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Trigonometric sums over primes III

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Résumé. On donne une majoration nouvelle de la somme trigonométrique

$$\sum_{P \leq p < 2P} e(\alpha p^k)$$

où $k \geq 5$, p désigne un nombre premier et $e(x) = \exp(2\pi i x)$.

Abstract. New bounds are given for the exponential sum

$$\sum_{P \leq p < 2P} e(\alpha p^k)$$

where $k \geq 5$, p denotes a prime and $e(x) = \exp(2\pi i x)$.

1. Introduction

Throughout this paper we write $e(x) = \exp(2\pi i x)$ and let p denote a prime variable. Sums of the form

$$\sum_{p \leq P} e(\alpha p^k) \tag{1}$$

arise in applications of the Hardy-Littlewood circle method to the Waring-Goldbach problem (for example see [9]), or in problems involving the distribution of αp^k modulo one (see [1, 12]). The aim of this paper is to give improved bounds for these sums when $k \geq 5$ (although Theorem 3 does give a better bound in certain ranges than any previously published explicit result for $k = 4$ as well). The first unconditional bounds for these sums were given by Vinogradov [10, 11]. The case $k = 1$ was required for his celebrated proof of the ternary Goldbach problem. In 1981 the author [4] showed that, if

$$|q\alpha - a| < q^{-1}, \quad q \geq 1, \quad (a, q) = 1, \tag{2}$$

then, for $k \geq 2$

$$\left| \sum_{p \leq P} e(\alpha p^k) \right| \ll P^{1+\epsilon} \left(\frac{1}{q} + \frac{1}{P^{\frac{1}{2}}} + \frac{q}{P^k} \right)^{\gamma} \tag{3}$$

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with $\gamma = 4^{1-k}, \epsilon > 0$, and constants implied by the $\ll$ notation will depend at most on k and ϵ. We shall state this dependence explicitly in the statements of theorems and lemmas. This improved the value $\gamma = (4^{k+1}(k+1))^{-1}$ obtained by Vinogradov, and generalised the case $k = 2$ given by Ghosh [3]. Improved bounds for

$$P^{1-1/k} \le q \le P^{k/2}, \quad |\alpha q - a| < P^{-k/2}$$

were given in [5]. Then, in [1], it was shown (essentially) that if $\alpha = \frac{a}{q}$ with q near $P^{\frac{k}{2}}$ we have

$$\left|\sum_{p\le P} e(\alpha p^k)\right| \ll P^{1-\sigma(k)+\epsilon} \tag{4}$$

with $\sigma(k) = (3.2^{k-1})^{-1}$. This bound, in which the exponent is only a factor of 3 away from the classical bound for a Weyl sum, was used to prove new results on the distribution of αp^k modulo one. The new idea introduced for this result was to estimate double sums

$$\sum e\left(\frac{a(mn)^k}{q}\right)$$

by approximating $\frac{am^k}{q}$ (or $\frac{a}{q}(m_1^k - m_2^k)$) with another rational $\frac{r}{s}$ where one could control the behaviour of $s = s(m)$ (or $s(m_1, m_2)$) on average over m (or m_1, m_2). Since then various authors have given more general estimates for these sums using similar ideas. For technical reasons it is often better in these results to restrict the range of summation to $p \sim P$ where $a \sim A$ throughout this paper means $A \le a < 2A$. The culmination of this work is Lemma 3.3 of [7], which we state as follows.

Theorem 1.1. *Let $k \in \mathbb{N}, k \ge 4, \epsilon > 0$ and let $P \in \mathbb{R}, P \ge 2$. Suppose that α is a real number, and there exist integers a, q with*

$$|q\alpha - a| < P^{-k/2}, \quad 1 \le q \le P^{k/2}, \quad (a,q) = 1.$$

Then one has

$$\left|\sum_{p\sim P} e(\alpha p^k)\right| \ll_{k,\epsilon} P^{1-2^{-k-1}+\epsilon} + \frac{q^\epsilon w_k(q)^{\frac{1}{2}} P(\log P)^4}{(1 + P^k|\alpha - a/q|)^{\frac{1}{2}}}. \tag{5}$$

In the above $w_k(q)$ is the multiplicative function whose value for a prime power p^c is given by

$$w_k(p^{uk+v}) = \begin{cases} p^{-u-1} & \text{when} \quad u \ge 0,\ 2 \le v \le k; \\ kp^{-u-\frac{1}{2}} & \text{when} \quad u \ge 0,\ v = 1. \end{cases}$$

This gives $w_k(q) \ll q^{-1/k}$, while, on average, $w_k(q) \ll q^{-1/2}$ is nearer the truth.

