122 (1997) MATHEMATICA BOHEMICA No. 1, 13–20

IDEAL BANACH CATEGORY THEOREMS AND FUNCTIONS

Zbigniew Piotrowski, Youngstown

(Received July 31, 1995)

Abstract. Based on some earlier findings on Banach Category Theorem for some "nice" σ-ideals by J. Kaniewski, D. Rose and myself I introduce the h operator (h stands for "heavy points") to refine and generalize kernel constructions of A. H. Stone. Having obtained in this way a generalized Kuratowski's decomposition theorem I prove some characterizations of the domains of functions having "many" points of h-continuity. Results of this type lead, in the case of the σ-ideal of meager sets, to important statements of Abstract Analysis such as Blumberg or Namioka-type theorems.

Keywords: Banach Category Theorem, categorical almost continuity, Blumberg space, separate and joint continuity

MSC 1991: 54E52, 54A25, 54B15, 54C08

1. Introduction

Given a topological space (X,τ), let $\mathcal{T} \subseteq \mathcal{P}(X)$ be an ideal of subsets of X. For any subset $A \subseteq X$, let $A^*(\tau,\mathcal{T})$ or simply A^* if τ and $\mathcal{T}$ are understood, be the adherence of A modulo $\mathcal{T}$. In particular, $A^* = \{x \in X\colon x \in U \in \tau$ implies $U \cap A \notin \mathcal{T}\}$. Observe that A^* is a closed subset of cl$A$. For convenience $A^0(\tau,\mathcal{T})$ or simply A^0, if τ and $\mathcal{T}$ are understood, denote the set $A \setminus A^*$. In the terminology of A. M. Stone, $A \cap A^*(\tau,\mathcal{T})$ is the kernel of the subspace $(A,\tau|A)$ (relative to the ideal $\mathcal{T}|A = \mathcal{T} \cap \mathcal{P}(A)$).

The following three conditions have been intensively studied in [Kaniewski, Piotrowski and Rose, 5].

B_1: Let $D \subset X$. Suppose that for every $\emptyset \neq U_{\text{open}}$ there is a nonempty open $V \subset U$ such that $V \cap D$ is an I-set in X. Then D is an I-set in X.

B_2: Let $D \subset X$. The set of all points x of D for which there is an open nbd $U \ni x$ such that $(U \cap D) \in I$, is an I-set. Actually, B_2 may be formulated as follows:

$$\forall D \subset X, \text{ the set } K_I(D) = D \cap \bigcup \{U_{\text{open}} \subset X \colon (U \cap D) \in I\} \in I \tag{1}$$

B_3: In a topological space X, the union of any family of open I-sets is an I-set in X, or equivalently $K_I(X) \in I$.

2. Kuratowski's Decomposition Theorem and the h operator

As was already noticed in Introduction, the set A^* is a closed subset of cl A, the closure of A in X. So, if $A = X$, we derive that the set X^*, the kernel of X, is closed in X. Now, since X^0, the co-kernel of X, is defined as the complement of X^* in X, we conclude that X^0 is an open subset of X.

Summarizing, we have

Generalized Kuratowski's Decomposition Theorem—first version. *Let (X, τ) be a topological space and let $\mathcal{I} \subseteq \mathcal{P}(X)$ be a σ-ideal of subsets of X satisfying B_2. Then X can be uniquely decomposed into a closed subspace A^*, possibly empty, which is the adherence of A modulo $\mathcal{I}$, and an open subspace A^0, possibly empty, where $A^0 \in \mathcal{I}$.*

The original decomposition theorem, due to K. Kuratowski, formulated for the σ-ideal $\mathcal{M}(\tau)$ of meager sets, can be easily obtained from the following version of the Banach Category Theorem [Blumberg, 1].

($\star$) If X is a metric space, $D \subseteq X$ and $S \subseteq D$ is the set of points in D which are meager relative to D, then S is meager in X.

It is easy to see that ($\star$) above is a special case of B_2, see Introduction. Also, it can be easily checked that B_2 *suffices* to derive *this version* of the decomposition theorem.

We shall now exhibit an example from which the reader will see the need of further refinement of the notion of the $(-)^*$ operator.

