A GENERALIZED MAXIMUM PRINCIPLE FOR BOUNDARY VALUE PROBLEMS FOR DEGENERATE PARABOLIC OPERATORS WITH DISCONTINUOUS COEFFICIENTS

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Dedicated to Professor Alois Kufner on the occasion of his 65th birthday

Abstract. We prove a generalized maximum principle for subsolutions of boundary value problems, with mixed type unilateral conditions, associated to a degenerate parabolic second-order operator in divergence form.

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1. Introduction

In [14] M. G. Platone Garroni has extended the classical generalized maximum principle (see, for instance, [15]), when the coefficients of the operator are discontinuous, to subsolutions of elliptic linear second order equations with mixed type boundary unilateral conditions, that is, on a portion of the boundary $\partial\Omega$ of Ω, the values of the solution are assigned, while on the other part a unilateral condition on the solution and its conormal derivative is given. In the present paper we will establish a similar result (see Theorem 5.1) for degenerate parabolic equations, using a technique different from that of [14]. As a corollary, we obtain a comparison theorem (see Theorem 6.1). Our procedure, rather similar to that followed in [12] and in [13] allows us to obtain more general results. Other sufficient conditions for the boundedness of weak subsolutions of Cauchy-Dirichlet problem, in the non degenerate case, may be obtained from [6] and [17], while in the degenerate case some results are announced in [3] and in [4].

Let $\mathbb{R}^m$ be the Euclidean space ($m > 2$) with a generic point $x = (x_1, x_2, \ldots, x_m)$, Ω a bounded open subset of $\mathbb{R}^m$ whose boundary satisfies locally a Lipschitz condition, T a real positive number. Let us denote by $Q(\tau_1, \tau_2)$ ($0 \leqslant \tau_1 < \tau_2 \leqslant T$) the cylinder $\Omega\times]\tau_1, \tau_2[$ and let $Q = Q(0,T)$; Γ is the parabolic boundary of Q, that is $\Gamma = \bigl(\Omega \times \{t = 0\}\bigr) \cup \bigl(\partial\Omega\times]0, T[\bigr)$.

Let $\partial\Omega_2$ be a closed subset of $\partial\Omega$, $\Gamma_2 = \partial\Omega_2 \times [0, T]$, and let us set $\partial\Omega_1 = \partial\Omega\setminus\partial\Omega_2$, $\Gamma_1 = \partial\Omega_1 \times [0, T]$.

The symbol meas_x will henceforth denote the m-dimensional measure.

If $u(x)$ is a measurable function defined in Ω, we will denote by $|u|_p$ ($1 \leqslant p \leqslant \infty$) the usual norm in the space $L^p(\Omega)$.

Hypothesis 2.1. Let $\nu(x)$ be a positive function defined in Ω such that

$$\nu(x) \in L^{\frac{g(m-1)}{g-m}}(\Omega), \quad \nu^{-1}(x) \in L^g(\Omega), \ \ g > m.$$

The symbol $\widetilde{H}^1(\nu, \Omega)$ stands for the completion of $C^1\bigl(\overline{\Omega}\bigr)$ with respect to the norm

$$\|u\|_1 = \biggl(|u|_2^2 + \sum_{i=1}^m \nu\Bigl|\frac{\partial u}{\partial x_i}\Bigr|_2^2\biggr)^{\frac{1}{2}};$$

$C^*(\Omega)$ denotes the following linear subspace of $C^\infty\bigl(\overline{\Omega}\bigr)$:

$$C^*(\Omega) = \bigl\{u \in C^\infty\bigl(\overline{\Omega}\bigr) : \ u = 0 \text{ on } \partial\Omega_2\bigr\}.$$

$H^*(\nu, \Omega)$ denotes the closure of $C^*(\Omega)$ in $\widetilde{H}^1(\nu, \Omega)$.

If $u(x,t)$ is a measurable real function in Q, we will denote by $|u|_{p,q}$ ($1 \leqslant p$, $q \leqslant +\infty$) the usual norm in the space $L^{p,q}(Q)$, with the obvious modification if p or q are $+\infty$.

Hypothesis 2.2. Let $\psi(t)$ be a positive monotone nondecreasing function defined in $]0, T[$ such that

$$\psi(t) \in L^1(0, T).$$

The symbol $\widetilde{H}^{1,0}\bigl(\nu\psi, Q(\tau_1, \tau_2)\bigr)$ ($0 \leqslant \tau_1 < \tau_2 \leqslant T$) stands for the completion of $C^1\bigl(\overline{Q(\tau_1, \tau_2)}\bigr)$ with respect to the norm

$$\|u\|_{1,0,(\tau_1,\tau_2)} = \biggl(\int_{Q(\tau_1,\tau_2)} \biggl(|u|^2 + \sum_{i=1}^m \nu\psi\Bigl|\frac{\partial u}{\partial x_i}\Bigr|^2\biggr) \,\mathrm{d}x \,\mathrm{d}t\biggr)^{\frac{1}{2}};$$

