According to (5) for all $\sigma \in S$ one has

$$f_\sigma(z) = \frac{d}{d_n} \sum_{1\le j\le q_n} \sum_{m\ge 1} \frac{A_{\sigma,m}^{(j)}}{(z - \sigma(\alpha_n^{(j)}))^m},$$

where $A_{\sigma,m}^{(j)} = \sigma(A_{e,m}^{(j)})$. As a consequence of (4) there exists a positive real number M such that for any $m \ge 1$ one has:

$$|a_{\sigma,m}| \le M\varepsilon^m. \tag{8}$$

It follows from (7) that

$$|A_{e,m}^{(j)}| \le M\varepsilon^m \tag{9}$$

for any $n \ge 2$ and $1 \le j \le q_n$.

4.5. At this point we assume that t is a Lipschitzian element of $\mathbb{C}_p$, $\varepsilon > 0$ and $\alpha \in B(t,\varepsilon)$, $\alpha \in K_t$. Let $f \in A(E(t,\varepsilon))$, $f = \sum_{\sigma\in S} f_\sigma(z)$ with $f_\sigma(z)$ given by (4). For any $m \ge 1$ denote

$$h_m(\alpha) = a_{e,1}\alpha^{m-1} + \binom{m-1}{1} a_{e,2}\alpha^{m-2} + \ldots + \binom{m-1}{m-1} a_{e,m}.$$

Also, for $\sigma \in S$ consider the function $F_{m,n}^\sigma(t,z)$ defined in Section 2.

Theorem 4.2. *Let t be a Lipschitzian element of $\mathbb{C}_p$, $\varepsilon > 0$, $\alpha \in B(t,\varepsilon) \cap K_t$ and $f \in A(E(t,\varepsilon))$. Then for any $z \in E(t,\varepsilon)$ one has*

$$f(z) = \sum_{\sigma\in S} \sum_{m\ge 1} \sum_{0\le j<m} \frac{(-1)^j}{j!} \sigma(h_m^{(j)}(\alpha)) F_{j,m}^\sigma(t,z).$$

Proof. For any $m \ge 1$ let $A_m(x) = \sum_{1\le i\le m} a_{e,i} \binom{m-1}{i-1} (\alpha - x)^{m-i}$ and

$$A(x,z) = \sum_{m\ge 1} \frac{A_m(x)}{(z-x)^m}. \tag{10}$$

Step 1. Fix $z_0 \in E(t,\varepsilon)$. We assert that for any $z \in B(z_0,\varepsilon)$, the function $x \mapsto A(x,z)$ is defined and is Lipschitzian on $B(t,\varepsilon)$. Firstly we remark that for any $x \in B(t,\varepsilon)$ one has (see (8)):

$$\left|\frac{A_m(x)}{(z-x)^m}\right| \le \frac{\left|\sum a_{e,i} \binom{m-1}{i-1} (\alpha - x)^{m-i}\right|}{\varepsilon^m} \tag{11}$$

$$\leq \max\left(\frac{\left|\sum_{i=1}^{[m/2]} a_{e,i}\binom{m-1}{i-1}(\alpha-x)^{m-i}\right|}{\varepsilon^m}, \frac{\left|\sum_{i=[m/2]+1}^{m} a_{e,i}\binom{m-1}{i-1}(\alpha-x)^{m-i}\right|}{\varepsilon^m}\right)$$

Notice that

$$\varepsilon^{-m}\left|\sum_{i=1}^{[m/2]} a_{e,i}\binom{m-1}{i-1}(\alpha-x)^{m-i}\right| \leq \max_{1\leq i\leq [m/2]}\left(M\left(\frac{|\alpha-x|}{\varepsilon}\right)^{m-i}\right)$$

$$= M\left(\frac{|\alpha-x|}{\varepsilon}\right)^{m-[m/2]}$$

and

$$\varepsilon^{-m}\left|\sum_{i=[m/2]+1}^{m} a_{e,i}\binom{m-1}{i-1}(\alpha-x)^{m-i}\right| \leq \max_{[m/2]+1\leq i\leq m}\frac{|a_{e,i}|}{\varepsilon^i}.$$

Since $\frac{|\alpha-x|}{\varepsilon} < 1$, by (4) and the above considerations it follows that $\left|\frac{A_m(x)}{(z-x)^m}\right| \to 0$ when $m \to \infty$. Then the function $A(x,z)$ is defined on $B(t,\varepsilon)$, as claimed. Now let $x, y \in B(t,\varepsilon)$. For any $m \geq 1$ we have

$$\frac{A_m(y)}{(z-y)^m} = \frac{A_m(y)}{(z-x)^m}\left(1+\sum_{i\geq 1} D_i\left(\frac{y-x}{z-x}\right)^i\right)$$

where D_i are suitable natural numbers. Then one can write

$$\left|\frac{A_m(x)}{(z-x)^m}-\frac{A_m(y)}{(z-y)^m}\right| \leq \max_{i\geq 1}\left(\left|\frac{A_m(x)-A_m(y)}{(z-x)^m}\right|, \left|\frac{A_m(y)(y-x)^i}{(z-x)^{m+i}}\right|\right).$$

