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CONVEXITIES OF LATTICE ORDERED GROUPS

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Abstract. In this paper an injective mapping of the class of all infinite cardinals into the collection of all convexities of lattice ordered groups is constructed; this generalizes an earlier result on convexities of d-groups.

Keywords: lattice ordered group, convex ℓ-subgroup, direct product, convexity of lattice ordered groups

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The notion of convexity of lattices has been introduced by E. Fried ([9], p. 225; cf. also [4]). By applying analogous postulates we can define convexities also for other types of ordered algebraic structures.

In the present paper the collection $\mathcal{C}(\mathcal{L})$ of all convexities of lattice ordered groups will be investigated.

An injective mapping of the class of all infinite cardinals into the collection $\mathcal{C}(\mathcal{L})$ will be constructed; hence $\mathcal{C}(\mathcal{L})$ is a proper class. This generalizes a result from [6] concerning convexities of d-groups.

The notion of torsion class is due to J. Martinez [8]. For some torsion classes (which have been studied in literature) we shall deal with the question whether they are convexities.

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1. Preliminaries

We shall apply the standard notation for lattice ordered groups. The group operation in a lattice ordered group will be written additively; the commutativity of this operation will not be assumed.

Let $\mathcal{L}$ be the class of all lattice ordered groups. A nonempty subclass of $\mathcal{L}$ will be said to be a convexity of lattice ordered groups if it is closed under homomorphic images, convex ℓ-subgroups and direct products.

We denote by $\mathcal{C}(\mathcal{L})$ the collection of all convexities of all lattice ordered groups. This collection is partially ordered by inclusion. The least element of $\mathcal{C}(\mathcal{L})$ is the class X_0 consisting of all one-element lattice ordered groups.

Let $\emptyset \neq X \subseteq \mathcal{L}$. We denote by

HX—the class of all homomorphic images of elements of X;
CX—the class of all convex ℓ-subgroups of elements of X;
PX—the class of all direct products of elements of X.

1.1. Lemma. *Let $\emptyset \neq X \subseteq \mathcal{L}$. Then*

(i) *$HCPX \in \mathcal{C}(\mathcal{L})$;*
(ii) *for each $Y \in \mathcal{C}(\mathcal{L})$ with $X \subseteq Y$ the relation $HCPX \subseteq Y$ is valid.*

The proof will be omitted. For analogous results concerning convexities of lattices and convexities of d-groups cf. [9], p. 256 and [6].

In view of 1.1. the convexity $HCPX$ will be said to be generated by X.

The direct product of lattice ordered groups A and B will be denoted by $A \times B$. If I is any nonempty system of indices and $G_i \in \mathcal{L}$ for each $i \in I$, then $\prod_{i \in I} G_i$ denotes the direct product of the system $\{G\}_{i \in I}$. If $I = \emptyset$, then we put $\prod_{i \in I} G_i = \{0\}$.

When no confusion can occur, then for $j \in I$ the lattice ordered group G_j will be identified with the ℓ-subgroup of $\prod_{i \in I} G_i$ consisting of all elements g of the direct product under consideration such that $g(i) = 0$ for each $i \in I \setminus \{j\}$.

If $G \in \mathcal{L}$, $g \in G$ and if D is an ℓ-ideal of G, then we put $\overline{x} = x + D$; for $X \subseteq G$ we set $\overline{X} = \{\overline{x} \colon x \in X\}$.

We will apply below the following well-known results:

1.2. Lemma. *Let $G \in \mathcal{L}$, $G = A \times B$ and let D be a convex ℓ-subgroup of G. Then $D = (A \cap D) \times (B \cap D)$.*

1.3. Lemma. *Let G, A and B be as in* 1.2. *Let D be an ℓ-ideal of G. Then $\overline{G} = G/D = \overline{A} \times \overline{B}$.*

2. The lattice ordered groups G_α

For each infinite cardinal α we denote by J_α the first ordinal having the power α.

The additive group of all integers with the natural linear order will be denoted by Z. Let α be a fixed infinite cardinal and for each $j \in J_\alpha$ let $P_j = Z$. Now let $Q'(\alpha)$ be the lexicographic product of the system $\{P_j\}(j \in J_\alpha)$. The ℓ-subgroup of $Q'(\alpha)$ consisting of all elements q' such that the set $\{j \in J(\alpha)\colon q'(j) \neq 0\}$ is finite will be denoted by $Q(\alpha)$.