$$\|u\|_{1,0} = \|u\|_{1,0,(0,T)}.$$

– $\widetilde{H}^{1,0}\big(\nu\psi, Q(\tau_1,\tau_2)\big)$ is a Hilbert space with respect to the norm $\|u\|_{1,0,(\tau_1,\tau_2)}$.
– $C^*\big(Q(\tau_1,\tau_2)\big)$ $(0 \leqslant \tau_1 < \tau_2 \leqslant T)$ denotes the following linear subspace of $C^\infty\big(Q(\tau_1,\tau_2)\big) \cap C^0\big(\overline{Q(\tau_1,\tau_2)}\big)$:
– $C^*\big(Q(\tau_1,\tau_2)\big) = \big\{u \in C^\infty\big(Q(\tau_1,\tau_2)\big) \cap C^0\big(\overline{Q(\tau_1,\tau_2)}\big)\colon\ u = 0 \text{ on } \partial\Omega_2 \times [\tau_1,\tau_2]\big\}$.
– $\widetilde{H}^{1,0}_*\big(\nu\psi, Q(\tau_1,\tau_2)\big)$ $(0 \leqslant \tau_1 < \tau_2 \leqslant T)$ is the closure of $C^*\big(Q(\tau_1,\tau_2)\big)$ in $\widetilde{H}^{1,0}\big(\nu\psi, Q(\tau_1,\tau_2)\big)$.

Finally, we will denote by $V^{1,0}(\nu\psi, Q)$ the space of functions $u(x,t)$ belonging to $\widetilde{H}^{1,0}(\nu\psi, Q)$, continuous in $[0,T]$ with values in $L^2(\Omega)$.[1]

Definition 1. Given a real number h, if $u \in \widetilde{H}^{1,0}\big(\nu\psi, Q(\tau_1,\tau_2)\big)$ $(0 \leqslant \tau_1 < \tau_2 \leqslant T)$, we will say that $u(x,t) \leqslant h$ $(\geqslant h)$ on $\partial\Omega_i \times [\tau_1,\tau_2]$ $(i = 1,\,2)$ if there exists a sequence $\{u_n\}$ of functions from $C^1\big(\overline{Q(\tau_1,\tau_2)}\big)$ such that

$$u_n(x,t) \leqslant h \ (\geqslant h) \text{ on } \partial\Omega_i \times [\tau_1,\tau_2]$$

and

$$(*) \qquad \lim_{n\to\infty} \big\|u_n - u\big\|_{1,0,(\tau_1,\tau_2)} = 0.$$

If k is such that $u(x,t) \leqslant k$ on $\partial\Omega_i \times [\tau_1,\tau_2]$, we will say that $u(x,t)$ is bounded from above on $\partial\Omega_i \times [\tau_1,\tau_2]$.

Definition 2. If $u(x,t)$, $w(x,t)$ belong to $\widetilde{H}^{1,0}\big(\nu\psi, Q(\tau_1,\tau_2)\big)$ $(0 \leqslant \tau_1 < \tau_2 \leqslant T)$ and $w(x,t) \geqslant 0$ on $\partial\Omega_i \times [\tau_1,\tau_2]$ $(i = 1,\,2)$, let us denote[2]

$$\sup{}^* \frac{u}{w} = \inf\big\{h \in \mathbb{R}\colon\ u(x,t) - h w(x,t) \leqslant 0 \text{ on } \partial\Omega_i \times [\tau_1,\tau_2]\big\}.$$

We will consider the following generalized problem:

$$(2.1) \qquad \begin{cases} -\displaystyle\sum_{i=1}^m \frac{\partial}{\partial x_i}\Big(\sum_{j=1}^m a_{ij}\frac{\partial u}{\partial x_j} + d_i u\Big) + \Big(\sum_{i=1}^m b_i \frac{\partial u}{\partial x_i} + cu\Big) + \frac{\partial u}{\partial t} = 0 & \text{in } Q \\ \displaystyle\frac{\partial u}{\partial \nu} + \alpha\, u + \sum_{i=1}^m d_i u \cos n x_i \geqslant 0 & \text{on } \Gamma_1, \end{cases}$$

where

$$\frac{\partial u}{\partial \nu} = \sum_{i,j=1}^m a_{ij} \cos n x_j \frac{\partial u}{\partial x_i},$$

and $\cos n x_j$ is the j-th directional cosine of x, normal to Γ_1 and external to Q.

[1] For more details concerning hypotheses (2.1), (2.2) see also [5], [7], [8] and [9].
[2] We suppose that $\inf \emptyset = +\infty$.

