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$*$-Sturmian words and complexity

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Résumé. Nous définissons des notions analogues à la complexité $p(n)$ et aux mots Sturmiens qui sont appelées respectivement $*$-complexité $p_*(n)$ et mots $*$-Sturmiens. Nous démontrons que la classe des mots $*$-Sturmiens coïncide avec la classe des mots satisfaisant à $p_*(n) \leq n+1$ et nous déterminons la structure des mots $*$-Sturmiens. Pour une classe de mots satisfaisant à $p_*(n) = n+1$, nous donnons une formule générale et une borne supérieure pour $p(n)$. En utilisant cette formule générale, nous donnons des formules explicites pour $p(n)$ pour certains mots appartenant à cette classe. En général, $p(n)$ peut prendre des valeurs élevées, à savoir $p(n) \geq 2^{n^{1-\epsilon}}$ pour certains mots $*$-Sturmiens. Cependant l'entropie topologique de n'importe quel mot $*$-Sturmien est nulle.

Abstract. We give analogs of the complexity $p(n)$ and of Sturmian words which are called respectively the $*$-complexity $p_*(n)$ and $*$-Sturmian words. We show that the class of $*$-Sturmian words coincides with the class of words satisfying $p_*(n) \leq n+1$, and we determine the structure of $*$-Sturmian words. For a class of words satisfying $p_*(n) = n+1$, we give a general formula and an upper bound for $p(n)$. Using this general formula, we give explicit formulae for $p(n)$ for some words belonging to this class. In general, $p(n)$ can take large values, namely, $p(n) \geq 2^{n^{1-\epsilon}}$ holds for some $*$-Sturmian words; however the topological entropy of any $*$-Sturmian word is zero.

1. Introduction

We announced results about $*$-Sturmian words as analogs of Sturmian words in [11]. In this paper, we give proofs for all the results given there together with some additional results. We define some notations. Let L be an alphabet, i.e., a non-empty finite set of letters. We denote by L^n the set of all finite words of length n over L, L^* denotes the set $\bigcup_{n=0}^{\infty} L^n$, where $L^0 = \{\lambda\}$ and λ is the empty word. $L^{\boldsymbol{N}}$ (resp. $L^{-\boldsymbol{N}}$) is the set of right-sided (resp. left-sided) infinite words over L. We define an equivalence

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relation $\sim$ on the set $L^{\boldsymbol{Z}}$ by: $W_1 \sim W_2$ (where $W_1, W_2 \in L^{\boldsymbol{Z}}$) if there exists an integer y such that

$$W_1(x+y) = W_2(x) \quad \text{for all } x \in \boldsymbol{Z}.$$

We mean by a two-sided infinite word over L an element of the set $L^{\boldsymbol{Z}}/\sim$.

We say that $W \in L^{\boldsymbol{Z}}/\sim$ is purely periodic if $W(x+y) = W(x)$ for all $x \in \boldsymbol{Z}$ for some fixed positive integer y. If $W = w_1 w_2 \cdots \in L^{\boldsymbol{N}}$ (resp. $W = \cdots w_{-2} w_{-1} \in L^{-\boldsymbol{N}}$) satisfy

$$w_i = w_{i+n} \tag{1.1}$$

for any sufficiently large (resp. small) i with some fixed positive integer n, we say that W is ultimately periodic with period n. The least period n of W is called its fundamental period and for sufficiently large (resp. small) i, the word $w_{i+1} \cdots w_{i+n}$ is also called a fundamental period. Especially, if (1.1) holds for all integer $i > 0$ (resp. $i < -n$), we say that W is purely periodic.

For any $W = \cdots w_{i+1} \cdots w_{i+n} \cdots \in L^{\wedge} := L^* \cup L^{\boldsymbol{N}} \cup L^{-\boldsymbol{N}} \cup L^{\boldsymbol{Z}}/\sim$ (where $w_j \in L,\ n \geq 0$), the word $w_{i+1} \cdots w_{i+n}$ is called a subword of W.

Definition 1.1. *We define $D(W) := \{V; V \text{ is a subword of } W\}$ and $D(n; W) := D(W) \cap L^n$ $(n \geq 0)$.*

Definition 1.2. *The complexity of a word $W \in L^{\wedge}$ is the function that counts the number of elements of $D(n; W)$:*

$$p(n) = p(n; W) := \sharp D(n; W).$$

For $W = \cdots w_i \cdots \in L^{\boldsymbol{N}} \cup L^{-\boldsymbol{N}} \cup L^{\boldsymbol{Z}}/\sim$, we say that a subword $w = w_{i+1} \cdots w_{i+n}$ $(n \geq 0)$ of W is a $*$-subword of W if w occurs infinitely many times in W, i.e.,

$$w_{i_j+1} \cdots w_{i_j+n} = w_{i+1} \cdots w_{i+n}$$

for $i_1 < i_2 < \cdots$ (or $\cdots < i_2 < i_1$).

Definition 1.3. *We define $D_*(W) := \{V; V \text{ is a } *\text{-subword of } W\}$ and $D_*(n; W) := D_*(W) \cup L^n$.*

Definition 1.4. *The $*$-complexity of a word $W \in L^{\wedge}$ is the function that counts the number of elements of $D_*(n; W)$:*

$$p_*(n) = p_*(n; W) := \sharp D_*(n; W).$$

In general, $p(n; W) \geq p_*(n; W)$ holds for all $W \in L^{\boldsymbol{N}} \cup L^{-\boldsymbol{N}} \cup L^{\boldsymbol{Z}}/\sim$. We remark that $p(n; W) = p_*(n; W)$ holds for billiard words W (also called cutting sequences) of dimension s, which are defined by billiards in the cube of dimension s with totally irrational direction $v \in \boldsymbol{R}^*$ (for the definition of these words, see [1] and [3]). This fact follows from Kronecker's theorem related to the distribution of the sequence $\{vn \bmod 1\}_{n=1,2,\ldots}$. It is well known that billiard words of dimension $s = 1$ coincide with Sturmian words defined below with some exceptions (for example, see [8]).

In what follows we assume that $L = \{0, 1\}$.

Definition 1.5. *A word $W \in L^{\boldsymbol{N}} \cup L^{-\boldsymbol{N}} \cup L^{\boldsymbol{Z}}/\sim$ is Sturmian, if W satisfies*

$$||A|_1 - |B|_1| \leq 1$$

for any $A, B \in D(n; W)$ for all $n \geq 0$, where $|w|_1$ denotes the number of occurrences of the symbol 1 appearing in the word $w \in L^*$, cf. [10].

Remark 1.1. We should use the term "balanced" instead of "Sturmian" if we followed the usual terminology. Note that the terminology "Sturmian" is used in the recent literature for the words whose complexity function is $p(n) = n + 1$; however we follow the terminology given by Morse and Hedlund in [9, 10], since they started from Definition 1.5 and showed that any Sturmian word has complexity function $p(n) = n + 1$ under a minor condition, and since our results for $*$-Sturmian given in Section 2 will be parallel to their results.

A $*$-Sturmian word W is defined to be a word satisfying the condition with $D_*(n; W)$ in place of $D(n; W)$ in the definition above, i.e.,

Definition 1.6. *A $*$-Sturmian word is defined to be a word $W \in L^{\boldsymbol{N}} \cup L^{-\boldsymbol{N}} \cup L^{\boldsymbol{Z}}/\sim$ satisfying*

$$||A|_1 - |B|_1| \leq 1$$

for any $A, B \in D_*(n; W)$ for all $n \geq 0$.

$*$-Sturmian words have been considered by a number of authors (see [4] and its references). There are some classical and well-known results on Sturmian words and words satisfying $p(n) \leq n + 1$ given by Morse, Hedlund and Coven, Hedlund. It is known that $p(n_0; W) \leq n_0$ ($W \in L^{\boldsymbol{N}}$) for some n_0 implies that W is ultimately periodic and any $W \in L^{\boldsymbol{Z}}/\sim$ with $p(n_0; W) \leq n_0$ for some n_0 is always purely periodic (see [9]). The class of words satisfying $p(n) \leq n + 1$ coincides with the class of Sturmian words with some explicit exceptions, cf. Theorems 2.1, 2.4, 2.5 below. Furthermore the authors above give a concrete description of Sturmian words, cf. Theorems 2.2, 2.3.

In this paper, we show that the class of $*$-Sturmian words coincides with the class of words satisfying $p_*(n) \leq n + 1$, cf. Theorems 2.6, 2.8. We also describe the structure of $*$-Sturmian words in a constructive manner. In [13] Yasutomi introduced super Bernoulli sequences as a generalization of Sturmian words. Super Bernoulli sequences coincide with $*$-Sturmian words in specific cases. But in [13] super Bernoulli sequences are not given in a constructive manner.

For completeness, we give our results (Theorems 2.6–2.11) together with classical results (Theorems 2.1-2.5) in Section 2. In Section 3, we give the proofs of Theorems 2.6–2.11.

For a class of words given by

$$W = 10^{a_1}10^{a_2}10^{a_3}\cdots, \quad 0 < a_1 < a_2 < a_3 \cdots,$$

which satisfy $p_*(n;W) = n+1$, we give a general formula for $p(n;W)$ and an upper bound: $p(n;W) \le \frac{n^2}{4} + \frac{n}{2} + \frac{17}{8} + \frac{(-1)^{n+1}}{8} - \lfloor(\frac{3}{4} + \frac{n}{4})^{-1}\rfloor$ $(n \ge 0)$, cf. Theorems 4.1, 4.2 in Section 4. Using the general formula, we give explicit formulae $p(n;W) = kn + c$ for some words belonging to this class and sufficiently large n, where k and c are constants, cf. Theorem 4.4. Also, using the general formula, there exist a word W and constants c_1 and c_2, such that $c_1 n^{1+1/\alpha} < p(n;W) < c_2 n^{1+1/\alpha}$ for any given $\alpha \ge 1$, cf. Theorem 4.3.

For a more general class of words given by (4.11), we can also give a general formula for $p(n;W)$, cf. Theorem 4.5 in Section 4.

In general, for W satisfying $p_*(n;W) \le n+1$, $p(n;W)$ can take large values, namely, $p(n;W) \ge 2^{n^{1-\epsilon}}$ holds for some W, cf. Theorem 5.2. On the other hand, any $*$-Sturmian word W is deterministic, i.e., the topological entropy $\lim_{n\to\infty} \dfrac{\log p(n;W)}{n}$ of W is zero, cf. Theorem 5.1. We give Theorems 5.1-5.2 together with their proofs in Section 5.

2. Characterization of Sturmian words and $*$-Sturmian words

2.1. Sturmian words. We put

$$\sigma(n;W) := \max_{A\in D(n;W)} |A|_1 \quad \text{and} \quad \sigma'(n;W) := \min_{A\in D(n;W)} |A|_1 \quad (W \in L^{\wedge}).$$

Theorem 2.1 (Morse and Hedlund [10]). *If W is a one-sided or two-sided infinite Sturmian word, then $p(n;W) \le n+1$, and the density $\alpha = \lim_{n\to\infty} \dfrac{\sigma(n,W)}{n} = \lim_{n\to\infty} \dfrac{\sigma'(n,W)}{n}$ exists.*

Now, we classify one-sided or two-sided infinite Sturmian words as follows:

(Type I) : α is irrational,
(Type II) : α is rational and W is purely periodic,
(Type III) : α is rational and W is not purely periodic.

It is known that each case can occur. The words of Type III are referred to as skew Sturmian words.

Definition 2.1. *Let $0 \leq \alpha \leq 1$ and β be real numbers. We define $G(n;\alpha,\beta) := \lfloor (n+1)\alpha+\beta\rfloor - \lfloor n\alpha+\beta\rfloor$ and $G'(n;\alpha,\beta) := \lceil (n+1)\alpha+\beta\rceil - \lceil n\alpha+\beta\rceil$, where $\lfloor x\rfloor$ is the greatest integer which does not exceed x and $\lceil x\rceil$ is the least integer which is not smaller than x. Obviously $G(n;\alpha,\beta)$, $G'(n;\alpha,\beta) \in \{0,1\}$. A word $G(\alpha,\beta)$ is defined by*

$$G(\alpha,\beta) := G(0;\alpha,\beta)G(1;\alpha,\beta)G(2;\alpha,\beta)\cdots G(n;\alpha,\beta)\cdots .$$

Similarly, $G'(\alpha,\beta)$ is defined by using $G'(n;\alpha,\beta)$. We set $G(\alpha) := G(\alpha,0)$, $G'(\alpha) := G'(\alpha,0)$, $G(n;\alpha) := G(n;\alpha,0)$ and $G'(n;\alpha) := G'(n;\alpha,0)$.

If α is rational, $G(\alpha,\beta)$ is obviously purely periodic.

Definition 2.2. *Let α be a rational number with $0 \leq \alpha \leq 1$. For $\alpha \neq 0,1$ we define $S(\alpha), S'(\alpha) \in L^{\boldsymbol{Z}}/\sim$ as follows:*

$$S(\alpha) = \ldots s(\alpha)_{-1}s(\alpha)_0\ldots s(\alpha)_n\ldots,$$

where

$$s(\alpha)_n = \begin{cases} \lceil (n+1)\alpha\rceil - \lceil n\alpha\rceil & \text{if } (n+1)\alpha < 1, \\ \lfloor (n+1)\alpha\rfloor - \lfloor n\alpha\rfloor & \text{if } (n+1)\alpha \geq 1, \end{cases}$$

and

$$S'(\alpha) = \ldots s'(\alpha)_{-1}s'(\alpha)_0\ldots s'(\alpha)_n\ldots,$$

where

$$s'(\alpha)_n = \begin{cases} \lfloor (n+1)\alpha\rfloor - \lfloor n\alpha\rfloor & \text{if } (n+1)\alpha < 1, \\ \lceil (n+1)\alpha\rceil - \lceil n\alpha\rceil & \text{if } (n+1)\alpha \geq 1. \end{cases}$$

For $\alpha = 0,1$ we define $S(0), S'(1)$ by

$$S(0) = \ldots s(0)_{-1}s(0)_0\ldots s(0)_n\ldots,$$
$$S'(1) = \ldots s(1)_{-1}s(1)_0\ldots s(1)_n\ldots,$$

where

$$s(0)_n = \begin{cases} 0 & \text{if } n \neq 0, \\ 1 & \text{if } n = 0, \end{cases}$$

$$s'(1)_n = \begin{cases} 1 & \text{if } n \neq 0, \\ 0 & \text{if } n = 0, \end{cases}$$

and we define $S'(0) = S(0)$ and $S(1) = S'(1)$.

Theorem 2.2 (Morse and Hedlund [10])**.** *If α is irrational (resp. rational), then $G(\alpha,\beta)$ and $G'(\alpha,\beta)$ are Sturmian words of Type I (resp. Type II). Conversely, if $W \in L^{\boldsymbol{N}}$ is a Sturmian word of type I or II with density $\alpha = \lim_{n\to\infty} \dfrac{\sigma(n,W)}{n}$, there exists a real number β such that $W = G(\alpha,\beta)$ or $W = G'(\alpha,\beta)$.*

Theorem 2.3 (Morse and Hedlund [10]).

(I) Let W be a two-sided infinite skew Sturmian word with density $\alpha = \frac{p}{q}$ $(p,\ q \in \boldsymbol{Z},\ (p,q) = 1)$. Then W is represented by $W = \cdots AA \cdots AACBB \cdots BB \cdots$ $(A, B, C \in L^q)$ with $|A|_1 = |B|_1 = p$, and $|C|_1 = p-1$ or $p+1$.