Let G_α be the set of all triples (x, y, z) such that $x, y \in Q(\alpha)$ and $z \in Z$. For $(x_1, y_1, z_1), (x_2, y_2, z_2) \in G_\alpha$ we put $(x_1, y_1, z_1) \leqslant (x_2, y_2, z_2)$ if either

(i) $z_1 < z_2$,

or

(ii) $z_1 = z_2$ and $x_1 \leqslant x_2$, $y_1 \leqslant y_2$.

Next we define the binary operation $+$ in G_α as follows.

a) If z_1 is even, then we put

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2).$$

b) If z_1 is odd, then we define

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + y_2, y_1 + x_2, z_1 + z_2).$$

Then G_α is a non-abelian lattice ordered group. Clearly $\operatorname{card} G_\alpha = \alpha$.

The class of all infinite cardinals will be denoted by J.

2.1. Lemma. *Let $\alpha, \beta \in J$, $\beta < \alpha$. Then G_β does not belong to the class $HCP\{G_\alpha\}$.*

P r o o f. By way of contradiction, assume that G_β belongs to $HCP\{G_\alpha\}$. Thus there exist $B \in CP\{G_\alpha\}$ and an ℓ-ideal D of B such that B/D is isomorphic to G_β. Next, there is an indexed system $\{A_i\}_{i \in I}$ of lattice ordered groups such that $A_i = G_\alpha$ for each $i \in I$ and B is a convex ℓ-subgroup of the lattice ordered group $A = \prod_{i \in I} A_i$.

For $a \in A$ we denote by a_i the component of a in the direct factor A_i. Let $b \in B \setminus D$, $i \in I$, $b_i = (x_i, y_i, z_i)$. If for each such b and each $i \in I$ the relation $z_i = 0$ is valid, then B/D is commutative; since G_β fails to be commutative, we obtain a contradiction. Therefore there exists $b \in B \setminus D$ such that $z_i \neq 0$ for some $i \in I$. Denote

$$I_1 = \{i \in I\colon z_i \neq 0\}, \quad I_2 = I \setminus I_1.$$

Hence $I_1 \neq \emptyset$. Put

$$A^1 = \prod A_i \quad (i \in I_1), \qquad A^2 = \prod A_i \quad (i \in I_2),$$
$$B^1 = B \cap A^1 \qquad\qquad B^2 = B \cap A^2.$$

Then $A = A^1 \times A^2$, hence in view of 1.2 and 1.3,

$$B/D = \overline{B}_1 \times \overline{B}_2. \tag{1}$$

There exist elements b' and b'' in A such that

$$b' = (x_i, y_i, 0)_{i \in I}, \quad b'' = (0, 0, z_i)_{i \in I}.$$

For each $i \in I$ we have

$$-|b_i| \leqslant b'_i \leqslant |b_i|, \quad -2|b_i| \leqslant b''_i \leqslant 2|b_i|,$$

hence both b' and b'' belong to B.

For each $t \in B$ we put $\overline{t} = t + D$. Clearly $b = b' + b''$ and $\overline{b} = \overline{b'} + \overline{b''}$. If $\overline{b''} = \overline{0}$ (i.e., $\overline{b} = \overline{b'}$) for all $b \in B$ with the above mentioned properties, then B/D would be abelian, which is impossible. Hence without loss of generality we can suppose that $b = b''$. Further, we can suppose that $b' > 0$.

We have $b \in B$ and $\overline{b} \neq \overline{0}$, whence $\overline{B^1}$ is a nonzero lattice ordered group. It is obvious that G_β is directly indecomposable, thus so is B/D. Hence (1) yields that $B/D = \overline{B}_1$. Therefore we can assume without loss of generality that $I = I_1$, whence

$$z_i > 0 \quad \text{for each } i \in I. \tag{2}$$

The relation (2) yields that whenever $b^1 \in A$ such that $b^1_i = (x^1_i, y^1_i, 0)$ for each $i \in I$, then $-b < b^1 < b$, whence $b^1 \in B$.

If for each b^1 with the above mentioned properties the relation $b^1 \in D$ holds, then B/D is commutative, which is a contradiction. Hence among the elements b^1 under consideration there exists at least one with $b^1 \neq D$. Below we deal with this fixed b^1.