But (see (8)) for any $i \geq 1$ and $z \in B(z_0,\varepsilon)$ one has

$$\left|\frac{A_m(y)(y-x)^i}{(z-x)^{m+i}}\right| \leq \frac{|A_m(y)|}{|z-x|^m}\cdot\frac{|y-x|^i}{|z-x|^i} \leq M\,\frac{|y-x|}{\varepsilon}.$$

Also by an easy computation one sees that:

$$\left|\frac{A_m(x)-A_m(y)}{(z-x)^m}\right| \leq \frac{M\,|y-x|}{\varepsilon}.$$

Finally, one has $|A(x,y)-A(y,z)| \leq \frac{M}{\varepsilon}|x-y|$ i.e. $A(x,z)$ is Lipschitzian on $B(t,\varepsilon)$. The above considerations also show that for any $\delta > 0$ we have

$$\left|\frac{A_m(x)}{(z-x)^m}-\frac{A_m(y)}{(z-y)^m}\right| \leq \delta|x-y| \tag{12}$$

for all m large enough in terms of z and δ, uniformly for $x, y \in C_e(t,\varepsilon)$.

Step 2. Let us denote $D = C_e(t,\varepsilon) = B(t,\varepsilon) \cap C(t)$. Then D is a compact Lipschitzian subset of $\mathbb{C}_p$ and we consider the integral

$$F(z) = \int_D A(x,z) d\pi_t(x), \; z \in B(z_0,\varepsilon).$$

Here we use the definition of the integral with respect the p-adic measure π_t as in [APZ2]. We assert that

$$f_e(z) = F(z), \; z \in B(z_0,\varepsilon),$$

where e is the identity element of G.

To see this, consider the sequences $\{\varepsilon_n\}_n$ and $\{\alpha_n\}_n$ from Proposition 4.1. Let $H(t,\varepsilon_n)$, d_n, S_n be as above. In particular $\varepsilon_1 = \varepsilon$, $\alpha_1 = \alpha$, $d_1 = d$. For any $n \geq 1$ let $B(\alpha_n^{(i)},\varepsilon_n)$, $1 \leq i \leq q_n$ be the open balls of radius ε_n which cover D. Then one has:

$$F(z) = \int_D A(x,z) d\pi_t(x) = \lim_n \Phi[A(x,z),\alpha_n^{(i)},\varepsilon_n]$$

where

$$\Phi[A(x,z),\alpha_n^{(i)},\varepsilon_n] = \frac{d}{d_n} \sum_{1\leq i\leq q_n} A(\alpha_n^{(i)},z)$$

is the Riemann sum associated to $(A,\alpha_n^{(i)},\varepsilon_n)$ (see[APZ2]). We have

$$\frac{d}{d_n} A(\alpha_n^{(i)},z) = \frac{d}{d_n} \sum_{m\geq 1} A_m(\alpha_n^{(i)})(z-\alpha_n^{(i)})^m.$$

From (6) it now follows that

$$\Phi[A,\alpha_n^{(i)},\varepsilon_n] = f_e(z).$$

Since this equality is valid for any n we conclude that

$$F(z) = \int_D A(x,z) d\pi_t(x) = f_e(z).$$

Step 3. We now apply formula (2) to obtain another expression for $A_m(x)$. One has:

$$\begin{aligned} A_m(x) &= h_m(\alpha) - \frac{1}{1!} h_m^{(1)}(\alpha)x + \ldots + \frac{(-1)^{m-1}}{(m-1)!} h_m^{(m-1)}(\alpha)x^{m-1} \\ &= \sum_{j=0}^{m-1} \frac{(-1)^j}{j!} h_m^{(j)}(\alpha) \frac{x^j}{(z-x)^m} \,. \end{aligned}$$

Therefore

$$\begin{aligned} \int_D \frac{A_m(x)}{(z-x)^m} d\pi_t(x) &= \sum_{j=0}^{m-1} \frac{(-1)^j}{j!} h_m^{(j)}(\alpha) \int_D \frac{x^j}{(z-x)^m} d\pi_t(x) \\ &= \sum_{j=0}^{m-1} \frac{(-1)^j}{j!} h_m^{(j)}(\alpha) F_{j,m}^e(t,z). \end{aligned} \tag{13}$$

We claim that

$$F(z) = f_e(z) = \sum_{m\geq 1} \sum_{j=0}^{m-1} \frac{(-1)^j}{j!} h_m^{(j)}(\alpha) F_{j,m}^e(t,z). \tag{14}$$

In order to prove this formula we need the following result:

Lemma 4.3. *Let t be a Lipschitzian element of $\mathbb{C}_p, \varepsilon > 0$ a real number, $g : B(C(t), \varepsilon) \to \mathbb{C}_p$ a Lipschitzian function, and let c be a real number such that $|g(x) - g(y)| \leq c|x - y|$ for all $x, y \in C(t)$. Then there exists a real number k independent of g such that:*

$$\left| \int_{C(t)} g(x) \mathrm{d}\pi_t \right| \leq \max(||g||, ck)$$

when $||g|| = \sup\limits_{x\in C(t)} |g(x)|$.