(II) Let W be a two-sided infinite skew Sturmian word with density $\alpha = \frac{p}{q}$ $(p,\ q \in \boldsymbol{Z},\ (p,q) = 1)$. Then, $W = S(\alpha)$ or $W = S'(\alpha)$. Conversely, for each rational number α with $0 \le \alpha \le 1$, $S(\alpha)$ and $S'(\alpha)$ are skew Sturmian words.

(III) Let D be a finite word and assume that the one-sided infinite word $W = DD \cdots DD \cdots$ is Sturmian. Then W can be extended to a two-sided infinite skew Sturmian word.

The converse of the assertion given in Theorem 2.1 does not hold, but the words $W \in L^\wedge$ satisfying $p(n;W) \le n+1$ for all $n \in \boldsymbol{N}$ are characterized by Coven and Hedlund, in particular, they showed the following

Theorem 2.4 (Coven and Hedlund [5]). *Let W be a one-sided infinite word and $p(n;W) = n+1$ for all $n \ge 0$. Then W is a Sturmian word.*

Theorem 2.5 (Coven and Hedlund [5]). *Let W be a two-sided infinite word and $p(n;W) = n+1$ for all $n \ge 0$ that is not Sturmian. Then there exist a number $m \ge 0$ and a word $B \in D(m;W)$ such that*

(1) Both $0B0$ and $1B1$ belong to $D(m+2;W)$ and one and only one of $0B1$ and $1B0$ belongs to $D(m+2;W)$, so that $aBa' \in D(m+2;W)$ and $a'Ba \notin D(m+2;W)$ with $a \neq a'$ $(a, a' \in L)$.

(2) aBa' occurs exactly once in W.

(3) If $aBa' = x_i \cdots x_{i+m+1}$, then

(3a) $W_R = x_{i+1}x_{i+2} \cdots$ is purely periodic and Sturmian and $i+1$ is the least integer such that $x_{i+1}x_{i+2} \cdots$ is purely periodic.

(3b) $W_L = \cdots x_{i+m-1}x_{i+m}$ is purely periodic and Sturmian and $i+m$ is the greatest integer such that $\cdots x_{i+m-1}x_{i+m}$ is purely periodic.

(4) If l_R, l_L are the lengths of the shortest periods of W_R, W_L, respectively, then $l_R + l_L = m+2$ and $(l_R, l_L) = 1$.

2.2. $*$-Sturmian words. We give a characterization of $*$-Sturmian words in terms of the $*$-complexity together with a description of $*$-Sturmian words by which we can construct any $*$-Sturmian word in Theorems 2.6, 2.8 below. We need some definitions.

For $A, B \in L^*$ we denote by $\{A, B\}^*$ the set

$$\{A, B\}^* := \{w_1 \cdots w_n;\ w_i = A \text{ or } B,\ n \ge 0\}.$$

We say a word $W \in \{a, b\}^*$ is strictly over $\{a, b\}$ if both a and b eventually occur in W. The notation w^* (resp. ${}^*w$) $(\lambda \neq w \in L^*)$ stands for the word

$w^* := www\cdots \in L^{\boldsymbol{N}}$ (resp. ${}^*w := \cdots www \in L^{-\boldsymbol{N}}$), w^n ($n \in \boldsymbol{N}$, $w \in L^*$) is the word $w^n := v_1v_2\cdots v_n$ where $v_i = w$ for all i. We mean by ${}^*vw$ (resp. vw^*) the word $({}^*v)w$ (resp. $v(w^*)$).

Definition 2.3. *We define the substitutions δ_0, δ_1 by*

$$\delta_0 : \begin{cases} 0 & \to & 0 \\ 1 & \to & 01 \end{cases}, \quad \delta_1 : \begin{cases} 0 & \to & 01 \\ 1 & \to & 1 \end{cases}.$$

δ_k can be extended to $L^\wedge$ by

$$\delta_k(W) := \cdots \delta_k(w_i)\cdots$$

for $W = \cdots w_i \cdots \in L^\wedge$. The map $\delta_k : L^\wedge \to L^\wedge$ is injective. Hence we can write $B = \delta_k^{-1}(A)$ if $A = \delta_k(B)$, $(A, B \in L^\wedge)$.

It is known that Sturmian words have deep relations with substitutions (see [4]).

Definition 2.4. *For $k_1, \dots, k_i \in \{0,1\}$, we define $A_i = A(k_1,\dots,k_i) := \delta_{k_1} \circ \cdots \circ \delta_{k_i}(0)$, $B_i = B(k_1,\dots,k_i) := \delta_{k_1} \circ \cdots \circ \delta_{k_i}(1)$ $(A_0 := 0,\ B_0 := 1)$.*

Let x be a rational number with $0 \le x \le 1$. The two-sided infinite words $\underline{G}(x)$ and $\overline{G}(x)$ are defined in [13] as follows ($\underline{G}(0)$ and $\overline{G}(1)$ are not defined). Recall the definition of $G(\cdot)$, cf. Definition 2.1.

Definition 2.5. *If x is a rational number, $0 < x = n/m \le 1$, $((n,m) = 1)$, let $\underline{x} = \underline{n}/\underline{m}$ denotes the greatest number satisfying $x > \underline{x}$ and $\underline{m} \le m$. We define*

$$\underline{G}(x) = \cdots G(-3;x)G(-2;x)G(-1;x)G(0;\underline{x})G(1;\underline{x})\cdots G(\underline{m}-1;\underline{x})G(x).$$

Definition 2.6. *If x is a rational number, $0 \le x = n/m < 1$, $((n,m) = 1)$, let $\overline{x} = \overline{n}/\overline{m}$ denotes the least number satisfying $x < \overline{x}$ and $\overline{m} \le m$. We define*

$$\overline{G}(x) = \cdots G(-3;x)G(-2;x)G(-1;x)G(0;\overline{x})G(1;\overline{x})\cdots G(\overline{m}-1;\overline{x})G(x).$$

We see that $S(x) = \overline{G}(x)$ and $S'(x) = \underline{G}(x)$ with some exceptions (see Section 3).

Definition 2.7. *For $W \in L^{\boldsymbol{N}} \cup L^{-\boldsymbol{N}} \cup L^{\boldsymbol{Z}}/\sim$ and $x \in [0,1]$, we define the following conditions*

(C1) $D_*(W) = D(G(x))$,

(C2) $D_*(W) = D(\underline{G}(x))$ *with* $x \in \mathbf{Q}$ *and* $x \ne 0$,

(C3) $D_*(W) = D(\overline{G}(x))$ *with* $x \in \mathbf{Q}$ *and* $x \ne 1$,

where $D(\cdot)$ and $D_*(\cdot)$ are defined in Definition 1.1 and Definition 1.3 respectively.

Remark 2.1. Words which satisfy one of the conditions (C1)–(C3) are super Bernoulli words. Super Bernoulli words play an important role for Markov spectra as shown in [13]. For any $x \in [0,1]$, $G(x)$ is a super Bernoulli word that satisfies Condition (C1).

Theorem 2.6. *Let $W \in L^{\mathbf{N}}$. Then the following four conditions are equivalent:*

(i) W is $*$-Sturmian.
(ii) $p_*(n;W) \leq n+1$ for all $n \geq 0$.
(iii) There exists a finite or infinite sequence $\kappa = \{k_1, k_2, \dots, k_i \dots\}$, $k_i \in \{0,1\}$ which satisfies the following equation,

$$W = \begin{cases} u_0 u_1 \cdots u_i \cdots, & (\kappa \textit{ is infinite}), \\ u_0 A_i^* \quad \textit{or} \quad u_0 B_i^*, & (\kappa \textit{ is finite}), \end{cases}$$

where $A_i = A(k_1, \cdots, k_i)$, $B_i = B(k_1, \cdots, k_i)$ are the words given in Definition 2.4, $u_0 \in L^*$, and each u_i is a certain finite word strictly over $\{A_i, B_i\}$ for all $i > 0$.
(iv) W satisfies one of the conditions (C1), (C2) or (C3) in Definition 2.7.

Remark 2.2. In condition (iii), if $p_*(m;W) = m+1$ for any m, then $W = u_0 u_1 \cdots u_i \cdots$. If $p_*(m;W) < m+1$ for some m, then $W = u_0 A_i^*$ or $u_0 B_i^*$ for some i and $p_*(n;W)$ is bounded. In condition (iv), if x is an irrational number or W satisfies Condition (C2) or (C3) in Definition 2.7, then $p_*(n;W) = n+1$ for all n. If x is a rational number and W satisfies condition (C1) in Definition 2.7, then $p_*(n;W)$ is bounded.

Theorem 2.7. *Let $W = w_1 w_2 \ldots \in L^{\mathbf{N}}$ be $*$-Sturmian. Then there exists $\alpha = \lim\limits_{n\to\infty} \frac{\sigma(n;W)}{n} = \lim\limits_{n\to\infty} \frac{\sigma'(n;W)}{n}$, and one of the conditions (C1)-(C3) in Definition 2.7 holds with $x = \alpha$.*

We give an example.

Example 2.1. Let $W = 010^2 10^3 10^4 10^5 1 \cdots$. Then, we see that

$$D_*(n;W) = \{0^l 10^m | l, m \geq 0, l+m = n-1\} \cup \{0^n\}. \tag{2.1}$$

By Definition 1.6 we see that W is $*$-Sturmian and $p_*(n;W) = n+1$. Let $k_i = 0$ for $i = 1, 2, \ldots$ and $A_i = A(k_1, \cdots, k_i)$, $B_i = B(k_1, \cdots, k_i)$ be the words given in Definition 2.4. Then, we have $A_i = 0$, $B_i = 0^{i-1}1$. Thus, we have

$$W = A_1 B_1 A_2 B_2 A_3 B_3 \cdots.$$

On the other hand, we have $\lim_{n\to\infty} \frac{\sigma(n;W)}{n} = \lim_{n\to\infty} \frac{\sigma'(n;W)}{n} = 0$. Using (2.1) and $\overline{G}(0) = \cdots 0001000 \cdots$, W satisfies Condition (C3) with $x = 0$ in Definition 2.7.

Theorem 2.8. *Let $W \in L^{\mathbf{Z}}$. Then the following three conditions are equivalent:*

(i) W is $*$-Sturmian.

(ii) There exists a finite or infinite sequence $\kappa = \{k_1, k_2, \ldots, k_i, \ldots\}$, $k_i \in \{0,1\}$ such that W has one of the following representations.

(1) $W = \cdots u_{-i} \cdots u_{-1} u_0 u_1 \cdots u_i \cdots$, κ is an infinite sequence,

(2) $W = \cdots u_{-i} \cdots u_{-1} u_0 A_j^*$, κ is an infinite sequence and $k_i = 0$ for all $i > j$,

(3) $W = {}^*A_j u_0 u_1 \cdots u_i \cdots$, κ is an infinite sequence and $k_i = 0$ for all $i > j$,

(4) $W = \cdots u_{-i} \cdots u_{-1} u_0 B_j^*$, κ is an infinite sequence and $k_i = 1$ for all $i > j$,

(5) $W = {}^*B_j u_0 u_1 \cdots u_i \cdots$, κ is an infinite sequence and $k_i = 1$ for all $i > j$,

(6) $W = {}^*A_j u_0 A_j^*$, κ is a finite sequence and k_j is its final term,

(7) $W = {}^*B_j u_0 B_j^*$, κ is a finite sequence and k_j is its final term,

where $A_i = A(k_1, \cdots, k_i)$, $B_i = B(k_1, \cdots, k_i)$ are the words given in Definition 2.4 and u_i and u_{-i} are certain finite words strictly over $\{A_i, B_i\}$ for $i > 0$ and $u_0 \in L^*$.

(iii) W satisfies one of the conditions (C1), (C2) or (C3) in Definition 2.7.

Theorem 2.9. *Let $W = \ldots w_{-1} w_0 w_1 w_2 \ldots \in L^{\mathbf{Z}}$ be $*$-Sturmian. Then there exists $\alpha = \lim_{n\to\infty} \frac{\sigma(n;W)}{n} = \lim_{n\to\infty} \frac{\sigma'(n;W)}{n}$, and one of the conditions (C1)-(C3) holds with $x = \alpha$.*

Theorem 2.10. *Let $W \in L^{\mathbf{Z}}$ be a $*$-Sturmian word. Then, $p_*(n;W) \leq n+1$ for all $n \geq 0$.*

We give an example.

Example 2.2. Let $W_1 = {}^*0010^2 10^3 1 \cdots$ and $W_2 = \cdots 0^3 10^2 1010^2 10^3 1 \cdots$. Then, we see that

$$D_*(n;W) = \{0^l 1 0^m \mid l, m \geq 0, l+m = n-1\} \cup \{0^n\}. \tag{2.2}$$

By Definition 1.6 we see that W_1 and W_2 are $*$-Sturmian and $p_*(n;W_1) = p_*(n;W_2) = n+1$. Let $k_i = 0$ for $i = 1, 2, \ldots$ and $A_i = A(k_1, \cdots, k_i)$, $B_i = B(k_1, \cdots, k_i)$ be the words given in Definition 2.4. Then, we have $A_i = 0$, $B_i = 0^{i-1}1$. Thus, we have

$$W_1 = {}^*A_1 A_1 B_1 A_2 B_2 A_3 B_3 \cdots, \; W_2 = \cdots A_3 B_3 A_2 B_2 A_1 A_1 B_1 A_2 B_2 A_3 B_3 \cdots.$$

On the other hand, we have $\lim_{n\to\infty} \frac{\sigma(n;W_i)}{n} = \lim_{n\to\infty} \frac{\sigma'(n;W_i)}{n} = 0$ for $i = 1, 2$. Using (2.2) and $\overline{G}(0) = \cdots 0001000 \cdots$, W_i $(i = 1, 2)$ satisfies Condition (C3) with $x = 0$ in Definition 2.7.

Theorem 2.11. *Let $W \in L^{\mathbf{Z}}$. Suppose that $p_*(n;W) \le n+1$ for all $n \ge 0$ and W is not a $*$-Sturmian word. Then, there exists a finite sequence $\{k_i\}_{i=1}^{j}$, and a word $u_0 \in L^*$ such that*

$$W = {}^*A_j u_0 B_j^*, \quad or \quad {}^*B_j u_0 A_j^*,$$

where $A_j = A(k_1, \cdots, k_j)$, $B_j = B(k_1, \cdots, k_j)$ are the words given in Definition 2.4.

We give an example.

Example 2.3. Let $W = {}^*01^*$. Then, we have

$$D_*(n;W) = \{0^n, 1^n\}.$$

We see easily that $p_*(n;W) = 2$ and W is not $*$-Sturmian.

3. Lemmas and the proof of Theorems 2.6-2.11

3.1. A dynamical system. We need some definitions to state lemmas.

Definition 3.1. *Let $I_0 := [0,1/2]$, $I_1 := (1/2,1]$, $\phi_0(x) := \frac{x}{x+1} \in I_0$, $\phi_1(x) := \frac{1}{2-x} \in I_1 \cup \{1/2\}$ $(x \in [0,1])$. Let T denotes the transformation on $[0,1]$ defined by*

$$T(x) := \begin{cases} \phi_0^{-1}(x) & \text{if } \ x \in I_0, \\ \phi_1^{-1}(x) & \text{if } \ x \in I_1. \end{cases}$$

The above ϕ_0, ϕ_1 and T have an important role in our paper. The following lemma gives a connection between ϕ_i and δ_i for $i = 0, 1$. We give a proof of Lemma 3.1 for completeness.