Example 2.1. Let (X, τ) be the closed upper half-plane with the Euclidean topology and $\mathcal{I} = \mathcal{M}(\tau)$. Further, let

$$\begin{aligned} A_+ &= \{(x,y) \colon (x \in \mathbb{Q} \wedge x > 0) \wedge (y \in \mathbb{Q} \wedge y > 0)\}, \\ A_- &= \{(x,y) \colon x < 0 \wedge y > 0 \wedge [(x \in \mathbb{R} \setminus \mathbb{Q}) \vee (y \in \mathbb{R} \setminus \mathbb{Q})]\}. \end{aligned} \tag{2}$$

Define $A = A_+ \cup A_-$.
Observe that

$$A^0 = \{(x,y) \colon x > 0 \wedge y \geqslant 0\} \text{ and } A^* = \{(x,y) \colon x \leqslant 0 \wedge y \geqslant 0\}.$$

The fact that A^* (resp. A^0) is a closed (resp. open) subset of the closed upper half-plane illustrates only the general situation, where A^* (resp. A^0) is a closed (resp. open) subset of $\mathrm{cl}A$, the closure of A in X.

What about the *open* subspace $H = \{(x, y)\colon x < 0 \wedge y \geqslant 0\}$ made of the "true," deep-inside non-meager points? Observe that while the points of H *do* have the property $(\mathcal{P})$ below, *none* of the points of the "border"—the non-negative part of the y-axis, has the property $(\mathcal{P})$.

First, we need the following definition: Let X be a space and let $A \subseteq X$. A point $x \in X$ is said to be *non-meager relative to* A if every open neighborhood U of x contains a subset B of A which is non-meager in X.

And now the promised property:

$(\mathcal{P})$ Let X be a space and let $A \subseteq X$. Let $x \in X$ be arbitrary. Then there is an open neighborhood V of x such that every point $y \in V$ is non-meager relative to A.

We shall now formulate a general case of a "good" property of all points of H.

As in Section 1, (X, τ) denotes a topological space, $\mathcal{I} \subseteq \mathcal{P}(X)$ is a σ-ideal of subsets of X. Let $A \subseteq X$; associate with A the set A^h, the *heavy part of* A defined by

$A^h = \{x \in X\colon \exists U \in \tau$ such that $x \in U$ and $U \subseteq A^*\}$.

The reader will notice

F a c t 2.1. *A^h is an open (possibly empty) subspace.*

Using quite elementary properties of the relative topology we deduce that A^h is the *maximal* open subset contained in A^*, that is :

Proposition 2.1. $A^h = \mathrm{Int}A^*$.

Let us turn to another version of the decomposition theorem.

Observe that $A^* \setminus A^h = A^* \setminus$ Int A^* is nowhere dense in X. So, if we assume that nowhere dense sets are in the σ-ideal $\mathcal{I}$, then we can "shift" $A^* \setminus A^h$ to A^0(!)

In view of Theorem 1 of Part 1, an ideal $\mathcal{I}$ satisfies B_1 if and only if $\mathcal{N}(\tau) \subseteq \mathcal{I}$ and $\mathcal{I}$ satisfies B_2.

We are now ready for

Generalized Kuratowski's Decomposition Theorem—second version.
Let (X, τ) be a topological space and let $\mathcal{I} \subseteq \mathcal{P}(X)$ be a σ-ideal of subsets of X satisfying B_1. Then X can be uniquely decomposed into an open subspace A^h, a possibly empty, the heavy part of A, and the closed subspace $A^0 \cup (A^* \setminus A^h)$ which is an element of $\mathcal{I}$.

P r o o f. Consider the first version of Generalized Kuratowski's Decomposition Theorem. By earlier remarks, the set A^h is open, and it is the maximal set with this property. So, $A^* \setminus A^h$ is nowhere dense.

Now, by Theorem 1 of Part 1—the equivalence of B_1—the set $A^* \setminus A^h \in \mathcal{I}$. Thus $A^0 \cup (a^* A^h) \in \mathcal{I}$, the subspace $A^0 \cup (A^* \setminus A^h)$ being closed in X, as the complement of A^h. □

Remark 2.1. Observe that the second version of the decomposition theorem requires *the stronger condition* B_1 rather than B_2.

3. Functions on $(X, \tau, \mathcal{I})$

Let (X, τ) be a space and let $\mathcal{I}$ be a σ-ideal of subsets of X.

Throughout this section we will need the following definitions.

Given a set $A \subseteq X$. Every element $x \in A^h$, the heavy part of A, will be called a *heavy point relative to* A.

In other words, $x \in X$ is a heavy point relative to A if and only if there is an open neighborhood U of x such that $U \subseteq A^*$.

Let $f\colon X \to Y$ be a function. We say that f *is h-continuous at* x_0, if for every open set G containing $f(x_0)$, the set $f^{-1}(G)$ has x as a heavy point relative to X.