Theorem 1.1 was applied to obtain new results for the Waring-Goldbach problem. The purpose of this note is to improve the first term on the right hand side of (5). We are also able to consider more relaxed conditions on α. Our main result is as follows.

Theorem 1.2. *Let $k \in \mathbb{N}, k \geq 5, \epsilon > 0$ and let $P, H \in \mathbb{R}, P \geq 2$,*

$$2P^{k\sigma(k)} \leq H \leq P^{k/2+2k\sigma(k)}. \tag{6}$$

Suppose that α is a real number, and there exist integers a, q with

$$|q\alpha - a| < H^{-1}, \quad 1 \leq q \leq H, \quad (a,q) = 1. \tag{7}$$

Then

$$\left|\sum_{p\sim P} e(\alpha p^k)\right| \ll_{k,\epsilon} P^{1-\sigma(k)+\epsilon} + \frac{q^\epsilon w_k(q)^{\frac{1}{2}} P(\log P)^{20}}{(1+P^k|\alpha - a/q|)^{\frac{1}{2}}}. \tag{8}$$

The restriction to $k \geq 5$ is caused by parts of the proof which require $\sigma(k) \leq (8k+2)^{-1}$. For larger k one should be able to obtain an exponent $\sigma(k) \simeq (4.5k^2 \log k)^{-1}$ by using [13] in place of Weyl's inequality (used implicitly in Lemma 3.1 below). As applications of this result we note that the case $k = 5$ leads to a major simplification of the proof of Theorem 2 in [7], and, for general $k \geq 5$ this result should lead to improved bounds for solving inequalities involving powers of primes. The main factor in our strengthening of Theorem 1 is our estimation of Type I sums (Lemma 2 below), which in some ways is less sophisticated than the corresponding result in [7], but the injection of a simple idea from [1] enlarges the permissible range of parameters.

Before stating our final result we will describe what is known when α is rational. It turns out that one such result combines well with Theorem 1.2 to produce a strong bound for trigonometric sums over primes. In [3] the author remarked that the bound

$$\left|\sum_{p\leq P} \log p \; e\left(\frac{ap^k}{q}\right)\right| \ll_{k,\epsilon} Pq^\epsilon (\log P)^{\frac{7}{2}} \left(q^{-\frac{1}{2}} + q^{\frac{1}{2}}P^{-\frac{1}{2}} + P^{-\frac{1}{4}}q^{\frac{1}{8}}\right) \tag{9}$$

follows from work of Vaughan [8]. This is non-trivial for $(\log p)^{7+\epsilon} < q < P^{1-\epsilon}$, and greatly improves on Theorem 1.2 for small q. Fouvry and Michel [2] proved quite a general result which shows, as one example, that, for q a prime, $q > P, k \geq 2$,

$$\left|\sum_{p\leq P} \log p \; e\left(\frac{ap^k}{q}\right)\right| \ll_{k,\epsilon} q^{3/16+\epsilon} P^{25/32}.$$

This result is only non-trivial for the short range $P < q < P^{\frac{7}{6}}$, but it improves our results in a narrow range. It is interesting to note that the exponents of both the above results for rational α are independent of k. The bound (9) can be combined with Theorem 1.2 to produce the following.

Theorem 1.3. *Suppose that $k \geq 5, P \geq 2, \epsilon > 0$ and $\alpha \in \mathbb{R}$ such that (2) holds. Then*

$$\left|\sum_{p\sim P} e(\alpha p^k)\right| \ll_{k,\epsilon} P^{1+\epsilon-\sigma(k)} + \tag{10}$$
$$+\ P(\log P)^{17}\left(q^{-\tau_1(k)+\epsilon} + (q/P^k)^{\tau_2(k)-\epsilon}\right).$$

Here

$$\tau_1(k) = \frac{1}{6} + \frac{1}{3k}, \qquad \tau_2(k) = \frac{1}{8} + \frac{4}{8k-2}.$$

Moreover, the bound (10) remains valid for $k = 4$ with $\sigma(k)$ replaced by $\frac{1}{32}$.

Remark. This improves on all previous bounds, not just by virtue of improving the term independent of q, but also in the way the bound behaves for small or large q. Indeed, for very small or very large q, (10) gives a better bound than can be obtained for the standard Weyl sum. The fact that $\tau_1(k)$ and $\tau_2(k)$ do not tend to zero with increasing k is very unusual. Using Theorem 1.2 alone we would only obtain $\tau_1(k) = \tau_2(k) = \frac{1}{2k}$. The explanation for this phenomenon would seem to lie in the underlying model complete sum with which the exponential sum is compared, either

$$\sum_{\substack{n=1\\(n,q)=1}}^{q} e\left(\frac{an^k}{q}\right) \qquad \text{or} \qquad \sum_{n=1}^{q} e\left(\frac{an^k}{q}\right),$$

for which we know very different bounds hold in general.