Lemma 3.1 (Ito, Yasutomi [7])**.** *For any $x \in [0,1]$, the equality $G(\phi_i(x)) = \delta_i G(x)$ $(i \in \{0,1\})$ holds.*

Proof. First, let us show $G(\phi_0(x)) = \delta_0 G(x)$. If $x = 0, 1$, then we see easily that the equality holds. Let $x \ne 0, 1$. Let $U = [-1, 0)$, $U_0 = [-1, -x)$ and $U_1 = [-x, 0)$. We define a transformation F on U as follows: for $y \in U$

$$F(y) := \begin{cases} y + x & \text{if } \ y \in U_0, \\ y + x - 1 & \text{if } \ y \in U_1. \end{cases}$$

We define an infinite word $j_0 j_1 \ldots j_n \ldots$ by

$$j_n := \begin{cases} 1 & \text{if } \ F^n(-1) \in U_1, \\ 0 & \text{if } \ F^n(-1) \in U_0. \end{cases}$$

Let us show that $G(m;x) = 1$ if and only if $F^m(-1) \in U_1$ for some non-negative integer m. First, we suppose $G(m;x) = 1$. Then, from Definition (2.1) we see that $mx < \lfloor (m+1)x \rfloor \le (m+1)x$. Therefore, we have $mx - \lfloor (m+1)x \rfloor \in U_1$. On the other hand, it is not difficult to see that $F^m(-1) - mx \in \mathbf{Z}$. Therefore, $F^m(-1) = mx - \lfloor (m+1)x \rfloor$ and $F^m(-1) \in$

U_1. Next, we suppose $G(m;x)=0$. Then, similarly, we have $F^m(-1)\in U_0$. Thus, we have $G(x)=j_0j_1\ldots$.

Let $V=[-1,x)$, $V_0=[-1,0)$ and $V_1=[0,x)$. We define a transformation h_0 on V as follows: for $y\in V$

$$h_0(y):=\begin{cases} y+x & \text{if} \quad y\in V_0,\\ y-1 & \text{if} \quad y\in V_1.\end{cases}$$

We define an infinite word $j_0^0j_1^0\ldots j_n^0\ldots$ by

$$j_n^0:=\begin{cases} 1 & \text{if} \quad h_0^n(-1)\in V_1,\\ 0 & \text{if} \quad h_0^n(-1)\in V_0.\end{cases}$$

Then, we see that

(1) if $y\in U_0$, then $y\in V_0$ and $F(y)=h_0(y)$,
(2) if $y\in U_1$, then $y\in V_0$, $h_0(y)\in V_1$ and $h_0^2(y)=F(y)$.

Therefore, we have $\delta_0(j_0j_1\ldots)=j_0^0j_1^0\ldots$. On the other hand, by using a map Θ from V to $[-1,0)$ defined by $\Theta(y)=\frac{y-x}{x+1}$, the dynamical system (V,h_0) is equivalent to the dynamical system $([-1,0),h_0')$ where the transformation h_0' on $\Theta(V)$ is defined as follows: for $y\in V$

$$h_0'(y):=\begin{cases} y+a & \text{if} \quad y\in\Theta(V_0)=[-1,-a),\\ y+a-1 & \text{if} \quad y\in\Theta(V_1)=[-a,0),\end{cases}$$

where $a=\frac{x}{x+1}$. Similarly we have $G(a)=j_0^0j_1^0\ldots$. Thus, we have $\delta_0(G(x))=G(\frac{x}{x+1})$.

Secondly, let us show $G(\phi_1(x))=\delta_1G(x)$. Let $V'=[-1,1-x)$, $V_0'=[-1,-x)$ and $V_1'=[-x,1-x)$. We define a transformation h_1 on V' as follows: for $y\in V'$

$$h_1(y):=\begin{cases} y+1 & \text{if} \quad y\in V_0',\\ y+x-1 & \text{if} \quad y\in V_1'.\end{cases}$$

We define an infinite word $j_0^1j_1^1\ldots j_n^1\ldots$ by

$$j_n^1:=\begin{cases} 1 & \text{if} \quad h_0^n(-1)\in V_1',\\ 0 & \text{if} \quad h_0^n(-1)\in V_0'.\end{cases}$$

We see that

(1) if $y\in U_0$, then $y\in V_0'$, $h_1(y)\in V_1'$ and $h_1^2(y)=F(y)$,
(2) if $y\in U_1$, then $y\in V_1'$ and $h_1(y)=F(y)$.

Therefore, we have $\delta_1(j_0j_1\ldots)=j_0^1j_1^1\ldots$. Similarly we have $j_0^1j_1^1\ldots=G(\frac{1}{2-x})$. Thus, we have $G(\phi_1(x))=\delta_1G(x)$. $\square$

The following Lemma 3.2 is obtained from Lemma 3.1.

Lemma 3.2 (Ito, Yasutomi [7]). *The following diagrams commute for* $k = 0, 1$*;*

$$\begin{tikzcd} {[0,1]} \arrow[r, "G"] \arrow[d, "\phi_k"'] & \mathcal{W} \arrow[d, "\delta_k"'] \\ I_k \arrow[r, "G"] & \mathcal{W}_k \end{tikzcd} \qquad \begin{tikzcd} {[0,1]} \arrow[r, "G"] & \mathcal{W} \\ I_k \arrow[r, "G"] \arrow[u, "T"] & \mathcal{W}_k \arrow[u, "\delta_k^{-1}"] \end{tikzcd},$$

where $\mathcal{W}$ *(resp.* $\mathcal{W}_k$*) is the image of* $[0,1]$ *(resp.* I_k*) by* G*.*

The assertion obtained by replacing, respectively, I_k and T by $\tilde{I}_k$ and $\tilde{T}$ in Lemma 3.2 can be shown in the same way as in [7].

Definition 3.2 (Itinerary of a real number). *We define the itinerary of* $x \in [0,1]$ *to be the sequence* $\{i_n\}_{n=1}^{\infty}$ *given by*

$$i(x) := \{i_n\}_{n=1}^{\infty},\ i_n = i_n(x) := \begin{cases} 0 & \text{if} \quad T^{n-1}(x) \in I_0, \\ 1 & \text{if} \quad T^{n-1}(x) \in I_1. \end{cases}$$

Lemma 3.3. *If* $x \in [0,1]$ *is an irrational number, then* 0 *and* 1 *occur infinitely many times in its itinerary. If* $x \neq 0$ *is a rational number, then there exists a natural number* j *such that* $T^l(x) = 1$ *and* $i_l(x) = 1$ *for any natural number* $l > j$*.*

Proof. Let $x \in [0,1]$ be an irrational number. We suppose that 0 or 1 does not occur infinitely many times in its itinerary. First, we suppose that 1 does not occur infinitely many times in its itinerary. Then, there exists an integer $k > 0$ such that $i_n = 0$ for each $n \geq k$. Let $n \geq k$. On the other hand, from Definition 3.2 and Lemma 3.2

$$T^{k-1}(x) = \phi_{i_k} \circ \cdots \circ \phi_{i_{n-1}} \circ \phi_{i_n} T^n(x).$$

Therefore, $T^{k-1}(x) \in \phi_0^{n-k+1}([0,1])$. We see easily that $\phi_0^{n-k+1}([0,1]) = [0, \frac{1}{n-k+2}]$. Since $T^{k-1}(x) \in \cap_{n=k}^{\infty}[0, \frac{1}{n-k+2}]$, $T^{k-1}(x) = 0$. From Definition 3.2 and Lemma 3.2 we have

$$x = \phi_{i_1} \circ \cdots \circ \phi_{i_{k-1}} T^{k-1}(x).$$

Therefore, $x \in \mathbf{Q}$. But this contradicts the assumption. Thus, 1 occurs infinitely many times in the itinerary of x. Similarly we see that 0 occurs infinitely many times in the itinerary of x. Secondly, let $x \in (0,1]$ be a rational number. We set $x = \frac{p}{q}$, where $p, q \in \mathbf{Z}$ and $p \geq 0, q > 0$ and p and q are relatively prime. We shall prove the lemma by induction on q. Let $q = 1$. Then, $x = 1$. We see easily that $i_n = 1$ for any integer $n > 0$. Next, we suppose that $q > 1$ and the lemma holds for each $y \in (0,1]$ whose denominator is less than q. Let x be in I_0. Then, $T(x) = \frac{y}{1-y} = \frac{q}{q-p}$. Since the denominator of $T(x)$ is less than x, from the induction hypothesis there exists an integer j such that $i_l(T(x)) = 1$ for any integer l with $l > j$. Therefore, we have $i_l(x) = 1$ for any integer l with $l > j + 1$. Secondly, let x be in I_1. Then, $T(x) = \frac{2y-1}{y} = \frac{2p-q}{p}$. Since the denominator of $T(x)$

is less than x, from the induction hypothesis there exists an integer j such that $i_l(T(x)) = 1$ for any integer l with $l > j$. Therefore, we have $i_l(x) = 1$ for any integer l with $l > j + 1$. Thus, we have the lemma. □

Lemma 3.4. *For any sequence $\{i_n\}_{n=1}^{\infty}$ $(i_n = 0, 1)$ in which 0 and 1 occur infinitely many times, there exists a unique irrational number x such that $\{i_n\}_{n=1}^{\infty} = i(x)$.*

Proof. For any positive integer n we set $\Delta_n = \phi_{i_1} \circ \cdots \circ \phi_{i_n}[0, 1]$. Then, $\Delta_1 \supset \Delta_2 \supset \Delta_3 \supset \cdots$. Since $[0, 1]$ is a compact set, there exists an x such that $x \in \cap_{n=1}^{\infty} \Delta_n$. It is not difficult to see that $i_n(x) = i_n$ for $n = 1, 2, \ldots$. Let $y \in \cap_{n=1}^{\infty} \Delta_n$. Let us show that $x = y$. We suppose that $x \neq y$. We suppose that $x < y$ without loss of generality. Then, apparently, for any $z \in [x, y]$, $i(z) = i(x)$. Since $\mathbf{Q} \cap [0, 1]$ is dense in $[0, 1]$, there exists a rational number z' such that $z' \in [x, y]$. Then Lemma 3.3 implies that there exists a natural number j such that $i_l(z') = 1$ for any natural number $l > j$. But this is a contradiction. Thus, we have the lemma. □

Lemma 3.5. *For any sequence $\{i_n\}_{n=1}^{\infty}, (i_n = 0, 1)$ in which 0 occurs finitely many times, there exists a rational number $x \neq 0$ such that $\{i_n\}_{n=1}^{\infty} = i(x)$.*

Proof. If for all $n \geq 1$, $i_n = 1$, then $i_n(1) = i_n$ for all $n \geq 0$. We suppose that 0 occurs in $\{i_n\}_{n=1}^{\infty}$. Then, there exists an integer $j > 1$ such that $i_{j-1} = 0$ and for any integer $l \geq j$, $i_l = 1$. We set $x = \phi_{i_1} \circ \cdots \circ \phi_{i_{j-1}}(1)$. Then we see that $\{i_n\}_{n=1}^{\infty} = i(x)$. □

Let $\tilde{I}_0 = [0, 1/2)$, $\tilde{I}_1 = [1/2, 1]$. We define $\tilde{T}(x)$ as $T(x)$ in Definition 3.1 with $\tilde{I}_i$ in place of I_i $(i = 0, 1)$ and $\tilde{i}(x) = \{\tilde{i}_n\}_{n=1}^{\infty}$, $(\tilde{i}_n = \tilde{i}_n(x))$ is defined in the same manner as $i(x)$ in Definition 3.2 with $\tilde{T}$ in place of T. Noting $T(x) = \tilde{T}(x), (x \neq 1/2)$, we can show the following

Lemma 3.6. *If x is irrational, then $i(x) = \tilde{i}(x)$. If $x \neq 1$ is a rational number, then there is a natural number j such that $\tilde{T}^l(x) = 0$ and $\tilde{i}_l(x) = 0$ for any natural number $l > j$.*

We remark that $i(x) \neq \tilde{i}(x)$ if x $(\neq 0, 1)$ is a rational number. The proof of the following lemma is similar to the proof of Lemma 3.5.

Lemma 3.7. *For any sequence $\{i_n\}_{n=1}^{\infty}$ in which 1 occurs finitely many times, there exists a rational number $x \neq 1$ such that $\tilde{i}(x) = \{i_n\}_{n=1}^{\infty}$.*

Lemma 3.8. *Let x be a rational number with $0 \leq x \leq 1$. Let $\{i_n\}_{n=1}^{\infty} = i(x)$ and $\{i'_n\}_{n=1}^{\infty} = \tilde{i}(x)$. Let for any integer $n > 0$*

$$A_n = \delta_{i_1} \circ \cdots \circ \delta_{i_n}(0), \qquad B_n = \delta_{i_1} \circ \cdots \circ \delta_{i_n}(1),$$
$$A'_n = \delta_{i'_1} \circ \cdots \circ \delta_{i'_n}(0), \qquad B'_n = \delta_{i'_1} \circ \cdots \circ \delta_{i'_n}(1),$$

and $A_0 = 0, B_0 = 1, A'_0 = 0$ *and* $B'_0 = 1$. *Let* $j \geq 0$ *be the least integer such that* $i_l = 1$ *for any* l *with* $l > j$. *Let* $j' \geq 0$ *be the least integer such that* $i'_l = 0$ *for any* l *with* $l > j'$. *Then, if* $x \neq 0$, $\underline{G}(x) = {}^*B_jA_jB_j^*$; *and if* $x \neq 1$, $\overline{G}(x) = {}^*A'_{j'}B'_{j'}A'_{j'}{}^*$.

Proof. Let $x \neq 0$. We set $x = \frac{p}{q}$ where $p \geq 0$ and $q > 0$ are integers and $(p, q) = 1$. Let us show that $(|A_n|, |A_n|_1) = 1$ and $(|B_n|, |B_n|_1) = 1$. From

$$A_n = \delta_{i_1} \circ \cdots \circ \delta_{i_n}(0), \qquad B_n = \delta_{i_1} \circ \cdots \circ \delta_{i_n}(1),$$

it follows

$$\begin{pmatrix} |A_n| & |B_n| \\ |A_n|_1 & |B_n|_1 \end{pmatrix} = M_0 M_{i_1} \cdots M_{i_n},$$

where

$$M_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad M_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Therefore, we get

$$|A_n||B_n|_1 - |B_n||A_n|_1 = 1. \tag{3.1}$$

On the other hand, Lemma 3.2 implies $G(x) = \delta_{i_1} \circ \cdots \circ \delta_{i_n}(G(T^n(x))$ for any integer $n \geq 0$. Since $T^k(x) = 1$ for $k > j$, we see that

$$G(x) = \delta_{i_1} \circ \cdots \circ \delta_{i_j}(G(1)) = \delta_{i_1} \circ \cdots \circ \delta_{i_j}(1^*) = B_j^*. \tag{3.2}$$

On the other hand, from the definition of $G(x)$

$$G(x) = G(0; x)G(1; x) \cdots = (G(0; \frac{p}{q})G(1; \frac{p}{q}) \cdots G(q-1; \frac{p}{q}))^*, \tag{3.3}$$

and

$$|G(0; \frac{p}{q})G(1; \frac{p}{q}) \cdots G(q-1; \frac{p}{q})|_1 = \sum_{j=0}^{q-1} \lfloor (j+1)\frac{p}{q} \rfloor - \lfloor j\frac{p}{q} \rfloor = p. \tag{3.4}$$

Using (3.2), (3.3), (3.4) and the fact $(|B_j|, |B_j|_1) = 1$ which is a consequence of (3.1), we see that $B_j = G(0; \frac{p}{q})G(1; \frac{p}{q}) \cdots G(q-1; \frac{p}{q})$. Thus, $|B_j| = q$ and $|B_j|_1 = p$. On the other hand,

$$A_j^* = \delta_{i_1} \circ \cdots \circ \delta_{i_j}(0^*) = G(\phi_{i_1} \circ \cdots \circ \phi_{i_j}(0)).$$

We set $\frac{p'}{q'} = \phi_{i_1} \circ \cdots \circ \phi_{i_j}(0)$, where $p' \geq 0$ and $q' > 0$ are integers with $(p', q') = 1$. Similarly, we have $A_j = G(0; \frac{p'}{q'})G(1; \frac{p'}{q'}) \cdots G(q'-1; \frac{p'}{q'})$. Thus, we see that $|A_j| = q'$, $|A_j|_1 = p'$. Since $j > 0$ and $i_j = 0$ for $x \neq 1$, we see that

$$\begin{aligned}|B_j| &= |\delta_{i_1} \circ \cdots \circ \delta_{i_j}(1)|\\ &= |\delta_{i_1} \circ \cdots \circ \delta_{i_{j-1}}(01)|\\ &= |A_j \delta_{i_1} \circ \cdots \circ \delta_{i_{j-1}}(1)|\\ &> |A_j|.\end{aligned}$$

Since $j = 0$ for $x = 1$, we have

$$|B_j| = |A_j| = 1.$$

Thus, we get that $p \geq p'$ and $pq' - qp' = 1$. Therefore, we obtain $\underline{G}(x) = {}^*B_j A_j B_j^*$. Similarly, we see that $\overline{G}(x) = {}^*A'_{j'} B'_{j'} A'_{j'}{}^*$ for $x \neq 0$. □

We show that $\overline{G}(x) = S(x)$ and $\underline{G}(x) = S'(x)$ hold with some exceptions in the following lemma.