Example 3.1. The "salt & pepper" function, $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 0$, if x is rational and $f(x) = 1$, if x is irrational, is h-continuous at every irrational $\mathcal{I}$ being the σ-ideal of countable sets.

This type of "almost continuity" is very strong. For a fairly large class of spaces and σ-ideals (or just ideals) on them, if f is h-continuous at *every* point of the domain space, then f is continuous—see [Kaniewski and Piotrowski, 5] for appropriate generalizations.

If both the domain and the range of f is the set of reals, then h-continuity of f at x_0 can be characterized by: $\exists S\colon X \setminus S \in \mathcal{I}$ and $\lim_{x \in S, x \to x_0} f(x) = f(x_0)$.

The following result is found in [Thomson, 10], see [Kaniewski and Piotrowski, 5] for generalizations.

Proposition 3.1. ([Thomson, 10], Thm 34.1, p. 78) *Let $\mathcal{I}$ be an ideal of sets in $\mathbb{R}$ which does not contain an open nonempty set. If a function $f\colon \mathbb{R} \to \mathbb{R}$ is h-continuous at every point of f, then f is continuous.*

In what follows a *network* $\mathcal{N}$ for a space X is a collection of subsets N of X such that whenever $x \in U$ with U open, then there exists $N \in \mathcal{N}$ with $x \in N \subset U$.

So, a network is like a base, but its elements need not be open. In some cases a network can be "fattened up" to a base for a space, e.g., a compact space with a countable network has a countable base [Engelking, 4].

Given a space Y having a network $\mathcal{N}$, let $f\colon X \to Y$ be a function.

Define $C_h(f, \mathcal{N})$ (or $C_n(f, \mathcal{N})$) to be the set of all points $x \in X$ such that for every $N \in \mathcal{N}$ we have:

(3) $\quad f(x) \in N$ implies $x \in [f^{-1}(N)]^h$ (or $x \in \text{Int cl } f^{-1}(N)$, respectively).

Since for every $S \subseteq X$ we have $S^* \subseteq$ cl S we have

$$C_h(f,\mathcal{N}) \subseteq C_n(f,\mathcal{N}). \tag{4}$$

When $\mathcal{N}$ is a base for τ then $C_h(f,\mathcal{N})$ (or $C_n(f,\mathcal{N})$) stands for the set $C_h(f)$ (or $C_n(f)$) of points of *h-continuity* (or *near continuity*) of f.

For any network $\mathcal{N}$ the following relations are true:

$$C_h(f,\mathcal{N}) \subset C_h(f) \text{ and } C_n(f,\mathcal{N}) \subseteq C_n(f). \tag{5}$$

Theorem 3.1. *Let $\mathcal{I}$ be a σ-ideal satisfying B_1 and let $\mathcal{N}$ be countable. Then $C_n(f,\mathcal{N})$ is "almost everywhere" in X, i.e.,*

$$X \setminus C_h(f,\mathcal{N}) \in \mathcal{I}. \tag{6}$$

P r o o f. By the definition of $C_h(f,\mathcal{N})$ we have

$$X \setminus C_h(f,\mathcal{N}) = \bigcup_{N\in\mathcal{N}} \{f^{-1}(N) \setminus [f^{-1}(N)]^h\}.$$

By the Decomposition Theorem—second version, $f^{-1}(N) \setminus [f^{-1}(N)]^h \in \mathcal{I}, N \in \mathcal{N}$. So, the countable union of elements $f^{-1}(N) \setminus [f^{-1}(N)]^h$ of the σ-ideal is in the σ-ideal, so $X \setminus C_h(f,\mathcal{N}) \in \mathcal{I}$. □

The same assertion for $C_n(f,\mathcal{N})$ does not require the strength of the Banach Category Theorem, since for any $S \subseteq X, S\setminus$ Int cl S is nowhere dense. Clearly, $C(f) \subseteq C_n(f)$, where $C(f)$ stands for the set of points of continuity of f.

Proposition 3.2. *Assume $X = X^*$. Then $C(f) \subseteq C_h(f)$.*

P r o o f. The proof of this claim is routine and as such is left to the reader. The fact that $X = X^*$ is necessary in Proposition 3.2 easily follows from Example 3.2. Let $\mathcal{M}(\tau)$ be the σ-ideal of meager sets in $\mathbb{Q}$ (the rationals) and let $f\colon \mathbb{Q} \to \mathbb{R}$ be any constant function, i.e., $f(x) = c$. Clearly, $C(f) = \mathbb{Q}$, whereas $C_h(f) = \emptyset$. □

Before we elaborate on the consequences of Theorem 3.1 and its importance in General Topology and Real Analysis, let us consider a simple condition which implies the converse of the statement made in Proposition 3.1.