Let b^{11} be the element of A with $b^{11}_i = (x^1_i, 0, 0)$ for each $i \in I$; similarly, let $b^{12} \in A$ such that $b^{12}_i = (0, y^1_i, 0)$ for each $i \in I$. Then either b^{11} or b^{12} does not belong to D. Without loss of generality we can suppose that $b^{11} \notin D$ and that $b^{11} > 0$.

Let $I_{11} = \{i \in I \colon b^{11}_i \neq 0\}$, $I_{12} = I \setminus I_{11}$. Next, we put

$$B^{11} = \{t \in B \colon t_i = 0 \quad \text{for each } i \in I_{12}\},$$
$$B^{12} = \{t \in B \colon t_i = 0 \quad \text{for each } i \in I_{11}\}.$$

Then $B = B^{11} \times B^{12}$. Hence in view of 1.3,

$$B/D = \overline{B^{11}} \times \overline{B^{12}}.$$

Clearly $b^{11} \in B^{11} \setminus D$, therefore $\overline{b^{11}} \in \overline{B^{11}}$ and $\overline{b^{11}} \neq \overline{0}$. Thus $\overline{B^{11}} \neq \{\overline{0}\}$. From the fact that G_β is directly indecomposable we obtain that $B/D = \overline{B^{11}}$. Now it is obvious that instead of A and B it suffices to take the lattice ordered groups

$$\prod_{i \in I_{11}} A_i, \quad B \cap \prod_{i \in I_{11}} A_i,$$

respectively. This means that without loss of generality we can suppose the validity of the relation $I = I_{11}$. Hence $b_i^{11} > 0$ for each $i \in I$. Hence $x_i^1 > 0$ for each $i \in I$.

Now we apply the fact that x_i^1 belongs to Q_α. Let $j(i)$ be the least element of J_α with $x_i^1(j(i)) \neq 0$.

In view of the definition of J_α there exists a monotone injection ψ_i of J_α onto $\{j \in J_\alpha : j \geqslant j(i)\}$.

Let J_α^0 be the set of all elements of J_α which are distinct from the least element of J_α. We construct the elements b^j $(j \in J_\alpha^0)$ in A as follows. For each $i \in I$ and $j \in J_\alpha$ let $b_i^j = (x_i^j, 0, 0)$ where, for each $j(1) \in J_\alpha$, we have

$x_i^j(j(1)) = 1$ if $j(1) = \psi_i(j)$ and

$x_i^j(j(1)) = 0$ otherwise.

Thus $b^j \in B$ for each $j \in J_\alpha^0$.

If $j(1)$ and $j(2)$ are elements of J_α^0 with $j(1) < j(2)$, then

$$|b^1| < b^{j(2)} - b^{j(1)}. \tag{3}$$

Hence if $b^{j(2)} - b^{j(1)} \in D$, we would have $b^1 \in D$, which is a contradiction. Therefore $b^{j(1)} + D$ and $b^{j(2)} + D$ are distinct elements of B/D. Thus $\operatorname{card}(B/D) \geqslant \operatorname{card} J_\alpha^0 = \alpha$. On the other hand, $\operatorname{card}(B/D) = \operatorname{card} G_\beta = \beta$ and so we arrived at a contradiction. □

2.2. Theorem. *For each infinite cardinal α let $\varphi(\alpha) = HCP\{G_\alpha\}$, where G_α is as above. Then φ is an injective mapping of the class J of all infinite cardinals into the collection of all convexities of lattice ordered groups.*

P r o o f. This is a consequence of 1.1. and 2.1. □

We apply the notion of a d-group in the same sense as in the paper of Kopytov and Dimitrov [7]; cf. also [5]. Convexities of d-groups were investigated in [6].

Let $\mathcal{D}$ be the class of all d-groups and $\mathcal{C}(\mathcal{D})$ the collection of all convexities of d-groups. Since the class $\mathcal{L}$ of all lattice ordered groups is a variety in $\mathcal{D}$ (cf. [7])

and since each variety in $\mathcal{D}$ is an element of $\mathcal{C}(\mathcal{D})$, we conclude that each convexity of lattice ordered groups is, at the same time, a convexity of d-groups. In fact, $\mathcal{C}(\mathcal{L})$ is an interval of $\mathcal{C}(\mathcal{D})$. Thus 2.2 implies

2.3. Corollary. (Cf. [6].) *There exists an injective mapping of the class of all infinite cardinals into the collection $\mathcal{C}(\mathcal{D})$.*

3. Concluding remarks; radical classes and torsion classes

3.1. Each variety of lattice ordered groups is a convexity. This is an immediate consequence of the definition of convexity.