Lemma 3.9. *Let x be a rational number with $0 \leq x \leq 1$. Then, $\overline{G}(x) = S(x)$ $(x \neq 1)$, and $\underline{G}(x) = S'(x)$ $(x \neq 0)$.*

Proof. Let $x \neq 1$. Let us show $\overline{G}(x) = S(x)$. If $x = 0$, then we see easily that $\overline{G}(x) = S(x)$. Let $0 < x < 1$. We set $x = \frac{p}{q}$, where $p > 0$ and $q > 0$ are integers with $(p, q) = 1$. Let $p' \geq 0$ and $q' > 0$ be integers with $q \geq p'$ and $p'q - pq' = 1$. Then, from the definition of $\overline{G}(x)$

$$\overline{G}(x) = \cdots G(-3; x)G(-2; x)G(-1; x)G(0; \frac{p'}{q'})G(1; \frac{p'}{q'}) \cdots G(q'-1; \frac{p'}{q'})G(x).$$

First, let us show that for $n < q - q'$, $s(x)_n = G(n + q'; x)$. Let $n < q - q'$. Since, for any integer m, $\lceil (m+1)\frac{p}{q} \rceil = \lceil m\frac{p}{q} \rceil$ if and only if $\lfloor (m+1)\frac{p}{q} - \frac{1}{q} \rfloor = \lfloor m\frac{p}{q} - \frac{1}{q} \rfloor$, we see that $\lceil (m+1)\frac{p}{q} \rceil - \lceil m\frac{p}{q} \rceil = \lfloor (m+1)\frac{p}{q} - \frac{1}{q} \rfloor - \lfloor m\frac{p}{q} - \frac{1}{q} \rfloor$. On the other hand, using $p'q - pq' = 1$ we have

$$\begin{aligned}G(n+q'; x) &= \lfloor (n+q'+1)\frac{p}{q} \rfloor - \lfloor (n+q')\frac{p}{q} \rfloor\\ &= \lfloor (n+1)\frac{p}{q} - \frac{1}{q} \rfloor - \lfloor n\frac{p}{q} - \frac{1}{q} \rfloor.\end{aligned}$$

Therefore, we get $s(x)_n = G(n + q'; x)$ for x such that $(n + 1)x < 1$. Let $(n + 1)x \geq 1$. Since $0 < n < q - q'$ and $np \not\equiv 1 \mod q$, we have $n\frac{p}{q} - \lfloor n\frac{p}{q} \rfloor > \frac{1}{q}$. On the other hand, $(q - q')\frac{p}{q} - \lfloor (q - q')\frac{p}{q} \rfloor = \frac{1}{q}$. Therefore, we have $(n + 1)\frac{p}{q} - \lfloor (n + 1)\frac{p}{q} \rfloor \geq \frac{1}{q}$. Thus, we have

$$\begin{aligned}s(x)_n &= \lfloor (n+1)\frac{p}{q} \rfloor - \lfloor n\frac{p}{q} \rfloor\\ &= \lfloor (n+1)\frac{p}{q} - \frac{1}{q} \rfloor - \lfloor n\frac{p}{q} - \frac{1}{q} \rfloor\\ &= G(n+q'; x).\end{aligned}$$

Secondly, let us show that $s(x)_n = G(n+q'-q;\frac{p'}{q'})$ for n such that $q-q' \leq n < q$. Let $n = q-q'+m$ with $0 \leq m < q'$. If $n \neq q-1$, we get in the same way

$$s(x)_n = \lfloor (m+1)\frac{p}{q} + \frac{1}{q} \rfloor - \lfloor m\frac{p}{q} + \frac{1}{q} \rfloor.$$

On the other hand,

$$\begin{aligned} G(n+q'-q;\frac{p'}{q'}) &= \lfloor (m+1)\frac{p'}{q'} \rfloor - \lfloor m\frac{p'}{q'} \rfloor \\ &= \lfloor (m+1)(\frac{p'}{q'} - \frac{p}{q} + \frac{p}{q}) \rfloor - \lfloor m(\frac{p'}{q'} - \frac{p}{q} + \frac{p}{q}) \rfloor \\ &= \lfloor (m+1)\frac{p}{q} + \frac{m+1}{qq'} \rfloor - \lfloor m\frac{p}{q} + \frac{m}{qq'} \rfloor. \end{aligned}$$

Since $(m+1)\frac{p}{q} + \frac{1}{q}, m\frac{p}{q} + \frac{1}{q} \notin \mathbf{Z}$ for m with $0 \leq m < q'-1$, $s(x)_n = G(n+q'-q;\frac{p'}{q'})$ for m with $0 \leq m < q'-1$. Let $m = q'-1$. Then, $s(x)_n = s(x)_{q-1} = \lfloor q\frac{p}{q} \rfloor - \lfloor (q-1)\frac{p}{q} \rfloor = 1$. On the other hand, $G(n+q'-q;\frac{p'}{q'}) = G(q'-1;\frac{p'}{q'}) = 1$. Hence, $s(x)_n = G(n+q'-q;\frac{p'}{q'})$. We can prove $s(x)_n = G(n-q;x)$ for $n \geq q$ in the same way. Thus, we obtain $\overline{G}(x) = S(x)$ $(\in L^{\mathbf{Z}}/\sim)$. We get $\underline{G}(x) = S'(x)$ similarly. □

3.2. Combinatorial considerations.

Lemma 3.10. *Let $W \in L^{\mathbf{N}}$ be a word satisfying $p_*(m;W) = p_*(m+1;W)$ for some integer $m \geq 0$. Then we have*

(i) $p_*(n;W) = p_*(m;W)$ for any integer $n \geq m$,

(ii) W is ultimately periodic.

Proof. (i) We suppose $p_*(m;W) = p_*(m+1;W) = l$. Let $W_1, W_2, \ldots, W_l$ be all the words in $D_*(m,W)$. Then we can choose $a_i = 0$ or 1 such that $a_1W_1, a_2W_2, \ldots, a_lW_l$ are words in $D_*(m+1,W)$. On the other hand, $p_*(m;W) = p_*(m+1;W)$ yields that they are all the words in $D_*(m+1,W)$, so that the l-tuple $(a_1, a_2, \ldots, a_l)$ is uniquely determined.

Similarly, we can choose $b_i = 0$ or 1 such that $W_1b_1, W_2b_2, \ldots, W_lb_l$ are all the words in $D_*(m+1,W)$ and $(b_1, b_2, \ldots, b_l)$ is uniquely determined. Obviously $a_1W_1b_1, a_2W_2b_2, \ldots, a_lW_lb_l$ are all the words in $D_*(m+2,W)$. Then $p_*(m+2;W) = l$. By induction $p_*(n) = p_*(m)$ holds for any $n \geq m$.

Proof of Lemma 3.10, (ii). We assume that W satisfies $p_*(m;W) = p_*(m+1;W)$. Any subword V belonging to $D(m+1;W)$ but not belonging to $D_*(m+1,W)$ occurs in W finitely many times. Hence taking sufficiently large N, we may assume that V is not a subword of $U = w_Nw_{N+1}\cdots$, i.e., $p(m+1;U) = p_*(m+1;U) = l$, $p(m;U) = p_*(m;U) = l$. Let $U = U_0a_1a_2\cdots$, $U_0 \in D_*(m,U)$, then, by the proof of (i) above, $a_1, a_2, \ldots$ are uniquely determined by U_0 as subsequent symbols and U_0 occurs in $a_1a_2\cdots$

again, i.e., $U = U_0 a_1 a_2 \cdots U_0 a_1 a_2 \cdots$. Hence U is purely periodic and therefore W is ultimately periodic. $\square$

By Lemma 3.10 we have the following remark.

Remark 3.1. Condition (ii) in Theorem 2.6 implies that we have the following two cases:

(i) $p_*(n;W) = n+1$ for all $n \geq 1$.

(ii) There exists a number $m \geq 0$ such that

$$p_*(n;W) = \begin{cases} n+1 & n < m, \\ m & n \geq m. \end{cases}$$

In Case (ii), W is ultimately periodic and $D_*(m;W)$ coincides with the set of fundamental periods of W.

Lemma 3.11. *Let $p_*(1;W) = 2$ and $p_*(n;W) \leq n+1$ for all $n \geq 0$ for a word $W = w_1 w_2 \cdots \in L^{\boldsymbol{N}}$. Then there exists some number m such that there is an inverse image $\delta_k^{-1}(w_m w_{m+1} \cdots)$ and $p_*(n; \delta_k^{-1}(w_m w_{m+1} \cdots)) \leq n+1$ holds for all $n \geq 0$, where $k \in \{0,1\}$ is defined by*

$$k = \begin{cases} 0 & \text{if} \quad 00 \in D_*(2,W), \\ 1 & \text{if} \quad 11 \in D_*(2,W), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The assumption implies $p_*(2;W) \leq 3$. First, we consider the case $p_*(2;W) = 2$. Then $D_*(2,W) = \{01, 10\}$, which implies $w_m w_{m+1} \cdots = (01)^*$ and $\delta_0^{-1}(w_m w_{m+1} \cdots) = 00 \cdots = 0^*$ for some $m \geq 1$. Noting $p_*(n;0^*) = 1 \leq n+1$, we have the lemma in this case.

Secondly we consider the case $p_*(2;W) = 3$. Since $01, 10 \in D_*(2,W)$, only one of 00 or 11 belongs to $D_*(2,W)$. We suppose $00 \in D_*(2,W)$. Since 11 occurs finitely many times in W, we can choose m such that

$$w_m w_{m+1} \cdots = 0^{m_1} 1 0^{m_2} 1 \cdots 1 0^{m_i} 1 \cdots \ (m_i \geq 1).$$

Then

$$\delta_0^{-1}(w_m w_{m+1} \cdots) = 0^{m_1 - 1} 1 0^{m_2 - 1} 1 \cdots 1 0^{m_i - 1} 1 \cdots .$$

Put $S = 0^{m_1-1} 1 0^{m_2-1} 1 \cdots 1 0^{m_i-1} 1 \cdots$. Assuming $p_*(n_0;S) > n_0 + 1$ for some $n_0 \geq 0$, we shall obtain a contradiction.

If $p_*(m+1;S) - p_*(m;S) \leq 1$ for any positive integer $m < n_0$, then $p_*(n_0;S) \leq n_0 + 1$. We put $n_1 = \min\{m \geq 1; p_*(m+1;S) - p_*(m;S) > 1\}$, consequently $p_*(n_1;S) - p_*(n_1 - 1;S) = 1$, $p_*(n_1+1;S) - p_*(n_1;S) > 1$. So there are distinct words $A, B \in D_*(n_1, S)$ such that $A0, A1, B0, B1 \in D_*(n_1+1, S)$. We can write $A = eA'$, $B = fB'$, $e, f \in \{1, 0\}$. If $A' \neq B'$, then $p_*(n_1;S) - p_*(n_1-1;S) > 1$ follows from $A'0, A'1, B'0, B'1 \in D_*(n_1, S)$. Hence, we get $A' = B'$, $e \neq f$. We may suppose $e = 0, f = 1$ without loss of generality. Since $0A'00 \in D_*(n_1+3, S)$ or $0A'01 \in$

$D_*(n_1+3,S)$, we obtain $\delta_0(0A'00) = 0\delta_0(A')00 \in D_*(j+1,W)$ or $0\delta_0(A')001 \in D_*(j+2,W)$, where j is the length of $0\delta_0(A')0$. Consequently, $0\delta_0(A')00 \in D_*(j+1,W)$ follows. It can be shown similarly that $0\delta_0(A')01, 1\delta_0(A')00, 1\delta_0(A')01 \in D_*(j+1,W)$. Therefore $p_*(j+1;W) - p_*(j;W) > 1$, which contradicts that $p_*(n;W) \le n+1$ for all n. Next, we suppose $11 \in D_*(2,W)$. Since 00 occurs finitely many times in W, we can choose m such that

$$w_m w_{m+1}\cdots = 01^{m_1}01^{m_2}1\cdots 01^{m_i}1\cdots \ (m_i \ge 1).$$

Then

$$\delta_1^{-1}(w_m w_{m+1}\cdots) = 01^{m_1-1}01^{m_2-1}1\cdots 01^{m_i-1}1\cdots .$$

Put $S = 0^{m_1-1}10^{m_2-1}1\cdots 10^{m_i-1}1\cdots$. We can prove in the same way as above that $p_*(n;S) \le n+1$ for each $n \ge 1$. □

Lemma 3.12. *Let $p_*(n;W) = n+1$ for all $n \ge 0$ for $W = w_1w_2\cdots \in L^{\boldsymbol{N}}$. Then there exist a number $m > 1$ and a number $k \in \{0,1\}$ such that $p_*(n;\delta_k^{-1}(w_m w_{m+1}\cdots)) = n+1$ for all $n \ge 0$.*

Proof. In Lemma 3.11 the inequality $p_*(n;\delta_k^{-1}(w_m w_{m+1}\cdots)) \le n+1$ has been shown. If $p_*(l;\delta_k^{-1}(w_m w_{m+1}\cdots)) < l+1$ for some l, then $\delta_k^{-1}(w_m w_{m+1}\cdots)$ is ultimately periodic and so is W. Therefore $p_*(n;W)$ does not exceed the length of a fundamental period of W, which is a contradiction. □

Lemma 3.13. *Let $p_*(n;W) \le n+1$ ($W = w_1w_2\cdots \in L^{\boldsymbol{N}}$) for all $n \ge 0$ with $p_*(1;W) = 2$ and $p_*(l;W) < l+1$ for a number $l \ge 1$. Then $p_*(l;\delta_k^{-1}(w_m w_{m+1}\cdots)) < p_*(l;W)$ for a number $m \ge 0$.*

Proof. Let l' be the least positive integer satisfying $p_*(l';W) < l'+1$. Then $p_*(l';W) = l'$ and W is ultimately periodic with period l'. Let $00, 11 \notin D_*(2,W)$. Then, for some m, $w_m w_{m+1}\cdots = (01)^*$. Therefore, the lemma holds apparently. We assume $00 \in D_*(2,W)$. We may assume a word $V \in D_*(l',W)$ has 1 as its suffix. Then 0 is a prefix of V, because VV occurs in W infinitely many times and 11 occurs finitely many times in W. Since V has 0 (resp. 1) as its prefix (resp. suffix), and 11 is not a subword of V, $\delta_0^{-1}(V)$ does exist. Hence $\delta_0^{-1}(w_m w_{m+1}\cdots) = \delta_0^{-1}(V)^*$ for some m. Let q be the length of $\delta_0^{-1}(V)$. Then

$$p_*(l;\delta_0^{-1}(w_m w_{m+1}\cdots)) \le q < l' \le l.$$

We can prove the lemma similarly for the case $11 \in D_*(2,W)$. □

Definition 3.3 (itinerary of a word). *Let $p_*(n;W) \le n+1$ for any $n \ge 0$ ($W \in L^{\boldsymbol{N}}$). We define a finite or infinite sequence $\kappa = \kappa(W) := \{k_n\}_{n=1,2,\dots}$ ($k_n = 0,1$) inductively by the following algorithm:*

- *If $p_*(1;W) = 1$, then κ is defined to be the null sequence, i.e., $\kappa = \emptyset$.*

• *If* $p_*(1;W) = 2$, *then we set* $W_1 := W = w_{1,1}w_{2,1}\cdots$, $k_1 := \begin{cases} 1 & \textit{if} \quad 11 \in D_*(2,W_1) \\ 0 & \textit{otherwise} \end{cases}$, *and* $W_2 := \delta_{k_1}^{-1}(w_{m_1,1}w_{m_1+1,1}\cdots)$ *for a number* m_1 *such that* $\delta_{k_1}^{-1}(w_{m_1,1}w_{m_1+1,1}\cdots)$ *exists.*

Suppose $W_1,\ldots,W_t$, *and* $k_1,\ldots,k_{t-1}$ *are defined. Set* $W_t = w_{1,t}w_{2,t}\cdots$.