Proposition 3.3. *Let (Y,γ) be a regular space with a topology γ and a network $\mathcal{N}$. If for every $V \in \gamma$ and $y \in V$ there is $N \in \mathcal{N}$ such that $y \in N \subseteq V$ and $[f^{-1}(N)]^* \subset f^{-1}(\text{cl } V)$, then $C_h(f,\mathcal{N}) \subset C(f)$.*

P r o o f. Let $x \in C_h(f,\mathcal{N})$ and $y = f(x) \in V_0 \in \gamma$. Let $V \in \gamma$ be such that $y \in V \subseteq \text{cl } V \subseteq V_0$. Further, let N be as in the assumption of our Proposition. Then $x \in [f^{-1}(N)]^h \subseteq f^{-1}(\text{cl } V) \subseteq f^{-1}(V_0)$, which shows that $x \in C(f)$. □

The following result may be also viewed as a converse to Theorem 3.1.

Theorem 3.2. *Let Y contain a countably infinite discrete subset N. Then, if every function $f\colon X \to Y$ has a dense set of h-continuity, then $X = X^*$.*

P r o o f. Suppose $X \subseteq X^*$, that is, there is an open nonempty set U s.t. $U \in \mathcal{I}$, i.e., $U = \bigcup_{i=1}^{\infty} F_i$, $F_i \in \mathcal{I}$, $i = 1, 2, 3, \ldots$. Let $N = \{n_0, n_1, n_2, \ldots\}$. Define $f\colon X \to Y$ as follows:

$$f(x) = \begin{cases} n_i, & \text{if } \ x \in F_i \\ n_0, & \text{if } \ x \in X \setminus U. \end{cases} \tag{7}$$

We claim that f does not have a dense set D of h-continuity, more specifically f does *not* have a point of h-continuity in U.

If D is dense in X, then there is an $x \in U$ such that $x \in D \cap U$, U being open in X. Then $x \in F_i$ for some i. So, $f(x) \in \{n_i\}$. The set $\{n_i\}$ is open in Y, so let us consider $f^{-1}\{n_i\}$. Observe that $f^{-1}\{n_i\} = F_i$; recall F_i's belong to the σ-ideal $\mathcal{I}$. This indicates that f is not nowhere h-continuous in U. □

The following characterization of spaces that are equal to their kernels easily follows from Theorems 3.1 and 3.2, namely:

Theorem 3.3. *Let $\mathcal{I}$ be a σ-ideal of subsets of X satisfying B_1, and let Y be an infinite, second countable, Hausdorff space. Then $X = X^*$ if and only if every function $f\colon X \to Y$ has a dense set of points of h-continuity.*

P r o o f. The range space Y has a countable infinite discrete subset N as an infinite Hausdorff space; being second countable it has a countable network. So, the assumptions of Theorem 3.1 and 3.2 are met. The conclusion of Theorem 3.3 follows easily as a corollary from the above two theorems. □

Recall that $f\colon X \to Y$ is called *categorically almost continuous* at x_0 if f is h-continuous at x_0 and $\mathcal{M}(\tau)$ is the σ-ideal of meager sets in X.

Corollary 3.1. *A space X is Baire if and only if every function $f\colon X \to \mathbb{N}$ has a dense set of points of categorical almost continuity. Here $\mathbb{N}$ stands for the set of natural numbers with the Euclidean topology.*

4. Concerning Theorem 3.1

A special case of Theorem 3.1, namely the one for the σ-ideal of meager sets is known in literature since 1922 [Blumberg, 1]; we shall refer to this special case as the Lemma on the existence of a residual set of categorical almost continuity points.

Its original statement, see [Bradford and Goffman, 2], asserts that if X is *any* topological space and Y is second countable then *every* function $f\colon X \to Y$ has a residual set of points of categorical almost continuity.

In other words, if the domain space X is Baire and the range space is second countable, then *any* function has a "thick," in fact a dense G_δ set of points of almost continuity.

We now exhibit Example 4.1 showing that the assumption that Y has a countable network (or a weaker assumption that Y is second countable) in Theorem 3.1 cannot be weakened.

Example 4.1. Let $\mathcal{E}$ and $\mathcal{D}$ denote the Euclidean and the discrete topology, respectively. Consider the identity function $F\colon (\mathbb{R},\mathcal{E}) \to (\mathbb{R},\mathcal{D})$ defined by $f(x) = x$. With the σ-ideal $\mathcal{M}(\mathcal{E})$, we see that $C_h(f,\mathcal{N})$ is empty.