Proof. (Theorem 1.3) If $P^{\frac{1}{4}} \leq q \leq P^{\frac{k}{2}}$ the result follows immediately from Theorem 1.2. Otherwise we can pick u, v with

$$\left|\alpha - \frac{u}{v}\right| < \frac{1}{vP^{\frac{k}{2}}}, \quad (u,v) = 1, \quad v \leq P^{\frac{k}{2}}.$$

If $v \geq P^{\frac{1}{4}}$ the result again follows from Theorem 1.2 since $8k \leq \sigma(k)^{-1}$. We now suppose that $v < P^{\frac{1}{4}}$. Writing

$$S = \left|\sum_{p\sim P} e\left(\alpha p^k\right)\right|,$$

we can suppose that $S \gg P^{1-\sigma+\epsilon}$. Let $R = 1 + P^k|\alpha - u/v|$. Then, by (9) with partial summation,

$$S \ll RP(\log P)^{\frac{7}{2}} v^{\epsilon-\frac{1}{2}}, \tag{11}$$

while (8) (or (5) if $k = 4$) gives

$$S \ll \frac{Pv^{\epsilon - \frac{1}{2k}}}{R^{\frac{1}{2}}} (\log P)^{20}. \tag{12}$$

From (11) and (12) we deduce that, for any $\beta \in [0,1]$,

$$S \ll Pv^{\epsilon} \left(\frac{(\log P)^{\frac{7}{2}} R}{v^{\frac{1}{2}}} \right)^{1-\beta} \left(\frac{(\log P)^{20} v^{-\frac{1}{2k}}}{R^{\frac{1}{2}}} \right)^{\beta}. \tag{13}$$

We first consider the case $q < P^{\frac{1}{2}}$. Now we cannot have $v < q/2$ for otherwise we get the contradictory sequence of inequalities:

$$\frac{2}{q^2} \le \left| \frac{a}{q} - \frac{u}{v} \right| \le \left| \alpha - \frac{a}{q} \right| + \left| \alpha - \frac{u}{v} \right| < \frac{1}{q^2} + \frac{1}{vP^{k/2}} < \frac{2}{q^2}.$$

We choose $\beta = \frac{2}{3}$ in (13) (to eliminate R) and then

$$S \ll Pv^{\epsilon - 1/6 - 1/3k} (\log P)^{15} \ll Pq^{\epsilon - \tau_1(k)} (\log P)^{15}.$$

This establishes (10) for this case, so we henceforth suppose that $q > P^{\frac{k}{2}}$. From the first case (with q replaced by v) we may also suppose that $v < P^k/q$, and so $v < \min(P^k/q, P^{\frac{1}{4}})$. It follows that (noting $q > P^2 > 2P^{\frac{1}{4}} > 2v$)

$$\left| \alpha - \frac{u}{v} \right| \ge \left| \frac{a}{q} - \frac{u}{v} \right| - \left| \alpha - \frac{a}{q} \right| \ge \frac{1}{qv} - \frac{1}{q^2} > \frac{1}{2qv}.$$

Hence $vR \gg P^k q^{-1}$.

We now choose $\alpha = 3k/(4k-1)$ in (13) (to equalise the powers of R and v) and deduce that

$$S \ll Pv^{\epsilon} (\log P)^{17} (RV)^{-\tau_2(k)} \ll P \left(\frac{P^k}{q} \right)^{\epsilon} (\log P)^{17} \left(\frac{P^k}{q} \right)^{-\tau_2(k)}.$$

This establishes (10) for this case and so completes the proof. □

2. Theorem 1.2 deduced from 3 lemmas

We write $\Lambda(n)$ for the von Mangoldt function. By a familiar argument (partial summation) it suffices to show that, for $P' \sim P$,

$$\sum_{P \le n \le P'} \Lambda(n) e(\alpha n^k) \ll P^{1-\sigma(k)+\epsilon} + \frac{q^{\epsilon} w_k(q)^{\frac{1}{2}} P \log^{20} P}{(1 + P^k |\alpha - a/q|)^{\frac{1}{2}}}. \tag{14}$$

An application of Heath-Brown's generalised Vaughan identity [6] as on page 172 of [1] gives the following decomposition of the left hand side of (14). In the following we write $\tau_k(n)$ for the number of ways of writing n as a product of k positive integers with the convention that $\tau_1(n) \equiv 1$. Also, whenever we write $\log x$, we mean $\max(1, \log x)$.