• *If* $p_*(1;W_t) = 1$, *then the algorithm terminates.*

• *If* $p_*(1;W_t) = 2$, *then we define* $k_t := \begin{cases} 1 & \textit{if} \quad 11 \in D_*(2,W_t) \\ 0 & \textit{otherwise} \end{cases}$ *and* $W_{t+1} := \delta_{k_t}^{-1}(w_{m_t,t}w_{m_t+1,t}\cdots)$ *for a number* m_t *such that* $\delta_{k_t}^{-1}(w_{m_t,t}w_{m_t+1,t}\cdots)$ *exists.*

We call $\kappa(W)$ *the itinerary of* W $(W \in L^{\boldsymbol{N}})$.

In the definition of W_{t+1} above, note that Lemma 3.11 implies the existence of a number m_t such that $\delta_{k_t}^{-1}(w_{m_t,t}w_{m_t+1,t}\cdots)$ exists. We remark that the sequence $\kappa(W)$ is uniquely determined for any W satisfying $p_*(n;W) \le n+1$ for all $n \ge 0$; while, in general, the sequence of words W_t is not uniquely determined.

We can define the itinerary for $W \in L^{-\boldsymbol{N}}$ by Definition 3.3 with ${}^RW$ in place of W, where ${}^RW = w_1w_2w_3\cdots$ for $W = \cdots w_3w_2w_1$.

Lemma 3.14. $\kappa(G(x)) = i(x)$ *for all irrational* $x \in [0,1]$.

Proof. Let $x \in [0,1]$ be an irrational number. It is not difficult to see that if $x < \frac{1}{2}$, then 00 occurs in $G(x)$ and if $x > \frac{1}{2}$, then 11 occurs in $G(x)$. Therefore, $k_1 = i_1$. By Lemma 3.10, for any natural number n, $G(T^{n-1}(x)) = W_n$. By induction we get the lemma. □

3.3. Proof of Theorems.

Proof of Theorem 2.6. The proof of (i)⇒(ii) is similar to that of Theorem 2.1. Let us prove (ii)⇒(iii). First, let us suppose $p_*(n,W) = n+1$ for any $n \ge 0$. Then, it follows from Lemma 3.12 that $\{k_n\}_{n=1,2,\ldots} = \kappa(W)$ is an infinite sequence. We can choose a sequence $\{W_n\}_{n=1,2,\ldots}$ of infinite words and a sequence $\{m_n\}_{n=1,2,\ldots}$ of positive integers as in Definition 3.3, so that

$$W_n = w_{1,n}w_{2,n}\cdots w_{m_n-1,n}\delta_{k_n}(W_{n+1}) \tag{3.5}$$

where $w_{1,n}, w_{2,n},\ldots,w_{m_n-1,n}$ are words strictly over L for $n > 1$. In view of (3.5), we have $W = W_1$, which can be written in the following form:

$$W = w_{1,1}\cdots w_{m_1-1,1}\delta_{k_1}(w_{1,2}\cdots w_{m_2-1,2})\delta_{k_1}\circ\delta_{k_2}(w_{1,3}\cdots w_{m_3-1,3})\cdots$$
$$\cdots\delta_{k_1}\circ\cdots\circ\delta_{k_{n-1}}(w_{1,n}\cdots w_{m_n-1,n}W_n).$$

It is clear that $\delta_{k_1} \circ \cdots \delta_{k_n}(w_{1,n} \cdots w_{m_n-1,n})$ is a word strictly over $\{A_n, B_n\}$ with $A_n = \delta_{k_1} \circ \cdots \circ \delta_{k_n}(0)$, $B_n = \delta_{k_1} \circ \cdots \circ \delta_{k_n}(1)$. Setting $\delta_{k_1} \circ \cdots \circ \delta_{k_n}(w_{1,n} \cdots w_{m_n-1,n}) = u_n$, we obtain the assertion (iii).

Secondly, let us suppose $p_*(n, W) < n+1$ for some $n > 0$. From Remark 3.1, $p_*(n, W)$ is bounded and W is ultimately periodic. We can choose a sequence $\{W_n\}_{n=1,2,\ldots}$ as in Definition 3.3. Lemma 3.13 implies that $\{W_n\}_{n=1,2,\ldots}$ is a finite sequence of infinite words, i.e., $\{W_n\}_{n=1,2,\ldots} = \{W_n\}_{n=1,2,\ldots,l+1}$. In view of Definition 3.3, we can write

$$W_{l+1} = w_{1,l+1} \cdots w_{m_1-1,l+1} V, \quad V = 1^* \text{ or } 0^*.$$

Hence, we get

$$\begin{aligned}
W &= w_{1,1} \cdots w_{m_1-1,1} \delta_{k_1}(w_{1,2} \cdots w_{m_2-1,2}) \delta_{k_1} \circ \delta_{k_2}(w_{1,3} \cdots w_{m_3-1,3}) \cdots \\
&\quad \cdots \delta_{k_1} \circ \cdots \circ \delta_{k_l}(w_{1,l+1} \cdots w_{m_l-1,l+1} V) \\
&= w_{1,1} \cdots w_{m_1-1,1} \delta_{k_1}(w_{1,2} \cdots w_{m_2-1,2}) \delta_{k_1} \circ \delta_{k_2}(w_{1,3} \cdots w_{m_3-1,3}) \cdots \\
&\quad \cdots \delta_{k_1} \circ \cdots \circ \delta_{k_{l-1}}(w_{1,l} \cdots w_{m_{l-1}-1,l} \delta_{k_l}(w_{1,l+1} \cdots w_{m_l-1,l+1})) \\
&\qquad \delta_{k_1} \circ \cdots \circ \delta_{k_l}(V)
\end{aligned}$$

Since $\delta_{k_1} \circ \cdots \circ \delta_{k_l}(V)$ is A_l^* or B_l^*, setting

$$\begin{aligned}
u_0 &= w_{1,1} \cdots w_{m_1-1,1} \delta_{k_1}(w_{1,2} \cdots w_{m_2-1,2}) \delta_{k_1} \circ \delta_{k_2}(w_{1,3} \cdots w_{m_3-1,3}) \cdots \\
&\quad \cdots \delta_{k_1} \circ \cdots \circ \delta_{k_{l-1}}(w_{1,l} \cdots w_{m_{l-1}-1,l} \delta_{k_l}(w_{1,l+1} \cdots w_{m_l-1,l+1})),
\end{aligned}$$

we obtain assertion the (iii).

(iii)⇒(iv): We divide the proof into the following four cases (Cases I-IV).

Case I: Suppose that $\{k_n\}_{n=1,2,\ldots}$ is an infinite sequence in which 0 and 1 occur infinitely many times. Let R be a $*$-subword of W. There exists an integer l such that the length of A_l and B_l is larger than the length of R. Choose an integer $j > 0$ such that 10 is a subword of $\delta_{k_{l+1}} \circ \cdots \circ \delta_{k_{l+j}}(0)$ and $\delta_{k_{l+1}} \circ \cdots \circ \delta_{k_{l+j}}(1)$. Then

$$\begin{aligned}
A_{l+j} &= \delta_{k_1} \circ \cdots \circ \delta_{k_l}(\delta_{k_{l+1}} \circ \cdots \circ \delta_{k_{l+j}}(0)) \\
&= \delta_{k_1} \circ \cdots \circ \delta_{k_l}(0 \cdots 10 \cdots 1) \\
&= A_l \cdots B_l A_l \cdots B_l, \qquad (3.6)\\
B_{l+j} &= \delta_{k_1} \circ \cdots \circ \delta_{k_l}(\delta_{k_{l+1}} \circ \cdots \circ \delta_{k_{l+j}}(1)) \\
&= \delta_{k_1} \circ \cdots \circ \delta_{k_l}(0 \cdots 10 \cdots 1) \\
&= A_l \cdots B_l A_l \cdots B_l. \qquad (3.7)
\end{aligned}$$

R occurs in $A_{l+j}, B_{l+j}, A_{l+j}A_{l+j}, A_{l+j}B_{l+j}, B_{l+j}A_{l+j}$ or $B_{l+j}B_{l+j}$.

Suppose that R occurs in $A_{l+j}A_{l+j}$. If R does not occur in A_{l+j}, then (3.6) implies that R occurs in $B_l A_l$ since the length of R is smaller than that of A_l, B_l. Therefore R occurs in A_{l+j}. Similarly, we can conclude that

R occurs in B_{l+j} by (3.7). Lemma 3.4 implies that $\{k_n\}_{n=1,2,\ldots} = i(x)$ for an irrational x. It follows from Lemma 3.2 that

$$G(x) = \delta_{k_1} \circ \cdots \circ \delta_{k_{l+j}}(G(T^{l+j}(x))).$$

Since 0 and 1 occurs in $G(T^{l+j}(x))$, both $A_{l+j} = \delta_{k_1} \circ \cdots \circ \delta_{k_{l+j}}(0)$ and $B_{l+j} = \delta_{k_1} \circ \cdots \circ \delta_{k_{l+j}}(1)$ occur in $G(x)$. Thus we obtain that $R \in D(G(x))$ and $D_*(W) \subset D(G(x))$.

Conversely, we suppose $V \in D(G(x))$. Then, there exist natural numbers l and m such that V occurs in A_{l+m} and B_{l+m} by an argument similar to the argument above. We have $V \in D_*(W)$, since $A_{l+m}, B_{l+m} \in D_*(W)$. Therefore, we obtain that $D_*(W) = D(G(x))$, so that W satisfies Condition (C1). Thus we have shown the assertion (iv) in Case I.

Case II: Suppose that there exists an integer j such that $k_n = 0$ for any $n > j$ and $k_j = 1$. Lemma 3.7 implies that $\{k_n\}_{n=1,2,\ldots} = \tilde{i}(x)$ for a rational x. Then if $n > j$,

$$A_n = \delta_{k_1} \circ \cdots \circ \delta_{k_n}(0) = A_j \tag{3.8}$$

$$B_n = \delta_{k_1} \circ \cdots \circ \delta_{k_n}(1) = (A_j)^{n-j} B_j. \tag{3.9}$$

We can write by (3.8) and (3.9)

$$\begin{aligned} W &= u_0 \cdots u_j \cdots \\ &= u_0 \cdots u_{j-1} A_j^{a_1} B_j A_j^{a_2} B_j A_j^{a_3} B_j \cdots . \end{aligned} \tag{3.10}$$

where $\{a_n\}_{n=1,2,\ldots}$ is an infinite sequence of non-negative integers with $\lim_{n\to\infty} a_n = \infty$. Therefore, any $*$-subword R in W occurs in $A_j^m B_j A_j^m$ for sufficiently large integer m. By Lemma 3.8,

$$\overline{G}(x) = \cdots A_j A_j B_j A_j A_j \cdots .$$

Therefore R occurs in $\overline{G}(x)$, i.e., $D_*(W) \subset D(\overline{G}(x))$.

Conversely if $V \in D(\overline{G}(x))$, then V occurs in $A_j^m B_j A_j^m$ for a sufficiently large integer m. (3.10) implies that V is a $*$-subword of W, i.e., $V \in D_*(W)$. Therefore, we get $D_*(W) = D(\overline{G}(x))$, so that W satisfies Condition (C3).

Case III: Suppose that there exists an integer j such that $k_n = 1$ for any $n > j$ and $k_j = 0$. Then W satisfies Condition (C2). The proof is similar to that given in Case II.

Case IV: Suppose that the sequence $\{k_n\}_{n=1,2,\ldots}$ is a finite sequence. In this case $D_*(n, W) = D(n; A_i^*)$ or $D_*(n, W) = D(n; B_i^*)$. There exist rational numbers u, v such that $A_i^* = G(u), B_i^* = G(v)$ by the proof of Lemma 3.8. Then W satisfies Condition (C1) for $x = u$ or $x = v$. We have completed the proof of (iii)$\Rightarrow$(iv).

(iv)$\Rightarrow$(i): By Definition 2.7, $D_*(W)$ coincides with one of the sets $D(G(x))$, $D(\underline{G}(x))$ or $D(\overline{G}(x))$. Since by Theorem 2.2, 2.3 and Lemma 3.9, $G(x), \underline{G}(x)$ and $\overline{G}(x)$ are Sturmian, W is a $*$-Sturmian word. $\square$

Proof of Theorem 2.7. Let $W = w_1w_2\ldots \in L^{\mathbf{N}}$ be $*$-Sturmian. By Theorem 2.6 there exists $\alpha \in [0,1]$ such that one of the following three conditions holds:

(1) $D_*(W) = D(G(\alpha))$,
(2) $D_*(W) = D(\underline{G}(\alpha))$ with $\alpha \in \mathbf{Q}$,
(3) $D_*(W) = D(\overline{G}(\alpha))$ with $\alpha \in \mathbf{Q}$.