We shall now exhibit Example 4.2 proving that the requirement of Theorem 3.1 that Y is second countable can *not* be relaxed to one such that Y has *both* an open-hereditarily countable pseudo-base and is hereditarily Lindelöf, see [Piotrowski, 8] for further results.

Example 4.2. Let $\mathcal{E}$ and $\mathcal{S}$ denote the Euclidean and Sorgenfrey topology, respectively. Consider the identity function $f\colon (\mathbb{R},\mathcal{E}) \to (\mathbb{R},\mathcal{S})$ given by $f(x) = x$. Again, with $\mathcal{M}(\mathcal{E})$ being the σ-ideal of meager sets in the domain space we see that $C_h(f,\mathcal{N}) = \emptyset$.

The Lemma on the existence of a residual set of categorical almost continuity points—not surprisingly—led in the past to two important results of Abstract Analysis:

Blumberg Theorem. (See [Bradford and Goffman, 2]; see also [White, 11].) *Let X be a metric Baire space and let $f\colon X \to \mathbb{R}$ be a function. Then there is a dense subset D of X such that $f|D$ is continuous on D (in the relative topology), and*

Separate and Joint Continuity Theorem. ([Ke], where one needs to prove an analogue of the lemma for multivalued functions ($\equiv$ relations) first.) *Let X be a space, let Y be second countable and let Z be a regular, second countable space. If $f\colon X \times Y \to Z$ is separately continuous, i.e., is continuous in one variable, while the other is fixed, then there is a residual subset A of X such that f is (jointly) continuous at every point of the set $A \times Y$.*

Remark 4.1. There have been studies [Brown, 3] or more recently [Recław, 9], of Blumberg theorem vis-à-vis various σ-ideals in the domain space.

An interested reader is especially urged to consult a work of B. S. Thomson [10] who proves

Theorem 4.2. ([10], Theorem 34.2, p. 79.) *Let $\mathcal{I}$ be a σ-ideal of sets that contains no interval. Then, if $f\colon \mathbb{R} \to \mathbb{R}$ is h-continuous except at the points of a set N_0 that belongs to $\mathcal{I}$, then there is a set M, whose complement is in $\mathcal{I}$ and f is continuous relative to M at each point of M.*

The proof clearly uses the Lindelöfness of the domain space. It would be interesting to get a generalization of the just quoted result of [Thomson, 10] to general topological spaces.

In conclusion, one can now apply Theorem 3.1 to obtain appropriate analogues of Blumberg's Theorem or Kenderov's theorem on separate and joint continuity.

The author acknowledges the support from Youngstown State University; when this article was written he was 1993–94 YSU Research Professor.

References

[1] *H. Blumberg*: New properties of all real functions. Trans. Amer. Math. Soc. *24* (1922), 113–128.
[2] *J. C. Bradford, C. Goffman*: Metric spaces in which Blumberg's theorem holds. Proc. Amer. Math. Soc. *11* (1960), 667–670.
[3] *J. B. Brown*: Variations on Blumberg's Theorem. Real Anal. Exchange *9* (1983), 123–137.
[4] *R. Engelking*: General Topology. Warszawa (1977).
[5] *J. Kaniewski, Z. Piotrowski*: Concerning continuity apart from a meager set. Proc. Amer. Math. Soc. *98* (1986), 324–328.
[6] *J. Kaniewski, Z. Piotrowski, D. A. Rose*: Ideal Banach category theorems. Rocky Mountain J. Math. (accepted).
[7] *P. S. Kenderov*: Multi-valued mappings and properties of them similar to continuity. Russian Math. Surveys *35* (1980), 246–249.
[8] *Z. Piotrowski*: Blumberg property versus almost continuity. Internat. J. Math. & Math. Sci. *10* (1987), 93–96.
[9] *I. Recław*: Restrictions to continuous functions and Boolean algebras. Proc. Amer. Math. Soc. *118* (1993), 791–796.
[10] *B. S. Thomson*: Real Functions. Lecture Notes in Mathematics 1170, Springer Verlag, 1985.
[11] *H. E. White, Jr.*: Topological spaces in which Blumberg's theorem holds. Proc. Amer. Math. Soc., *44* (1974), 454–462.

Author's address: Department of Mathematics and Statistics, Youngstown State University, Youngstown OH 44555-0001 USA, e-mail: `zpiotr@math.ysu.edu`.