3.2. A convexity of lattice ordered groups need not be closed with respect to ℓ-subgroups. For example, let G_α and G_β be as in Section 2 (α and β are infinite cardinals with $\beta < \alpha$). Then G_β is isomorphic to an ℓ-subgroup of G_α, but G_β does not belong to the convexity generated by G_α.

3.3. A nonempty class X of lattice ordered groups is said to be closed under joins of convex ℓ-subgroups if, whenever $G \in \mathcal{L}$ and $\{G_i\}_{i\in I}$ is a system of convex ℓ-subgroups of G such that G_i belongs to X for each $i \in I$, then the join $\bigvee\limits_{i\in I} G_i$ also belongs to X.

A nonempty class Y of lattice ordered groups is called a radical class [3] if it is closed under isomorphisms, convex ℓ-subgroups and joins of convex ℓ-subgroups.

A radical class which is closed under direct products is called a product radical class; this notion was studied by Dao Rong Ton [2]. Hence a product radical class which is closed under homomorphic images is a particular case of convexity.

A radical class of lattice ordered groups need not be a convexity. For example, the class of all archimedean lattice ordered groups is a radical class, but it fails to be a convexity (since it is not closed under homomorphic images).

3.4. A radical class which is closed under homomorphic images is called a torsion class (Martinez [8]). A torsion class is a convexity iff it is closed under direct products.

The main results of Conrad's paper [1] consist in a detailed investigation of torsion classes **A**, **F**, $\mathbf{F}_v$, **D**, **O**, **R** and **B** (for definitions of these classes cf. below; they have been studied also in other papers). Let us consider the question which of these torsion classes are convexities.

The torsion classes under consideration are defined as follows:

A—all hyperarchimedean lattice ordered groups;

F—all lattice ordered groups such that each bounded disjoint subset is finite;

$\mathbf{F}_v$—all finite valued lattice ordered groups;

D—all lattice ordered groups whose regular subgroups satisfy the descending chain condition;

O—all cardinal sums of linearly ordered groups;

R—all cardinal sums of archimedean linearly ordered groups;

B—all lattice ordered groups such that each prime exceeds a unique minimal prime.

Let Z be as above (cf. Section 2). Then Z belongs to each of the torsion classes under consideration. Let I be an infinite set and for each $i \in I$ let $G_i = Z$, $G = \prod_{i \in I} G_i$.

3.4.1. Suppose that $I = \mathbb{N}$ (the set of all positive integers). Let f and g be elements of G such that $f(n) = n$ and $g(n) = 1$ for each $n \in \mathbb{N}$. Then $f \wedge ng < f \wedge (n+1)g$ for each $n \in \mathbb{N}$. Hence in view of Theorem 1.1. in [1] the lattice ordered group G is not hyperarchimedean. Therefore **A** is not a convexity.

3.4.2. G does not belong to **F**, hence $\mathbf{F} \notin \mathcal{C}(\mathcal{L})$.

3.4.3. Let $g \in G$ be such that $g(i) = 1$ for each $i \in I$. Then g has infinitely many values in G, hence $G \notin \mathbf{F}_v \notin \mathcal{C}(\mathcal{L})$.

3.4.4. Let G be as in 3.4.1 and for each $j \in I$ let $G^j = \{g \in G\colon g(i) = 0$ for each $i \in I$ with $i < j\}$. Then $G^1 \supset G^2 \supset G^3 \supset \ldots$ and the set $\{G^n\}_{n \in I}$ has no minimal element. Also, all G^n are regular subgroups of G. Hence $G \notin \mathbf{D}$ and so $\mathbf{D} \notin \mathcal{C}(\mathcal{L})$.

3.4.5. The lattice ordered group G does not belong to **O**, hence $G \notin \mathbf{R}$. Therefore $\mathbf{O} \notin \mathcal{C}(\mathcal{L})$ and $\mathbf{R} \notin \mathcal{C}(\mathcal{L})$.

3.4.6. Now we will show that **B** is a convexity. We will apply the following result (cf. [1], p. 492):

($*$) For each lattice ordered group G the following are equivalent:

(i) $G \in \mathbf{B}$.

(ii) Each pair of incomparable primes in G generates G.