Proof. Let $\{\varepsilon_n\}_{n\geq 1}$ be a decreasing sequence of positive real numbers such that $\lim\limits_n \varepsilon_n = 0, \varepsilon_n/\varepsilon_{n+1} \leq 2$ and $C(t) \subseteq B(t, \varepsilon_1)$. Then one has: $\displaystyle\int_{C(t)} g(x)\mathrm{d}\pi_t = \lim_n \Phi(g, \tau(t), \varepsilon_n)$, where (see Section 2) $\mathrm{d}_n = [G : H(t, \varepsilon_n)]$, S_n is a system of right cosets of G with respect $H(t, \varepsilon_n)$ and $\Phi(g, \tau(t), \varepsilon_n] = \displaystyle\frac{1}{\mathrm{d}_n} \sum_{\tau\in S_n} g(\tau(t))$ is the Riemann sum associated to ε_n, S_n and g (see [APZ2]). In particular $\Phi(g, \tau(t), \varepsilon_1) = g(t)$.

Let $n \geq 1$. Then d_n divides d_{n+1} and for any $\tau \in S_{n+1}$ there exists exactly one element $\sigma \in S_n$ such that $\tau(t) \in B(\sigma(t), \varepsilon_n)$. Then we have $|g(\sigma(t)) - g(\tau(t))| \leq c\varepsilon_n$, and so $\displaystyle\left|\frac{1}{\mathrm{d}_n} \sum_{\sigma\in S_n} g(\sigma(t)) - \frac{1}{\mathrm{d}_{n+1}} \sum_{\tau\in S_{n+1}} g(\tau(t))\right| \leq \frac{c\varepsilon_n}{|\mathrm{d}_{n+1}|}$. Let n be large enough such that

$$|\int_{C(t)} g(x)\mathrm{d}\pi_t| = \left| \frac{1}{d_{n+1}} \sum_{\tau\in S_{n+1}} g(\tau(t)) \right|.$$

Then by the above considerations one has:

$$\left|\frac{1}{\mathrm{d}_{n+1}}\sum_{\tau\in S_{n+1}} g(\tau(t))\right| =$$

$$\left|\frac{1}{\mathrm{d}_{n+1}}\sum_{\tau\in S_{n+1}} g(\tau(t)) - \frac{1}{\mathrm{d}_n}\sum_{\tau\in S_n} g(\sigma(t)) + \frac{1}{\mathrm{d}_n}\sum_{\sigma\in S_n} g(\sigma(t)) + \dots\right.$$

$$\left.\dots + \frac{1}{\mathrm{d}_2}\sum_{\chi\in S_2} g(\chi(t)) - g(t) + g(t)\right| \leq \max_{1\leq i\leq n}\left||g|, c\frac{\varepsilon_i}{|d_{i+1}|}\right|.$$

Now let us take $k = \sup_n \frac{\varepsilon_n}{|\mathrm{d}_{n+1}|} = \sup \frac{\varepsilon_{n+1}}{|\mathrm{d}_{n+1}|}\cdot\frac{\varepsilon_n}{\varepsilon_{n+1}} < \infty$ since $\lim_n \frac{\varepsilon_n}{|\mathrm{d}_n|} = 0$, t being Lipschitzian by hypothesis. □

Let $\delta > 0$ be a real number. Then by (4), (11) and (12) it follows that for m large enough one has: $\left|\frac{A_m(x)}{(z-x)^m}\right| < \delta$ and $\left|\frac{A_m(x)}{(z-x)^m} - \frac{A_m(y)}{(z-y)^m}\right| < \delta|x-y|$ for any $x, y \in D$. Lemma 4.3 implies that $\left|\int_D \frac{A_m(x)}{(z-x)^m} d\pi_t(x)\right| \to 0$ as $m \to \infty$. Therefore

$$F(z) = \int_D \sum_{m\geq 1} \frac{A_m(x)}{(z-x)^m} d\pi_t(x) = \sum_{m\geq 1}\int_D \frac{A_m(x)}{(z-x)^m} d\pi_t(x)$$

and using (13) one obtains (14).

Step 4. Let $\sigma \in S$ and denote $D^\sigma = B(\sigma(\alpha), \varepsilon) \cap C(t) = C_\sigma(t, \varepsilon)$. Working as above, one gets:

$$f_\sigma(z) = \sum_{m\geq 1}\sum_{j=0}^{m-1} \frac{(-1)^j}{j!} h_m^{(j)}(\sigma(\alpha)) F_{j,m}^\sigma(t, z).$$

Finally by adding these equalities for $\sigma \in S$ one obtains the expression of $f(z)$ stated in Theorem 4.2 □

Corollary 4.4. *The notations and hypothesis are as in Theorem 4.2 Assume $\alpha \in \mathbb{Q}_p$. Then $S = \{e\}$ and one has:*

$$f(z) = \sum_{m\geq 1}\sum_{j=0}^{m-1} h_m^{(j)}(\alpha) F_{j,m}(t, z).$$

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