Lemma 2.1. *Let $0 < \gamma < \frac{1}{12}$ and suppose that we have the Type I estimate*

$$\sum_{m \le M} |a_m| \max_{N \le 2P/m} \left| \sum_{P/m \le n < N} e(\alpha m^k n^k) \right| \ll A, \tag{15}$$

for $M \le 32P^{\frac{1}{2}+\gamma}$, and the Type II estimate

$$\left| \sum_{m \sim M} a_m \sum_{P/m \le n < P'/m} b_n e(\alpha m^k n^k) \right| \ll B, \tag{16}$$

for $\frac{1}{8}P^{\frac{2}{3}} \le M \le 32P^{1-2\gamma}, P' \sim P$. Here

$$\begin{gathered} |a_m| \ll \tau_u(m)(\log m)^g, \quad |b_n| \ll \tau_v(n)(\log n)^{1-g}, \\ u+v \le 6,\ u,v \ge 1, \quad g \in \{0,1\}. \end{gathered} \tag{17}$$

Then

$$\sum_{P \le n < P'} \Lambda(n) e(\alpha n^k) \ll (\log P)^6 (A+B). \tag{18}$$

The following result, proved in section 4 contains the main improvement on [7].

Lemma 2.2. *Let $\epsilon > 0$. Suppose H, P, M are positive reals with $2 \le P$, (6) holds, and*

$$1 \le M \le 32P^{\frac{1}{2}+\sigma(k)}. \tag{19}$$

Let α be a real number such that there exist integers a, q such that (7) holds. Then, if $a_m \ll \tau_5(m)\log m$, we have

$$\begin{aligned} &\sum_{m \le M} |a_m| \max_{N \le 2P/m} \left| \sum_{P/m \le n < N} e(\alpha m^k n^k) \right| \\ &\qquad \ll P^{1-\sigma(k)+\epsilon} + \frac{q^\epsilon w_k(q) P (\log P)^5}{(1+P^k|\alpha - a/q|)}. \end{aligned} \tag{20}$$

Remark. In Lemma 3.2 of [7] a trilinear sum is considered and a bound is given which is stronger than (20). However, we can take M larger than in [7] and this enables us to utilize the result of Lemma 2.1. Our advantage is thus obtained by establishing a weaker result for a larger range of parameters. The reader will see that it is possible to improve the first term on the right hand side of (20), but this would complicate the proof yet not strengthen the result of our main theorem.

Our final component in the proof of Theorem 1.2 is the following slightly revised version of Lemma 3.1 in [7]. The proof will be given in section 3.

Lemma 2.3. *Let $\epsilon > 0$. Suppose H, P, M are positive reals with*

$$P^{\frac{1}{2}} \leq M \leq 32P^{1-2\sigma(k)}, \quad P^{2\sigma(k)} < H < P^{k-2k\sigma(k)}. \tag{21}$$

Suppose that a_m, b_n satisfy (17) and α, a, q satisfy (7). Then, for $P' \sim P$

$$\begin{aligned} &\left| \sum_{m \sim M} a_m \sum_{P/m \leq n \leq P'/m} b_n e(\alpha m^k n^k) \right| \\ &\qquad \ll P^{1-\sigma(k)+\epsilon} + P^{1+\epsilon} M^{-2^{-k}} + \frac{q^\epsilon w_k(q)^{\frac{1}{2}} P (\log P)^{14}}{(1 + P^k |\alpha - a/q|)^{\frac{1}{2}}}. \end{aligned} \tag{22}$$

Proof. (Theorem 1.2) We apply Lemma 2.1 with $\gamma = \sigma(k)$. We can use Lemma 2.2 to estimate (15) and Lemma 2.3 is applicable to (16) (replacing ϵ by $\epsilon/2$ in those lemmas). This establishes (18) with

$$A + B \ll P^{1-\sigma(k)+\epsilon/2} + \frac{q^\epsilon w_k(q)^{\frac{1}{2}} P \log^{14} P}{(1 + P^k |\alpha - a/q|)^{\frac{1}{2}}}.$$

and so (8) follows. □

3. Proof of Lemma 2.3

First we need an important lemma which is a more general version of Lemma 2.1 in [7] (where ρ is fixed as 2^{1-k}) but which follows in the same way from Lemmas 2.4, 6.1 and 6.2 of [9].