Let us show that $\alpha = \lim_{n\to\infty} \frac{\sigma(n;W)}{n} = \lim_{n\to\infty} \frac{\sigma'(n;W)}{n}$. First, we suppose that (1) holds. Let $\epsilon > 0$ be any small number. Let m be a natural number with $\frac{1}{m} < \frac{\epsilon}{3}$. Since $W = w_1w_2\ldots \in L^{\mathbf{N}}$ is $*$-Sturmian, there exists an integer $k > 0$ such that any subword of $w_{k+1}w_{k+2}\ldots$ with length m is in $D_*(W)$. Let c be a positive integer with $\frac{k}{c} < \frac{\epsilon}{3}$. Let n be a positive integer with $n > c$. Let $w = w_lw_{l+1}\ldots w_{l+n-1} \in D(n;W)$. Then, we have

$$\begin{aligned} |w|_1 &= |w_lw_{l+1}\ldots w_{l+k-1}|_1 + |w_{l+k}\ldots w_{l+n-1}|_1 \\ &= |w_lw_{l+1}\ldots w_{l+k-1}|_1 \\ &\quad + \sum_{j=0}^{d-1} |w_{l+k+m*j}\ldots w_{l+k+m*(j+1)-1}|_1 + |w_{l+k+md}\ldots w_{l+n-1}|_1, \end{aligned} \tag{3.11}$$

where $d = \lceil \frac{n-k}{m} \rceil - 1$. Since $w_{l+k+m*j}\ldots w_{l+k+m*(j+1)-1} \in D_*(W)$ for $j = 0,\ldots,d-1$, $w_{l+k+m*j}\ldots w_{l+k+m*(j+1)-1} \in D(G(\alpha))$. Therefore, there exist integers f_j for $j = 0,\ldots,d-1$ such that for $i = 0,\ldots,m-1$, $w_{l+k+m*j+i} = \lfloor (f_j+i)\alpha \rfloor - \lfloor (f_j+i-1)\alpha \rfloor$. Thus, $||w_{l+k+m*j}\ldots w_{l+k+m*(j+1)-1}|_1 - m\alpha| = |\lfloor (f_j+m-1)\alpha \rfloor - \lfloor (f_j-1)\alpha \rfloor - m\alpha| \le 1$. Similarly we have $|w_{l+k+md}\ldots w_{l+n-1} - (n-k-md)\alpha|_1 \le 1$. Therefore, by (3.11) we have

$$||w|_1 - n\alpha| \le ||w_lw_{l+1}\ldots w_{l+k-1}|_1 - k\alpha| + d + 1 \le k + d + 1.$$

Therefore, we have

$$|\frac{|w|_1}{n} - \alpha| \le \frac{k}{n} + \frac{d}{n} + \frac{1}{n} \le \epsilon.$$

Thus, we have $\alpha = \lim\limits_{n\to\infty} \frac{\sigma(n;W)}{n} = \lim\limits_{n\to\infty} \frac{\sigma'(n;W)}{n}$. For cases (2) and (3) we have a similar proof. $\square$

The proofs of Theorems 2.8 and 2.11 are similar to that of Theorem 2.6, so we give a sketch of the proofs. Let $W = W_1W_2$ be a two-sided infinite word satisfying $p_*(n;W) \le n+1$ with $W_1 \in L^{-\boldsymbol{N}}, W_2 \in L^{\boldsymbol{N}}$, then W_1 and W_2 are $*$-Sturmian words. There are four cases.

Case I: Suppose $p_*(n; W_1) = p_*(n; W_2) = n + 1$ for all n. Then there exists an irrational number $x \in (0,1)$ or a rational number $x' \in [0,1]$ such that $D_*(W_1) = D_*(W_2) = D(G(x))$, $D(\underline{G}(x'))$, or $D(\overline{G}(x'))$. Therefore, W_1 and W_2 have the same itinerary, and $W_1 = \cdots u_{-2}u_{-1}u$, $W_2 = vu_1u_2\cdots$ $(u, v \in L^*)$.

Case II: Suppose $p_*(n; W_1) = n + 1$ for all n and $p_*(m; W_2) < m + 1$ for some m. Then W_2 is ultimately periodic and there exists a rational number $x \in [0,1]$ such that $D_*(W_1) = D_*(\underline{G}(x)) \supset D_*(W_2)$ or $D_*(W_1) = D_*(\overline{G}(x)) \supset D_*(W_2)$. By the definition of $\underline{G}(x)$ (resp. $\overline{G}(x)$), $W_2 = vB_j^*$ (resp. vA_j^*) $(v \in L^*)$. Therefore $W = \cdots u_{-1}u_0vB_j^*$ or $W = \cdots u_{-1}u_0vA_j^*$.

Case III: Suppose $p_*(n; W_2) = n + 1$ for all n and $p_*(m; W_1) < m + 1$ for some m. Then $W =^* A_jvu_0u_1\cdots$ or $W =^* B_jvu_0u_1\cdots$. The proof is similar to Case II.

Case IV: Suppose $p_*(m; W_1) < m + 1$ and $p_*(m; W_2) < m + 1$ for some m. Then $W_1 =^* A_ju$ or ${}^*B_ju$ and $W_2 = vA_j^*$ or vB_j^*. It is easy to show that ${}^*A_ju_0A_j^*$ and ${}^*B_ju_0B_j^*$ are $*$-Sturmian and ${}^*A_ju_0B_j^*$ and ${}^*B_ju_0A_j^*$ are not $*$-Sturmian.

The proof of Theorem 2.10 is similar to that of Theorem 2.1 and the proof of Theorem 2.9 is similar to that of Theorem 2.7, so we omit these proofs.

4. Complexity of certain $*$-Sturmian words

Let us consider the complexity of an infinite word W:

$$W = 10^{a_1}10^{a_2}10^{a_3}\cdots, 0 \le a_1 \le a_2 \le a_3 \cdots. \tag{4.1}$$

It is clear that W is a $*$-Sturmian word. We write $u \prec_p v$ $(u, v \in D(W))$ if u is a prefix of v. The binary relation $\prec_p$ is reflexive, asymmetric, and transitive, so that $X = X(W) := (D(W), \prec_p)$ is a partially ordered set with the order $\prec_p$. For each element $v \in D(n+1; W)$, $(n \ge 0)$, there exists a unique element $u \in D(n; W)$ such that $u \prec_p v$. Hence, X can be regarded as a tree consisting of the nodes $w \in D(W)$ with λ as its root, where every edge is understood to be one of the segments connecting two nodes $u \in D(n; W)$, $v \in D(n+1; W)$ as far as $u \prec_p v$. For example, if W is the word (4.1) with $a_n = n - 1$, then $X(W)$ is the following tree.

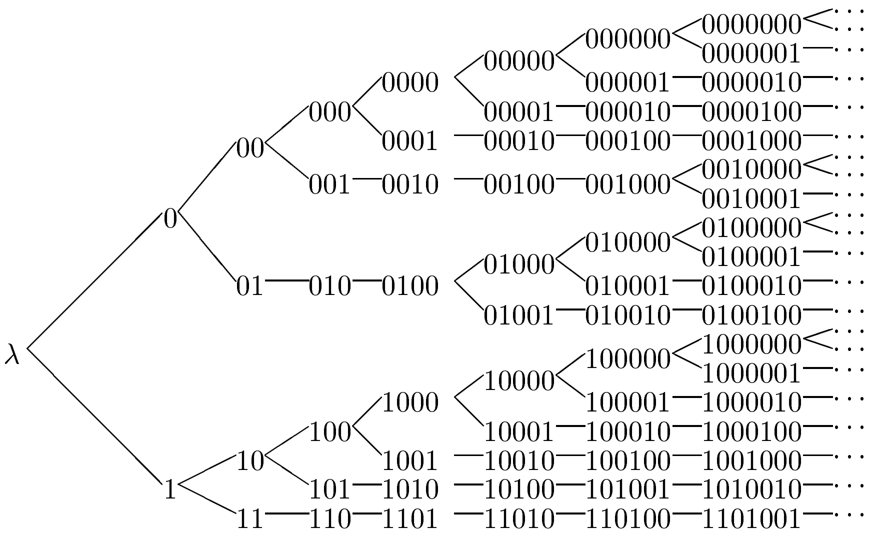

Fig. 1. $X(11010^2 10^3 10^4 \cdots)$

In Figure 1, only words of the form 0^k, $0^l 10^m$ $(l < m)$ are followed by two words:

$$0^k \begin{matrix} \nearrow & 0^{k+1} \\ \searrow & 0^k 1 \end{matrix}, \qquad 0^l 10^m \begin{matrix} \nearrow & 0^l 10^{m+1} \\ \searrow & 0^l 10^m 1 \end{matrix}.$$

Theorem 4.1. *Let W be a word given by (4.1) with $(a_0 :=)\ 0 \le a_1 < a_2 < a_3 \cdots$. Then*

$$p(n;W) = n + 1 + \sharp\{(i,j) \in N^2; j \le a_{i-1} + 1, a_i + j \le n - 1\},\ n \ge 0.$$

Proof. We put

$$B_n := B_n(W) = \{w \in D(n;W);\ w0 \in D(W), \text{ and } w1 \in D(W)\},$$
$$(4.2) \qquad B = B(W) := \bigcup_{n \ge 0} B_n(W).$$

Then, we have

$$(4.3) \qquad p(n) - p(n-1) = \sharp B_{n-1} \quad (n \ge 1;\ \ p(0) = 1,\ B_0 = \{\lambda\}).$$

Since $a_1 < a_2 < a_3 < \cdots$, w occurs only once in W if $|w|_1 \ge 2$, so that $w \notin B$. Hence, we get

$$w \in B \Rightarrow |w|_1 \le 1.$$

If $|w|_1 = 0$, i.e., $w = 0^n$, then $w \in B$, since a_n tends to infinity. If w belongs to B with $|w|_1 = 1$, then w can be written as follows, and vice versa:

$$w = 0^{j-1}10^{a_i}, \quad 1 \le j \le a_{i-1}+1, \quad i \ge 1. \tag{4.4}$$

Hence, in view of (4.3) and (4.4), we obtain

$$\begin{aligned} p(n) &= 1 + \sum_{m=0}^{n-1} \sharp B_m \\ &= n + 1 + \sharp\{w = 0^{j-1}10^{a_i};\ |w| \le n-1,\ 1 \le j \le a_{i-1}+1,\ i \ge 1\}, \end{aligned}$$

where $|w|$ is the length of w, which implies the theorem. □

Theorem 4.2. *Let W be as in Theorem 4.1. Then,*

$$p(n;W) \le \frac{n^2}{4} + \frac{n}{2} + \frac{17}{8} + \frac{(-1)^{n+1}}{8} - \lfloor (\frac{3}{4} + \frac{n}{4})^{-1} \rfloor \quad (n \ge 0). \tag{4.5}$$

The above estimate is sharp; the equality is attained by

$$W = W_0 := 11010^2 10^3 10^4 \cdots .$$

Proof. From the proof of Theorem 4.1, it follows that if $w \in B_n(W)$ with $w \ne 0^n$, then w is of the form (4.4) with $|w| = n$. On the other hand, all the words $w = 0^{j-1}10^{a_i}$ with

$$1 \le j \le a_i, \quad i \ge 1, \quad |w| = n$$

belong to $B_n(W_0)$. Hence $B_n(W) \subset B_n(W_0)$, so that $\sharp B_n(W) \le \sharp B_n(W_0)$, which implies $p(n;W) \le p(n;W_0)$.

Now, we consider

$$d_n := p(n;W_0) - p(n-1;W_0) = \sharp B_{n-1}(W_0) \quad (n \ge 1).$$

In view of Figure 1, we have $d_1 = 1$, $d_2 = 2$. For a given sequence $b = \{b_n\}_{n=1,2,\dots}$, we denote by $\int_1^n (b_1, b_2, b_3, \cdots)$ the number defined by

$$\int_1^n (b_1, b_2, b_3, \cdots) := 1 + \sum_{m=1}^{n} b_m, \qquad n \ge 0. \tag{4.6}$$

We use the notation $\int_1^n w$ also for a word w over $\boldsymbol{N}$ as far as its meaning is clear. For instance, $\int_1^n 12^3 3^2 5151 \cdots$ is the number $\int_1^n (1, 2, 2, 2, 3, 3, 5, 1, 5, 1, \cdots)$. Noting

$$\begin{aligned} B_{2n-2}(W_0) &= \{0^{2n-2},\ \ 0^i 10^{2n-i-3};\ \ 0 \le i \le n-2\}, \\ B_{2n-1}(W_0) &= \{0^{2n-1},\ \ 0^i 10^{2n-i-2};\ \ 0 \le i \le n-2\} \qquad (n \ge 2), \end{aligned}$$

we get $d_{2n-1} = d_{2n} = n$ $(n \ge 2)$. Therefore, we obtain

$$p(n;W_0) = \int_1^n 12^3 3^2 4^2 5^2 \dots . \tag{4.7}$$

By induction, we can show that the right-hand side of (4.7) coincides with that of (4.5), which completes the proof. □

Remark 4.1. If $\mathbf{N}\backslash\{a_n; n \geq 1\}$ is not the empty set, then

$$p(n; W) < p(n; W_0)$$

holds for all $n \geq 2 + \min(\mathbf{N}\backslash\{a_n; n \geq 1\})$.

We can give some examples.

Example 4.1.

(i)

$$\begin{aligned} p(n; 1010^2 10^3 10^4 \cdots) &= \int_1^n 1^2 2^2 3^2 4^2 \ldots \\ &= \frac{n^2}{4} + \frac{n}{2} + \frac{9}{8} + \frac{(-1)^{n+1}}{8} \quad (n \geq 0). \end{aligned}$$

(ii)

$$\begin{aligned} p(n; 10^2 10^4 10^6 10^8 \cdots) &= \int_1^n 1^3 212^2 323^2 434^2 545^2 6 \ldots \\ &= 2\lfloor\frac{n+1}{4}\rfloor(\lfloor\frac{n+1}{4}\rfloor + 1) \\ &\quad + \lfloor\frac{n+1}{4} + 1\rfloor(n + 1 - 4\lfloor\frac{n+1}{4}\rfloor) \\ &\quad + 1 - \lfloor\frac{n+3}{4}\rfloor \quad (n \geq 1). \end{aligned}$$

For a word W as in Theorem 4.1, it is easily seen that

$$p(n; W) = \begin{cases} n+1 & (0 \leq n \leq a_1 + 1), \\ n+2 & (a_1 + 2 \leq n \leq a_2 + 1). \end{cases}$$

In view of Theorem 4.1, for any $n \geq a_2 + 2$, we have

$$p(n; W) = n + 1 + \sum_{i=1}^{r(n)} \min\{a_{i-1} + 1, n - 1 - a_i\},$$

where

$$r(n) = r(n; W) := \max\{m;\ a_m \leq n - 2\}.$$

Hence, setting

$$s(n) = s(n; W) := \max\{m;\ a_{m-1} + a_m \leq n - 2\},$$

which does not exceed $r(n)$, we get

$$p(n;W) = n+1+\sum_{i=1}^{s(n)}(a_{i-1}+1)+\sum_{i=s(n)+1}^{r(n)}(n-1-a_i) \qquad (n \geq a_2+2). \tag{4.8}$$

Hence we obtain the following

Corollary 4.1. *Let W be as in Theorem 4.1, then*

$$p(n;W) = \begin{cases} n+1 & (0 \leq n \leq a_1+1), \\ n+2 & (a_1+2 \leq n \leq a_2+1), \\ (r(n)-s(n)+1)n+2s(n)-r(n) & \\ +1+\sum_{i=1}^{s(n)-1} a_i - \sum_{i=s(n)+1}^{r(n)} a_i & (n \geq a_2+2). \end{cases}$$

Remark 4.2. If $a_n + a_{n+1} \leq a_{n+2}$ holds for all $n \geq 1$, then

$$s(n) = r(n) \quad \text{or} \quad r(n)-1$$

holds for all $n \geq a_2+2$. In particular, if $a_n + a_{n+1} = a_{n+2}$ holds for all $n \geq 1$, then

$$s(n) = r(n) - 1 \quad (n \geq a_2+2).$$

Suppose that

$$a_n + a_{n+1} < a_{n+2} \quad \text{for all } n \geq n_0. \tag{4.9}$$

Then, $d_n(W) = 1$, or 2 $(n \geq a_{n_0+2}+2)$, so that

$$1 \leq \liminf_{n\to\infty} \frac{p(n;W)}{n} \leq \limsup_{n\to\infty} \frac{p(n;W)}{n} \leq 2.$$

Noting that $d_n(W) = 2$ if and only if

$$a_{i+2}+2 \leq n \leq a_{i+1}+a_{i+2}+2$$

holds for some $i \geq n_0$, we get

$$p(N_n) = N_n + a_1 + a_2 + \ldots + a_n + n + O(1),$$
$$p(M_n) = M_n + a_1 + a_2 + \ldots + a_n + n + O(1),$$

$$\limsup_{n\to\infty}\frac{p(n)}{n} = \limsup_{n\to\infty}\frac{p(N_n)}{N_n}, \quad \liminf_{n\to\infty}\frac{p(n)}{n} = \liminf_{n\to\infty}\frac{p(M_n)}{M_n},$$

where

$$N_n = a_n + a_{n+1} + 2, \quad M_n = a_{n+2} + 1.$$

It follows from (4.9) that

$$1 + \frac{a_{n+1}}{a_n} < \frac{a_{n+1}}{a_n}\frac{a_{n+2}}{a_{n+1}},$$

which implies

$$\liminf_{n\to\infty}\frac{a_{n+1}}{a_n} \geq \frac{1+\sqrt{5}}{2}.$$

Hence, we get

$$\frac{p(N_n)}{N_n} = 1 + \frac{a_1 + a_2 + \ldots + a_n}{a_n + a_{n+1}} + o(1),$$
$$\frac{p(M_n)}{N_n} = 1 + \frac{a_1 + a_2 + \ldots + a_n}{a_{n+2}} + o(1).$$

In view of the above formulae, we can show Remarks 4.3, 4.4.