Let I be a nonempty set of indices and for each $i \in I$ let B_i be a lattice ordered group belonging to **B** with $B_i \neq \{0\}$. Put $G = \prod_{i \in i} B_i$. We have to verify that G belongs to **B** as well.

First we consider the question what is the general form of primes in G. Let H^1 be a prime in G. Let $I(H^1)$ be the set of all $i \in I$ having the property that there is $g \in G \setminus H^1$ such that $g(i) \neq 0$. Then $I(H^1) \neq \emptyset$.

Suppose that $i(1)$ and $i(2)$ are distinct elements of $I(H^1)$. Put $\overline{G}_{i(1)} = \{g^1 \in G: g'(i(1)) = 0\}$, and let $\overline{G}_{i(2)}$ be defined analogously. Next, let

$$Q_1 = H^1 + G_{i(1)}, \quad Q_2 = H^1 + G_{i(2)}.$$

Then Q_1 and Q_2 are convex ℓ-subgroups of G and $H^1 \subseteq Q_j$ $(j = 1, 2)$. We have neither $Q_1 \subseteq Q_2$ nor $Q_2 \subseteq Q_1$. Nonetheless, since H^1 is prime, the system of all convex ℓ-subgroups Q of G with $H^1 \subseteq Q$ is linearly ordered; hence we arrive at a contradiction. Therefore $I(H^1)$ is a one-element set, $I(H^1) = \{i(1)\}$. Hence $\overline{G}_{i(1)} \subseteq H^1$ and thus

$$H^1 = (H^1 \cap G_{i(1)}) + \overline{G}_{i(1)}. \tag{4}$$

It is easy to verify that $H^1 \cap G_{i(1)}$ is a prime subgroup of H^1.

Conversely, if $H^1 \in C\{G\}$, $i(1) \in I$, $H^1 \cap G_{i(1)}$ is a prime in $G_{i(1)}$ and if (4) holds, then H^1 is a prime in G.

Let H^2 be a prime in G such that H^1 and H^2 are incomporable. There is $i(2) \in I$ such that $I(H^2) = \{i(2)\}$. Analogously as above we have

$$H^2 = (H^2 \cap G_{i(2)}) + \overline{G}_{i(2)}.$$

We distinguish two cases.

(i) First suppose that $i(1) \neq i(2)$. Then $\overline{G}_{i(1)} + \overline{G}_{i(2)} = G$, whence the pair H^1 and H^2 generates G.

(ii) Next suppose that $i(1) = i(2)$. Denote

$$H_0^1 = H^1 \cap G_{i(1)}, \quad H_0^2 = H^2 \cap G_{i(1)}.$$

Then H_0^1 and H_0^2 are incomparable primes in $G_{i(1)}$. Thus, since $G_{i(1)}$ belongs to $\mathbf{B}$, in view of $(*)$ the pair H_0^1 and H_0^2 generates $G_{i(1)}$. Therefore the pair H^1 and H^2 generates G.

By applying $(*)$ again we infer that G belongs to $\mathbf{B}$.

References

[1] *P. Conrad*: Torsion radicals of lattice ordered groups. Symposia Math. 31. Academic Press, New York-London, 1977, pp. 479–513.

[2] *Dao-Rong Ton*: Product radical classes of ℓ-groups. Czechoslovak Math. J. *42* (1992), 129–142.

[3] *J. Jakubík*: Radical classes and radical mappings of lattice ordered groups. Symposia Math. 31. Academic Press, New York-London, 1977, pp. 451–477.

[4] *J. Jakubík*: On convexities of lattices. Czechoslovak Math. J. *42* (1992), 325–330.

[5] *J. Jakubík*: On directed groups with additional operations. Math. Bohem. *118* (1993), 11–17.
[6] *J. Jakubík*: On convexities of d-groups. Czechoslovak Math. J. *44* (1994), 305–314.
[7] *V. M. Kopytov and Z. J. Dimitrov*: On directed groups. Sibirsk. Mat. Zh. *30* (1989), 78–86. (In Russian.)
[8] *J. Martinez*: Torsion theory of lattice ordered groups. Czechoslovak Math. J. *25* (1975), 284–299.
[9] *R. Mlitz (ed.)*: General algebra 1980. Proc. Internat. Conf. Krems 1988. North Holland, Amsterdam-New York-Tokyo-Oxford, 1990.

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