Lemma 3.1. *Let $\rho \in (0, 2^{1-k}], \epsilon > 0, X \geq 1$. Then, either*

$$S = \left| \sum_{n \sim X} e(\alpha n^k) \right| \ll_{\epsilon,k} X^{1-\rho+\epsilon},$$

or there exist integers a, q with

$$1 \leq q \ll X^{k\rho}, \quad |\alpha q - a| \ll X^{k\rho - k}, \quad (a, q) = 1,$$

and

$$S \ll_{\epsilon,k} \frac{w_k(q) X}{1 + X^k |\alpha - \frac{a}{q}|}.$$

Proof. (Lemma 2.3) We need to make two changes to the proof of Lemma 3.1 in [7]: first to allow for the changed conditions on α, a, q; second to allow for the changed conditions on a_m, b_n. We outline the steps in the proof to help the reader follow the argument.

Write

$$T = \min\left(P^{2\sigma(k)}, M^{2^{1-k}}\right).$$

We begin with a standard application of Cauchy's inequality to the left hand side of (20) together with a familiar shuffling of the orders of summation:

$$\begin{aligned}
&\left|\sum_{m\sim M} a_m \sum_{P/m\le n<P'/m} b_n e(\alpha m^k n^k)\right|^2 \\
&\le \sum_{m\sim M} |a_m|^2 \sum_{m\sim M}\left|\sum_{P/m\le n<P'/m} b_n e(\alpha m^k n^k)\right|^2 \\
&\ll M(\log M)^{u^2-1+2g} \sum_{N<n_1,n_2\le 4N} |b_{n_1} b_{n_2}| \left|\sum_{m\in I(n_1,n_2)} e((n_2^k-n_1^k)m^k\alpha)\right| \\
&\ll M(\log M)^{u^2+1} \sum_{N<n_1,n_2\le 4N} \tau_v(n_1)^2 \left|\sum_{m\in I(n_1,n_2)} e((n_2^k-n_1^k)m^k\alpha)\right|.
\end{aligned}$$

Here $N = P/2M$ and

$$I(n_1,n_2) = (M,2M)\cap(\max\{P/n_1,P/n_2\},\min\{P'/n_1,P'/n_2\}).$$

The terms with $n_1 = n_2$ can only be dealt with trivially and this leads to a $(PM)^{\frac{1}{2}+\epsilon}$ term which is $\ll P^{1-\sigma+\epsilon}$ in view of the upper bound for M contained in (21). We then assume that the sum over m above is $\ge MP^\epsilon/T$ and this shows, by Lemma 3.1, that we can approximate $(n_2^k - n_1^k)\alpha$ by a fraction $\frac{b}{r}$ where

$$1\le r\le TP^{-\epsilon/2},\quad |(n_2^k-n_1^k)\alpha r-b|<M^{-k}TP^{2-\epsilon/2},\quad (b,r)=1.$$

In the following we closely follow [6] and assume for the moment that in place of $\tau_v(n_1)$ we have b_n satisfying the condition ($b_n \ll \log n$) which occurs in that paper. We take out the factor $n_0 = (n_1, n_2)$, put $n = n_1/n_0$, $l = (n_2 - n_1)/n_0$ and write

$$D=\frac{(n+l)^k-n^k}{l},\qquad \frac{b}{rD}=\frac{c}{s},\quad (c,s)=1.$$

Terms with $n_0 > TP^{-\epsilon}$ can clearly be neglected, so we henceforth suppose $n_0 \le TP^{-\epsilon}$. Following the argument on page 11 of [7] we obtain a suitable bound, except possibly when (see (3.13) in [7])

$$1\le s\le T^k,\quad\text{and}\quad |sn_0^k l\alpha-c|\le \tfrac{1}{2}T^kM^{-1}P^{1-k}.$$

We deduce from Dirichlet's approximation theorem that there exist $d\in\mathbb{Z}$ and $t\in\mathbb{N}$ with

$$1\le t\le P^{k-1}MT^{-k},\quad |tn_0^k\alpha-d|\le T^kM^{-1}P^{1-k},\quad (d,t)=1.$$

As in [7], the sums to be estimated are of a suitable size, unless (see (3.18))

$$1 \le t \le \tfrac{1}{2}T^k \quad \text{and} \quad |tn_0^k\alpha - d| \le T^k P^{-k}.$$

Now the conditions on α, a, q are only needed in [7] (see page 13) to show that (see the display after (3.19)),

$$|n_0^k ta - dq| < 1 \qquad (\text{hence } n_0^k ta = dq). \tag{23}$$

In our situation we have

$$\begin{aligned} |n_0^k ta - dq| &= q\left|tn_0^k\alpha - d + n_0^k t\left(\frac{a}{q} - \alpha\right)\right| \\ &\le q|tn_0^k\alpha - d| + n_0^k tH^{-1} \\ &< \frac{1}{2} + \frac{1}{2} \end{aligned}$$

from (21). Hence (23) is satisfied in our case.