Remark 4.3. If there exists an infinite set $S \subset \mathbf{N}$ for a fixed positive number ϵ such that

$$(1+\epsilon)(a_n + a_{n+1}) < a_{n+2} \quad \text{for all } n \in S,$$

and

$$\frac{a_{n+2}}{a_1 + a_2 + \ldots + a_n} \quad \text{is bounded for all } n \in S,$$

then,

$$1 \le \liminf_{n\to\infty} \frac{p(n)}{n} < \limsup_{n\to\infty} \frac{p(n)}{n} \le 2.$$

Remark 4.4. If $\frac{a_{n+1}}{a_1+\ldots+a_n}$ tends to infinity, then $\lim_{n\to\infty} \frac{p(n)}{n} = 1$. For instance, the word over $\{0,1\}$ defined by the digits in the base 2 expansion of a Liouville number $\sum_{n=1}^{\infty} 2^{-n!}$ has this property.

Example 4.2.

(i)

$$p(n; 10^{2^0}10^{2^1}10^{2^2}\cdots) = \int_1^n 1^2 2^6 1^{2^1-1} 2^{2^2+1} 1^{2^2-1} 2^{2^3+1} 1^{2^3-1} 2^{2^4+1} \ldots,$$

which implies

$$\liminf_{n\to\infty} \frac{p(n)}{n} = \frac{3}{2}, \quad \limsup_{n\to\infty} \frac{p(n)}{n} = \frac{5}{3},$$

since the identities

$$p(2^n + 1) = 3 \cdot 2^{n-1} + n + 1,$$
$$p(3 \cdot 2^{n-1} + 2) = 5 \cdot 2^{n-1} + n + 3 \quad (n \ge 1)$$

hold. It is remarkable that the example shares a common phenomenon with a word different from $*$-Sturmian words: for the word generated by the catenative formula $B_{n+1} = B_n^2 1 B_n^2$, $(n \ge 0,\ B_0 := 0)$, $\liminf_{n\to\infty} \frac{p(n)}{n} = \frac{3}{2}$ and $\limsup_{n\to\infty} \frac{p(n)}{n} = \frac{5}{3}$ hold, cf. Proposition 11 in [6].

(ii) Let $W = 10^{a_1}10^{a_2}10^{a_3}\cdots$ with $a_1 = 1$, $a_2 = 2$, $a_3 = 4$, $a_{n+3} = a_{n+2} + a_{n+1} + a_n$ $(n \geq 1)$. Then,

$$p(n;W) = \int_1^n 1^2 2^{11} 1^{a_2-1} 2^{a_4+1} 1^{a_3-1} 2^{a_5+1} 1^{a_4-1} 2^{a_6+1} \ldots \ (n \geq 0).$$

Using the above formula, we can show

$$\limsup_{n\to\infty} \frac{p(n;W)}{n} = \frac{1}{13}\alpha^2 - \frac{1}{26}\alpha + \frac{43}{26} = 1.843333\cdots \notin \mathbf{Q},$$
$$\liminf_{n\to\infty} \frac{p(n;W)}{n} = \frac{3}{22}\alpha^2 - \frac{2}{11}\alpha + \frac{35}{22} = 1.717808\cdots \notin \mathbf{Q},$$

where

$$\alpha = \frac{1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}}{3}.$$

(iii) For any recurrence sequence $\{a_n\}_{n=1,2,\ldots}$ with $x^2 - x - 1$ as its characteristic polynomial and satisfying $0 \leq a_1 < a_2$,

$$p(n;W) = 2n + r(n;W) - a_2 - 1 \qquad (n \geq a_2 + 2)$$

holds for $W = 10^{a_1}10^{a_2}10^{a_3}\cdots$, so that

$$p(n;W) = 2n + \frac{\log n}{\log \alpha} + O(1), \quad \alpha = \frac{1+\sqrt{5}}{2}.$$

In particular,

$$p(n;W) = \int_1^n 1^2 2^2 3 2^{a_2-1} 3 2^{a_3-1} 3 2^{a_4-1} 3 2^{a_5-1} \ldots \ (a_1 = 1,\ a_2 = 2),$$
$$p(n;W) = \int_1^n 1^2 2 1^2 3 2^{a_2-1} 3 2^{a_3-1} 3 2^{a_4-1} 3 2^{a_5-1} \ldots \ (a_1 = 1,\ a_2 = 3).$$

We write $f(n) \asymp g(n)$ if $f(n) = O(g(n))$ and $g(n) = O(f(n))$.

Theorem 4.3. *Let W be a word given by (4.1) with $0 \leq a_1 < a_2 < \cdots$ and $a_n \asymp n^\alpha$ $(\alpha \geq 1)$. Then $p(n;W) \asymp n^{1+1/\alpha}$.*

Proof. We suppose $c_1 n^\alpha \le a_n \le c_2 n^\alpha$. (4.8) implies

$$\begin{aligned}
p(n;W) &\le \sum_{i=1}^{r(n)} (a_{i-1}+1) + n + 1 \\
&\le \sum_{a_i<n} a_i + O(n) \\
&\le \sum_{c_1 i^\alpha < n} c_2 i^\alpha + O(n) \\
&\le c_2 \int_0^{(\frac{n}{c_1})^{1/\alpha}} x^\alpha dx + O(n) \\
&= \frac{c_2}{(\alpha+1)c_1^{1+1/\alpha}} n^{1+1/\alpha} + O(n),
\end{aligned}$$

and

$$\begin{aligned}
p(n;W) &\ge \sum_{a_i+a_{i-1}\le n-2} (a_{i-1}+1) + O(n) \\
&\ge \sum_{2a_i \le n-2} a_{i-1} + O(n) \\
&\ge \sum_{c_2 i^\alpha \le (n-2)/2} c_1 i^\alpha + O(n) \\
&\ge c_1 \int_0^{(\frac{n-2}{2c_2})^{1/\alpha}} x^\alpha dx + O(n) \\
&= \frac{c_1}{\alpha+1} \left(\frac{n-2}{2c_2}\right)^{1+1/\alpha} + O(n).
\end{aligned}$$

Therefore $p(n;W) \asymp n^{1+1/\alpha}$ holds. $\square$

Theorem 4.4. *Let $k \ge 2$ be an integer, and $\{b_n\}_{n=1}^\infty$ a linear recurrence sequence with $x^k - x - 1$ as its characteristic polynomial, defined by the initial condition:*

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_k \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 2 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 1 \\ \vdots & \vdots & \vdots & & 2 & 1 \\ 1 & 1 & 2 & \cdots & 2 & 2 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ \vdots \\ \vdots \\ t_k \end{pmatrix}, \ (t_1, t_2, \ldots, t_k) \in \boldsymbol{N}^k.$$

Let W be the word defined by

$$W := 10^{a_1}10^{a_2}10^{a_3}\cdots, \quad a_n := b_n - 1.$$

Then $p(n;W)$ *is given by the following, so that*

$$p(n;W) = kn + c \quad \text{for all } n \geq b_k + 1,\ c \leq 0,$$

where c *is a non-positive constant, and* $c = 0$ *only if* $k = 2$, $t_1 = t_2 = 1$.

$$p(n;W) = \begin{cases} n+1 & (0 \leq n \leq b_1), \\ n+2 & (b_1 + 1 \leq n \leq b_2), \\ 2n - b_2 + 2 & (b_2 + 1 \leq n \leq b_3), \\ 3n - b_2 - b_3 + 2 & (b_3 + 1 \leq n \leq b_4), \\ \quad \cdots & \quad \cdots \quad , \\ jn - b_2 - b_3 - \cdots - b_j + 2 & (b_j + 1 \leq n \leq b_{j+1}), \\ \quad \cdots & \quad \cdots \quad , \\ kn - b_2 - b_3 - \cdots - b_k + 2 & (n \geq b_k + 1). \end{cases}$$

Proof. By definition

$$b_{n+k} = b_{n+1} + b_n \quad (n \geq 1).$$

Noting $1 \leq b_1 < b_2 < \cdots < b_k$, and $b_{k+1} - b_k = b_1 + b_2 - b_k = t_1 > 0$, we get $b_n < b_{n+1} \quad (n \geq 1)$ inductively, so that

$$a_n < a_{n+1} \quad (n \geq 1), \quad a_1 \geq 0.$$

Hence the word W satisfies the condition in Theorem 4.1. We denote by $[p,q)$ the interval $\{m \in \boldsymbol{N};\ p \leq m < q\}$. In view of Theorem 4.1, we have

$$\begin{aligned} d_n = d_n(W) &:= p(n;W) - p(n-1;W) \\ &= 1 + \sharp\{(i,j) \in \boldsymbol{N}^2;\ j \leq a_{i-1} + 1, a_i + j = n - 1\} \\ &= 1 + \sharp\{i \in \boldsymbol{N};\ a_i + 2 \leq n \leq a_i + a_{i-1} + 2\} \\ &= 1 + \sharp\{J_i;\ J_i \ni n,\ i \geq 1\} \quad (n \geq 1, a_0 := 0), \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} J_i &= [a_i + 2, a_i + a_{i-1} + 3) \\ &= [b_i + 1, b_i + b_{i-1} + 1) \\ &= [b_i + 1, b_{i+k-1} + 1) \qquad (i \geq 2), \\ J_1 &= [b_1 + 1, b_1 + 2). \end{aligned}$$

Hence, $J_{i+1} \cap J_{i+k} = \emptyset$, $\overline{J_{i+1}} \cap \overline{J_{i+k}} = \{b_{i+k-1} + 1\}$, where $\overline{X}$ denotes the closure of a set $X \subset \mathbf{R}$, cf. Fig. 2.

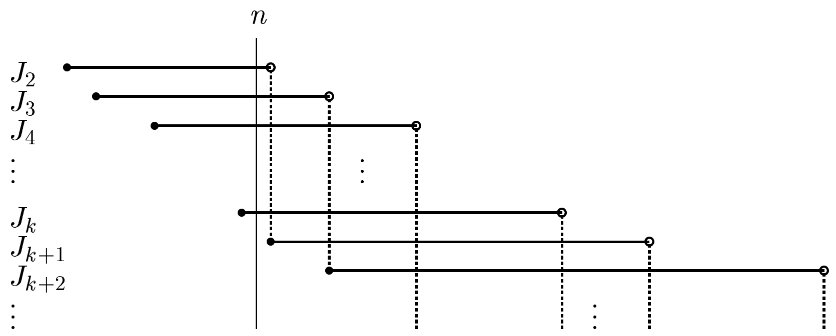

Fig. 2.

Thus, it is clear that the number of intervals J_i containing n is equal to $k-1$ for all $n \geq b_k+1$, which together with (4.10) implies $d_n = k$ $(n \geq b_k+1)$. Counting the number of intervals J_i that contain n with $1 \leq n \leq b_k$, we get

$$\begin{aligned} d_n &= 1 && (1 \leq n \leq b_1), \\ d_n &= 2 && (n = b_1+1), \\ d_n &= 1 && (b_1+2 \leq n \leq b_2), \\ d_n &= j && (b_j+1 \leq n \leq b_{j+1},\ 2 \leq j \leq k-1). \end{aligned}$$

Hence, we obtain $p(n;W) = n+1$ $(0 \leq n \leq b_1)$, $p(n;W) = n+2$ $(b_1+1 \leq n \leq b_2)$, so that the assertion

$$p(n;W) = jn - b_2 - b_3 - \cdots - b_j + 2 \qquad (b_j+1 \leq n \leq b_{j+1})$$

holds for $j=1$. By induction on j, we can easily show that the assertion holds for all j, $1 \leq j \leq k-1$. Recalling $d_n = k$ $(n \geq b_k+1)$, we obtain

$$\begin{aligned} p(n;W) &= kn + c \qquad (n \geq b_k+1), \\ c &= -b_2 - b_3 - \cdots - b_k + 2, \end{aligned}$$

so that $c \leq 0$, and $c = 0$ implies $k=2$, $t_1 = t_2 = 1$, which completes the proof. $\square$

Example 4.3. Let $W = W(k;t_1,\ldots,t_k)$ be as in Theorem 4.4. Then $p(n;W(2;1,1)) = 2n$ $(n \geq 1)$, $p(n;W(2;1,2)) = 2n-1$ $(n \geq 3)$, $p(n;W(2;2,1)) = 2n-1$ $(n \geq 2)$, $p(n;W(3;3,1,1)) = 3(n-3)$ $(n \geq 6)$. Related to linear complexity $p(n) = an+b$, there are some results. S. Ferenczi considered a class of words generated by a locally catenative formula $B_{n+1} = B_n^{r_n} 1 B_n^{s_n}$ $(n \geq 0, B_0 := 0)$, and considered the complexity of the word $\lim B_n$, cf. [6]. P. Arnoux and G. Rauzy investigated a class of words, having $p(n) = 2n+1$ as their complexity, given by an interval exchange of some intervals; G. Rote showed that a word $\beta_1\beta_2\cdots$ defined by $\beta_n = \chi_{[0,\phi]}(\{n\theta + c\})$ always has complexity $p(n) = 2n$, where $\{\cdot\}$ denotes the fractional part of a real number, and $\chi_J(x)$ the characteristic function equal to 1 (resp. 0) for $x \in J$ (resp. $x \notin J$), cf. [2], [12].