Now we deal with the changed conditions on b_n which, with the altered hypothesis for a_m eventually lead to the higher log powers in our result. This means that we must include a factor $\tau_v(n_1)^2$ in S_1 on page 9 of [7]. Now, if $q \ge P^\epsilon$ we can use $\tau_v(n_1)^2 \ll q^{\epsilon/2}$ and the proof goes through exactly as in [7]. If $q < P^\epsilon$ then we must persist with this factor. Thus the inequality (3.8) in [7] becomes, upon writing $N_0 = N/n_0$,

$$S_2 = \sum_{n_1,n_2} \frac{w_k(r)M\tau_v^2(n_1)}{1 + M^k|(n_2^k - n_1^k)\alpha - b/r|}$$

$$\ll \sum_{1\le n_0\le 4N}\sum_{1\le l\le 4N_0} \frac{M}{1 + M^k N_0^{k-1}|n_0^k l\alpha - c/s|} \sum_{\substack{1\le n\le 4N_0 \\ (n,l)=1}} w_k(s/(s,D))\tau_v^2(n_0 n).$$

We note that $\tau_v(n_0 n) \le \tau_v(n_0)\tau_v(n)$. Also,

$$\begin{aligned} \sum_{\substack{1\le n\le 4N_0 \\ (n,l)=1}} w_k(s/(s,D))\tau_v^2(n) &\le \sum_{s_0|s} w_k(s/s_0) \sum_{\substack{1\le n\le 4N_0 \\ (n,l)=1,\, D\equiv 0 \pmod{s_0}}} \tau_v^2(n) \\ &\ll s^\epsilon(w_k(s)N_0 + 1)(\log N)^{v^2-1} \end{aligned}$$

by making a suitable modification to the argument on page 10 of [7] to deal with the $\tau_v(n)$ factor. The factor $\tau_v(n_0)^2$ is then carried through the argument until at last one has to estimate the sum (see the un-numbered display before (3.20) in [7])

$$\sum_{1\le n_0\le T} w_k(q/(q,n_0^k))n_0^{-2}\tau_v(n_0)^2.$$

However, since $\tau_v(n_0) \ll n_0^{\frac{1}{4}}$ this sum is

$$\ll \sum_{1 \le n_0 \le T} w_k(q/(q, n_0^k)) n_0^{-\frac{3}{2}} \ll q^\epsilon w_k(q)$$

by modifying Lemma 2.3 of [7]. Combining the results we obtain

$$\left| \sum_{m \sim M} a_m \sum_{n \le P/m} b_n e(\alpha m^k n^k) \right|^2$$
$$\ll P^{2-2\sigma(k)+\epsilon} + P^2 M^{2\epsilon - 2^{1-k}} + \frac{q^\epsilon w_k(q) P^2 (\log P)^h}{1 + P^k |\alpha - a/q|}.$$

Here $h = 2 + v^2 + u^2$ and so the worst case has $u = 1, v = 5$ (or vice versa) giving $h = 28$ as required to establish (22). $\square$

4. Proof of Lemma 2.2

First we need a further two lemmas.

Lemma 4.1. *Let $P \ge R \ge 2, \epsilon > 0, k \in \mathbb{N}, k \ge 4, \delta > 0$. Then*

$$\sum_{\substack{w_k(r) > P^{-\delta} \\ r \le R}} w_k(r) \ll_{\epsilon,k} P^{\epsilon+\delta}. \tag{24}$$

Proof. We may assume that $\epsilon < \frac{1}{k}$. We have, since $w_k(r) \ge r^{-\frac{1}{2}}$,

$$\sum_{\substack{w_k(r) > P^{-\delta} \\ r \le R}} w_k(r) \le \sum_{r \le R} w_k(r)^{2+3\epsilon} P^\delta r^{3\epsilon/2} < P^{\delta+\epsilon} \sum_{r \le R} r^\epsilon w_k(r)^{2+3\epsilon} \ll P^{\delta+\epsilon}$$

by Lemma 2.4 of [7]. $\square$

Lemma 4.2. *Suppose $\eta, \epsilon, \theta > 0, d \ge 2, M \ge 1$. Say*

$$|d\alpha - c| < \eta, \quad (c, d) = 1.$$

Then, given $r < d$, the number of solutions to

$$||m^k r\alpha|| < \theta, \quad m \sim M, \ (m, d) = 1, \tag{25}$$

is

$$\ll_{\epsilon,k} d^\epsilon (d + M) \left(\theta + \frac{\eta M^k r}{d} \right). \tag{26}$$