Let us consider the word given by (4.1). If a_n is bounded, then W is an ultimately periodic word, which is not an interesting case. If a_n is unbounded, then without loss of generality, we can write

$$W = (10^{a_1})^{e_1}(10^{a_2})^{e_2}(10^{a_3})^{e_3}\cdots, \qquad (a_0 := 0) \le a_1 < a_2 < \cdots, \quad e_n \ge 1. \tag{4.11}$$

Theorem 4.5. *Let W be the word given by* (4.11). *Then*

$$\begin{aligned} p(n;W) = n+1 &\\ &+ \sharp\{(i,j,k) \in \mathbf{N}^3;\ j \le a_i + 1,\ k \le e_i - 1,\ k(a_i+1)+j \le n\} \\ &+ \sharp\{(i,j) \in \mathbf{N}^2;\ j \le a_{i-1}+1,\ e_i(a_i+1)+j \le n\} \quad (n \ge 0). \end{aligned}$$

Proof. For the set $B_n(W)$ defined by (4.2), we have the identity (4.3). We write $u \prec v$ (resp. $u \prec_s v$) if u is a subword (resp. a suffix) of v for $u, v \in D(W)$. Let $v \in D(W)$ with

$$10^{a_{i-1}}10^{a_i} \prec v \quad (i \ge 2) \text{ or } (10^{a_i})^{e_i}1 \prec v \quad (i \ge 1).$$

Then, v occurs only once in W, so that $v \notin B(W)$, where $B(W)$ is the set given by (4.2). Hence $w \in B(W)$ implies $w \prec 0^{a_{i-1}}(10^{a_i})^{e_i}$ for some $i \ge 1$. In addition, if $w \in B(W)$ with $|w|_1 \ge 1$, then $10^{a_i} \prec_s w$ for some $i \ge 1$. Hence,

$$w \in B(W), \quad |w|_1 \ne 0 \ \Rightarrow w \prec_s 0^{a_{i-1}}(10^{a_i})^{e_i} \quad \text{for some } \ i \ge 1,$$

which implies

$$\{w \in B(W); |w|_1 \ne 0\} \subset B^{(1)} \cup B^{(2)},$$

where

$$\begin{aligned} B^{(1)} = B^{(1)}(W) &:= \{0^{j-1}(10^{a_i})^k;\ 1 \le i,\ 1 \le j \le a_i+1,\ 1 \le k \le e_i - 1\}, \\ B^{(2)} = B^{(2)}(W) &:= \{0^{j-1}(10^{a_i})^{e_i};\ 1 \le i,\ 1 \le j \le a_{i-1}+1\}. \end{aligned}$$

It is easy to check $\{w \in B(W);\ |w|_1 \ne 0\} \supset B^{(1)} \cup B^{(2)}$. Hence we have $\{w \in B(W);\ |w|_1 \ne 0\} = B^{(1)} \cup B^{(2)}$. Since $0^n \in B(W)$ $(n \ge 0)$, we get

$$B(W) = \{0^n;\ n \ge 0\} \cup B^{(1)}(W) \cup B^{(2)}(W).$$

Noting that the right-hand side of the above equality is a disjoint union, we obtain by (4.3)

$$\begin{aligned}p(n) &= n+1\\ &+\sharp\{w=0^{j-1}(10^{a_i})^k;\ 1\le i,\ 1\le j\le a_i+1,\ 1\le k\le e_i-1,\ |w|\le n-1\}\\ &+\sharp\{w=0^{j-1}(10^{a_i})^{e_i};\ 1\le i,\ 1\le j\le a_{i-1}+1,\ |w|\le n-1\},\end{aligned}$$

which implies the theorem.

In view of Theorem 4.5, for $d_n(W) := p(n;W)-p(n-1;W)$, we have

$$\begin{aligned}d_n(W) &= 1+\sharp\{(i,j,k)\in\mathbf{N}^3;\ j\le a_i+1,\ k\le e_i-1,\ k(a_i+1)+j=n\}\\ &\quad+\sharp\{(i,j)\in\mathbf{N}^2;\ j\le a_{i-1}+1,\ e_i(a_i+1)+j=n\}\\ &= 1+\sharp\{(i,k)\in\boldsymbol{N}^2;\ 1\le n-k(a_i+1)\le a_i+1,\ k\le e_i-1\}\\ &\quad+\sharp\{i\in\boldsymbol{N};\ 1\le n-e_i(a_i+1)\le a_{i-1}+1\}\quad(n\ge 1).\end{aligned}\tag{4.12}$$

If $e_n=2$ for all $n\ge 2$, then we get by (4.12), (4.10)

$$d_n(W)=d_n(W')+t_n+u_n\quad(n\ge 1),$$

where $W'=10^{a_1}10^{a_2}10^{a_3}\cdots$, and

$$\begin{aligned}t_n = t_n(W) &:= \sharp\{i\in\mathbf{N};\ a_{i-1}+2\le n-a_i-1\le a_i+1\}\\ u_n = u_n(W) &:= \sharp\{i\in\mathbf{N};\ 1\le n-2a_i-2\le a_{i-1}+1\}.\end{aligned}$$

□

Example 4.4.

(i) W is the word (4.1) with $a_n=n-1$, $e_n=2$ for all $n\ge 1$. Then,

$$\begin{aligned}t_1t_2t_3\ldots &= 0^3101010\cdots\quad(t_n=1\ (n:\text{even}),\ t_n=0\ (n:\text{odd}),\ n\ge 3),\\ u_1u_2u_3\ldots &= 0^210101^4212^4323^4\cdots\quad(u_n=\lfloor(n-1/2)\rfloor-\lceil(n+1)/3\rceil+1,\\ &\qquad n\ge 6).\end{aligned}$$

From (4.7),

$$d'_1d'_2d'_3=12^33^24^25^2\cdots$$

follows, where $d'_n=d_n(W')$. Hence, we get

$$p(n;W)=\int_1^n 123^24^256^278^29\cdots.$$

(ii) Let W be the word (4.1) with $e_n = 2$ $(n \geq 1)$ and a_n as in Example 4.2.(iii) such that $a_1 = 1, a_2 = 2$. Then

$$t_1t_2t_3 \cdots = 0^3 1010^{a_1} 1^{a_1} 0^{a_2} 1^{a_3} 0^{a_3} \cdots ,$$
$$u_1u_2u_3 \cdots = 0^4 101^2 1^{a_2+1} 0^{a_2-1} 1^{a_3-1} 1^{a_4+1} 0^{a_4-1} \cdots ,$$

so that

$$p(n; W) = \int_1^n 1^2 23434^2 3^{a_3-2} 4^2 3^{a_4-2} 4^2 3^{a_5-2} \cdots .$$

5. Estimate of the complexity function of $*$-Sturmian words

Related to the bounds of the usual complexity of $*$-Sturmian words, we can show the following Theorems 5.1, 5.2.

Theorem 5.1. *Any $*$-Sturmian word W is deterministic, i.e.,*

$$\lim_{n \to \infty} \frac{\log(p(n; W))}{n} = 0.$$

Proof. If W is ultimately periodic, then $p(n)$ is bounded, so that we obtain the theorem. We suppose that W is not ultimately periodic. Then, Theorem 2.6 implies that there exists an infinite sequence $\{k_n\}_{n=1,2,\dots}$ such that

$$W = u_0 u_1 \cdots u_m \cdots ,$$

see the notation in (iii), Theorem 2.6. We put

$$W'_k := u_k u_{k+1} \cdots \in L^{\boldsymbol{N}} \ (k \in \boldsymbol{N}).$$

Since u_m is a finite word strictly over $\{A_m, B_m\}$ for $m \geq k$, we can write

$$W'_k = P_0 P_1 P_2 \cdots ,$$

where $P_0, P_1, \cdots \in \{A_k, B_k\}$. We suppose that $|B_k| \geq |A_k|$. (If $|B_k| \leq |A_k|$, we will have a similar proof.) We denote by Ψ the set of all finite sets of integers. Let n be a positive integer. We define a map $\Delta : D(n; W') \to \mathbf{N} \times \Psi$ as follows. For $A \in D(n; W')$, we denote by $\tau(A)$ the set

$$\tau(A) := \{(j, l) \in \mathbf{N}^2;\ A \text{ is a subword of } P_j \cdots P_l\}.$$

We choose a $(j', l') \in \tau(A)$ such that

$$j' = \min\{j; (j, l) \in \tau(A)\},$$
$$l' = \min\{l; (j', l) \in \tau(A)\},$$

and we put $h := \min\{|P|; P_{j'} \cdots P_{l'} = PAP'\}$. We define $\Delta(A)$ by

$$\Delta(A) := (h, \{1 + \sum_{s=j'}^{t} |P_s|; P_{t+1} = B_k, j' \leq t \leq l' - 1\} \cup \{\sum_{s=j'}^{l'} |P_s|\} \cup \iota),$$

where
$$\iota = \begin{cases} \{1\} & \text{if } P_{j'} = B_k, \\ \emptyset & \text{if } P_{j'} = A_k. \end{cases}$$
It is not difficult to show that the map Δ is injective and
$$\Delta(D(n;W')) \subset \{0,1,\ldots,|B_k|\} \times \Psi',$$
where
$$\Psi' := \left\{ \psi \in \Psi \,\middle|\, \begin{array}{l} \max\limits_{x\in\psi} x \le n + 2|B_k|, \\ t - s \ge |B_k| \text{ for } s,t \in \psi \text{ with } s < t \text{ and } t \ne \max\limits_{x\in\psi} x. \end{array} \right\}$$
The condition in the definition of Ψ' implies that
$$\sharp\Psi' \le (|B_k|+1)^{(\frac{n}{|B_k|}+4)}.$$
Therefore, we have
$$p(n;W') \le (|B_k|+1)^{(\frac{n}{|B_k|}+5)},$$
which implies
$$\frac{\log p(n;W')}{n} \le (\frac{1}{|B_k|} + \frac{5}{n}) \log(|B_k|+1).$$
On the other hand, we get
$$\frac{\log p(n;W)}{n} \le \frac{\log(p(n;W') + |u_0u_1\cdots u_{k-1}|)}{n}.$$
For $n \ge 2$, $k \ge 2$, $p(n;W') \ge 2$ and $|u_0u_1\cdots u_{k-1}| \ge 2$ hold, so that
$$\begin{aligned} \frac{\log p(n;W)}{n} &\le \frac{\log(p(n;W')}{n} + \frac{\log(|u_0u_1\cdots u_{k-1}|)}{n} \\ &\le (\frac{1}{|B_k|} + \frac{5}{n}) \log(|B_k|+1) + \frac{\log(|u_0u_1\cdots u_{k-1}|)}{n}. \end{aligned}$$
Noting $\max\{|A_k|,|B_k|\} \to \infty$, $(k \to \infty)$, we obtain the theorem. □

Theorem 5.2. *For any small positive number ϵ there exists a $*$-Sturmian word U such that $p(U;n) > 2^{n^{1-\epsilon}}$ holds for all sufficiently large integer n.*

We need a lemma for the proof of Theorem 5.2.

Lemma 5.1. *Let $k_1, k_2, \ldots, k_n \in \{0,1\}$, and A_n, B_n be as in Definition 2.4. If $k_n = 1$, then there exist words P, Q such that $A_n = P0Q, B_n = P1$, i.e., $A_n \not\prec_p B_n$ and $B_n \not\prec_p A_n$.*

Proof. The first letter of A_n is 0 and the last letter of B_n is 1. We shall prove the lemma by induction on n. When $n = 1$, we have $A_1 = 01, B_1 = 1$, so that the lemma holds with $P = \lambda$, $Q = 1$. Let $A' = \delta_{k_2} \circ \cdots \circ \delta_{k_n}(0), B' =$

$\delta_{k_2} \circ \cdots \circ \delta_{k_n}(1)$. By the induction hypothesis, there exist words P', Q' so that $A' = P'0Q', B' = P'1$. If $k_1 = 0$, then

$$A_n = \delta_0(A') = \delta_0(P')0\delta_0(Q'), \qquad B_n = \delta_0(B') = \delta_0(P')01.$$

Since the first letter of $\delta_0(Q')$ is 0, setting $P = \delta_0(P')0$, we get the lemma. If $k_1 = 1$, then

$$A_n = \delta_1(A') = \delta_1(P')01\delta_1(Q'), \qquad B_n = \delta_1(B') = \delta_1(P')1.$$

Setting $P = \delta_1(P')$, we get the lemma. □

Proof of Theorem 5.2. Let $w_1 = 0\cdots0, w_2 = 0\cdots01, \ldots, w_{2^n} = 1\cdots1$ be all elements of L^n ordered in lexicographic order. We put $W(n) = w_1 w_2 \cdots w_{2^n}$. Let $k_1, k_2, \ldots, k_m, \ldots \in \{0,1\}$ and $k_m = 1$ for infinitely many m. We set

$$U = \delta_{k_1}(W(N_1))\delta_{k_1} \circ \delta_{k_2}(W(N_2)) \cdots \delta_{k_1} \circ \cdots \circ \delta_{k_m}(W(N_m)) \cdots ,$$

where $N_m \in \boldsymbol{N}$ $(m \geq 1)$. By virtue of Theorem 2.6, U is a $*$-Sturmian word. We suppose $k_m = 1$. Since $|\delta_{k_1} \circ \cdots \circ \delta_{k_m}(1)| < |\delta_{k_1} \circ \cdots \circ \delta_{k_m}(0)| = |A_m|$,

$$|\delta_{k_1} \circ \cdots \circ \delta_{k_m}(w)| \leq N_m |A_m|$$

for any $w \in L^{N_m}$. By Lemma 5.1, for any $w_i, w_j \in L^{N_m}$, $\delta_{k_1} \circ \cdots \circ \delta_{k_m}(w_i) \not<_p \delta_{k_1} \circ \cdots \circ \delta_{k_m}(w_j)$ and $\delta_{k_1} \circ \cdots \circ \delta_{k_m}(w_j) \not<_p \delta_{k_1} \circ \cdots \circ \delta_{k_m}(w_i)$. Thus, we have shown that $k_m = 1$ implies $p(U; N_m|A_m|) \geq 2^{N_m}$. Taking a sufficiently large integer m with $k_m = 1$, we can choose integers $N = N_m$ and $n = n(m)$ such that $2^{1-\frac{\epsilon}{4}}|A_m|^{\frac{1}{\epsilon}-1} > N > 2^{1-\frac{\epsilon}{2}}|A_m|^{\frac{1}{\epsilon}-1}$, $2|A_m|^{\frac{1}{\epsilon}} > n > N|A_m|$; then $n^{1-\epsilon} < 2^{1-\epsilon}|A_m|^{\frac{1}{\epsilon}-1} < N$. We get

$$p(U; n) > p(U; N|A_m|) > 2^N > 2^{n^{1-\epsilon}}.$$

Since $k_i = 1$ for infinitely many i, $p(U; n) > 2^{n^{1-\epsilon}}$ holds for infinitely many n. In particular, if $k_i = 1$ for all i, then $|A_m| = m + 1$ for all m and $p(U; n) > 2^{n^{1-\epsilon}}$ for n satisfying $2(m+1)^{\frac{2}{\epsilon}} > n > 2^{1-\frac{\epsilon}{4}}(m+1)^{\frac{1}{\epsilon}}$. If $m > (2^{1+\epsilon+\epsilon^2/4} - 2^{\epsilon})/(2^{\epsilon} - 2^{\epsilon-\epsilon^2/4})$, then $2^{1-\frac{\epsilon}{4}}(m+2)^{\frac{1}{\epsilon}} < 2(m+1)^{\frac{1}{\epsilon}}$ and $(2(m+1)^{\frac{2}{\epsilon}}, 2^{1-\frac{\epsilon}{4}}(m+1)^{\frac{1}{\epsilon}}) \cap (2(m+2)^{\frac{2}{\epsilon}}, 2^{1-\frac{\epsilon}{4}}(m+2)^{\frac{1}{\epsilon}}) \neq \emptyset$, where (p, q) denotes the interval $\{x \in \boldsymbol{N}; p < x < q\}$. Therefore, there exists m such that $2(m+1)^{\frac{2}{\epsilon}} > n > 2^{1-\frac{\epsilon}{4}}(m+1)^{\frac{1}{\epsilon}}$ for any sufficiently large n, which implies the theorem. □

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