Proof. This simple bound follows by the method of Lemma 6 in [1]. To see this, note that when (25) is satisfied

$$\left\| m^k r \frac{c}{d} \right\| \le \theta + \frac{\eta (2M)^k r}{d}.$$

We then need to count the number of solutions to

$$m^k r \equiv a \pmod d, \quad (m,d)=1,\ m \le M,\ |a| \le 2^k \eta M^k r + \theta d.$$

This number is

$$\ll \left(\frac{M(d,r)}{d}+1\right) d^{\epsilon} \left(\frac{\eta M^k r}{(d,r)} + \theta \frac{d}{(d,r)}\right)$$

which gives (26). □

Proof. (Lemma 2.2) We start in a similar manner to the proof of Lemma 3.2 in [7]. Write $Q = P^{k\sigma(k)}$. By Lemma 3.1 with $X = P/m$ and ρ defined by $X^{1-\rho} = P^{1-\sigma(k)}/m$, for each m, either

$$\left|\sum_{P/m \le n < N} e(\alpha m^k n^k)\right| \ll \frac{P^{1-\sigma(k)+\epsilon/2}}{m}, \tag{27}$$

or there exist integers b, r with

$$(b,r)=1, \quad 1 \le r \le Q, \quad |rm^k\alpha - b| \le \tfrac{1}{2}(P/m)^{-k} Q. \tag{28}$$

Here we have changed a term $\ll P^{-\epsilon}$ to $< \frac{1}{2}$ since we may assume that P is sufficiently large, and we require an explicit constant less than 1 later in the proof. Evidently we can ignore those m for which (27) holds and concentrate our attention on those m for which we obtain (28). For each $r \le Q$ let $\mathcal{M}(r)$ be the set of all $m \sim M$ for which (28) holds. Then

$$\begin{aligned} &\sum_{m \sim M} |a_m| \max_{N \le 2P/m} \left|\sum_{P/m \le n < N} e(\alpha m^k n^k)\right| \\ &\ll P^{1-\sigma(k)+\epsilon} + \sum_{1 \le r \le Q} \sum_{m \in \mathcal{M}(r)} |a_m| \max_{N \le 2P/m} \left|\sum_{P/m \le n < N} e(\alpha m^k n^k)\right| \\ &= P^{1-\sigma(k)+\epsilon} + S_1 \quad \text{say.} \end{aligned} \tag{29}$$

Now, by Lemma 3.1,

$$S_1 \ll \sum_{r \le Q} \sum_{m \in \mathcal{M}(r)} |a_m| \frac{w_k(r) P}{m(1+(P/m)^k |m^k\alpha - \frac{b}{r}|)}.$$

Clearly we can absorb the contribution from pairs (m,r) with

$$w_k(r) < P^{-\sigma(k)} \tag{30}$$

into the first term on the right of (29). So henceforth we assume (30) is violated and denote the corresponding sum by S_2. Now there exist c, d with

$$(c,d)=1, \quad d \le 2P^{\frac{k}{2}} Q^2, \quad |d\alpha - c| < \tfrac{1}{2} P^{-\frac{k}{2}} Q^{-2}.$$

Moreover, if $d < H/2$, we have

$$\begin{aligned}|da - cq| &= |d\alpha q - cq + (dq(\tfrac{a}{q} - \alpha))| \\ &\le q|d\alpha - c| + d|q\alpha - a| \\ &< 1,\end{aligned}$$

and so $d = q, c = a$ in this case. Now write

$$\mathcal{M}(r) = \sum_{e|d} \mathcal{M}(r,e)$$

where $\mathcal{M}(r,e)$ counts those $m \in \mathcal{M}(r)$ with $(m,d) = e$. Clearly we can dispense with those e satisfying $e > \frac{1}{2}P^{\sigma(k)}$. We now split the argument into two cases.

(i) $e^k r < d$. We can apply Lemma 4.2 to show that

$$\begin{aligned}\mathcal{M}(r,e) &\ll d^\epsilon \left(d + \frac{M}{e}\right)\left(\left(\frac{P}{M}\right)^{-k} Q + \frac{M^k r}{dP^{\frac{k}{2}}Q^2}\right) \\ &\ll M^k d^\epsilon \left(\frac{dQ}{P^k} + \frac{1}{QP^{\frac{k}{2}}} + \frac{M}{Q^2P^{\frac{k}{2}}}\right).\end{aligned}$$

The contribution to S_2 from case (i) is therefore

$$\begin{aligned}&\ll \sum_{\substack{r \le Q \\ w_k(r) > P^{-\sigma(k)}}} w_k(r) P M^{k-1}\left(\frac{dQ}{P^k} + \frac{1}{QP^{\frac{k}{2}}} + \frac{M}{Q^2P^{\frac{k}{2}}}\right) \\ &\ll P^{1-\sigma(k)+\epsilon}\left(M^{k-1}P^{2\sigma(k)}\left(\frac{dQ}{P^k} + \frac{1}{QP^{\frac{k}{2}}} + \frac{M}{Q^2P^{\frac{k}{2}}}\right)\right) \\ &\ll P^{1-\sigma(k)+\epsilon}\left(QP^{\frac{k}{2}-\frac{1}{2}+\sigma(k)}\left(\frac{Q^3}{P^{\frac{k}{2}}} + \frac{1}{QP^{\frac{k}{2}}} + \frac{P^{\frac{1}{2}+\sigma(k)}}{Q^2P^{\frac{k}{2}}}\right)\right) \\ &\ll P^{1-\sigma(k)+\epsilon}\end{aligned}$$

where we have used Lemma 4.1, $M \le P^{\frac{1}{2}+\sigma(k)}$, and $d \le Q^2P^{\frac{k}{2}}$. We note that we also required

$$(8k+2)\sigma(k) \le 1$$

for the final inequality, and it is this point which limits our method to the case $k \ge 5$.

(ii) $e^k r \geq d$. For this case we have $d \leq \frac{1}{2}Q^2$ and thus $d = q, c = d$. Also

$$\begin{aligned}|rm^k a - bq| &= \left| qrm^k \left(\alpha - \frac{b}{m^k r}\right) - qrm^k \left(\alpha - \frac{a}{q}\right)\right| \\ &\leq q\left(\frac{P}{M}\right)^{-k} + \tfrac{1}{2}M^k Q P^{-\frac{k}{2}} Q^{-2} \\ &< 1.\end{aligned}$$

Hence $\frac{b}{rm^k} = \frac{a}{q}$, and so $w_k(r) = w_k(q/(m^k, q))$. We can therefore estimate this case as

$$\begin{aligned}&\ll \sum_{m \leq M} \frac{|a_m| P w_k(q/(m^k, q))}{m(1 + (P/m)^k |\alpha m^k - b/r|)} \\ &= \frac{P}{1 + P^k |\alpha - a/q|} \sum_{m \leq M} \frac{|a_m|}{m} w_k\left(\frac{q}{(m^k, q)}\right).\end{aligned}$$

We now consider

$$S(M, q) = \sum_{m \leq M} \frac{|a_m|}{m} w_k\left(\frac{q}{(m^k, q)}\right).$$

If $q > M^\epsilon$ then $|a_m| \ll q^{\epsilon/2}$ and we obtain

$$S(M, q) \ll q^\epsilon w_k(q) \log P$$

from Lemma 2.3 in [7]. On the other hand, for $q < M^\epsilon$ we first define the multiplicative function $v_k(n)$ by $v_k(p^{uk+v}) = p^{u+1}$ for $u \geq 0$ and $1 \leq v \leq k$. We then have

$$\begin{aligned}S(M, q) &\ll \sum_{m \leq M} \frac{\tau_5(m)}{m} w_k\left(\frac{q}{(m^k, q)}\right) \\ &\ll \sum_{d|q} w_k\left(\frac{q}{d}\right) \sum_{\substack{m \leq M \\ m^k \equiv 0 \bmod d}} \frac{\tau_5(m)}{m} \\ &\ll \sum_{d|q} w_k\left(\frac{q}{d}\right) \sum_{\substack{m \leq M \\ m \equiv 0 \bmod v_k(d)}} \frac{\tau_5(m)}{m} \\ &\ll \sum_{d|q} w_k\left(\frac{q}{d}\right) \frac{\tau_5(v_k(d))}{v_k(d)} (\log M)^5 \ll q^\epsilon w_k(q) (\log P)^5\end{aligned}$$

by Lemma 2.2 of [7] since $\tau_5(v_k(d)) \ll q^{\epsilon/2}$. We thus have an estimate

$$\ll P w_k(q) q^\epsilon (\log P)^5$$

in this case. Combining (i) and (ii) then establishes (20) and completes the proof. □

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