By an L_{Γ_1}-subsolution (L_{Γ_1}-supersolution) of problem (2.1) we mean any function $u \in V^{1,0}(\nu\psi, Q)$ satisfying the following conditions:

$$\tilde{a}(u,\varphi) = \int_Q \Bigg(\sum_{i,j=1}^m a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \varphi}{\partial x_i} + \sum_{i=1}^m b_i \frac{\partial u}{\partial x_i}\varphi + cu\varphi + \sum_{i=1}^m d_i u \frac{\partial \varphi}{\partial x_i} - u\frac{\partial \varphi}{\partial t} \Bigg) \mathrm{d}x\,\mathrm{d}t + \int_{\Gamma_1} \alpha u \varphi \, \mathrm{d}\sigma\,\mathrm{d}t \leqslant 0 \quad (\geqslant 0) \tag{2.2}$$

for any $\varphi \in C^*(Q)$ such that $\varphi(x,t) \geqslant 0$ a.e. in Q, $\varphi(x,0) = \varphi(x,T) = 0$ for a.e. $x \in \Omega$.

Of particular interest are L_{Γ_1}-subsolutions (L_{Γ_1}-supersolutions) such that

$$\int_Q \Bigg(\sum_{i,j=1}^m a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \varphi}{\partial x_i} + \sum_{i=1}^m b_i \frac{\partial u}{\partial x_i}\varphi + cu\varphi + \sum_{i=1}^m d_i u \frac{\partial \varphi}{\partial x_i} - u\frac{\partial \varphi}{\partial t} \Bigg) \mathrm{d}x\,\mathrm{d}t + \int_{\Gamma_1} \alpha u \varphi \, \mathrm{d}\sigma\,\mathrm{d}t \leqslant 0 \quad (\geqslant 0) \tag{2.3}$$

for any $\varphi \in C^*(Q)$, $\varphi(x,t) \geqslant 0$ on Γ_1, $\varphi(x,0) = \varphi(x,T) = 0$ for a.e. $x \in \Omega$.

In fact, problem (2.3) is equivalent, at least "formally," to the problem

$$\begin{cases} -\sum\limits_{i=1}^m \dfrac{\partial}{\partial x_i}\Bigg(\sum\limits_{j=1}^m a_{ij}\dfrac{\partial u}{\partial x_j} + d_i u\Bigg) + \Bigg(\sum\limits_{i=1}^m b_i \dfrac{\partial u}{\partial x_i} + cu\Bigg) + \dfrac{\partial u}{\partial t} = 0 & \text{in } Q \\ \dfrac{\partial u}{\partial \nu} + \alpha u + \sum\limits_{i=1}^m d_i u \cos nx_i \leqslant 0 \quad (\geqslant 0) & \text{on } \Gamma_1. \end{cases}$$

Let us consider the problem

$$\begin{cases} \displaystyle\int_Q \Bigg(\sum_{i,j=1}^m a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \varphi}{\partial x_i} + \sum_{i=1}^m b_i \tfrac{\partial u}{\partial x_i}\varphi + cu\varphi + \sum_{i=1}^m d_i u \frac{\partial \varphi}{\partial x_i} - u\frac{\partial \varphi}{\partial t} \Bigg) \mathrm{d}x\,\mathrm{d}t \\ \qquad + \displaystyle\int_{\Gamma_1} \alpha u \varphi \, \mathrm{d}\sigma\,\mathrm{d}t = \int_Q f\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_{\Gamma_1} g_1 \varphi\,\mathrm{d}\sigma\,\mathrm{d}t \\ \qquad\qquad \text{for any } \varphi \in C^*(Q),\ \varphi(x,T) = 0 \text{ in } \Omega \\ u(x,t) - g_2(x,t) \in \widetilde{H}_*^{1,0}(\nu\psi, Q) \\ u(x,0) = 0 \text{ in } \Omega. \end{cases} \tag{2.4}$$

The problem (2.4) is formally equivalent to the problem

$$\begin{cases} -\sum\limits_{i=1}^m \dfrac{\partial}{\partial x_i}\Bigg(\sum\limits_{j=1}^m a_{ij}\dfrac{\partial u}{\partial x_j} + d_i u\Bigg) + \Bigg(\sum\limits_{i=1}^m b_i \dfrac{\partial u}{\partial x_i} + cu\Bigg) + \dfrac{\partial u}{\partial t} = f & \text{in } Q \\ \dfrac{\partial u}{\partial \nu} + \alpha u + \sum\limits_{i=1}^m d_i u \cos nx_i = g_1 & \text{on } \Gamma_1 \\ u = g_2 & \text{on } \Gamma_2 \\ u(x,0) = 0 & \text{in } \Omega. \end{cases}$$

3. Hypotheses on coefficients

Let us denote by A the set of pairs (α^*, α) with $2 \leqslant \alpha^*, \alpha \leqslant +\infty$, such that there exists a positive constant β for which

$$\|u\|_{\alpha^*,\alpha} \leqslant \beta\big(\|u\|_{2,\infty} + \|u\|_{1,0}\big) \tag{3.1}$$

for any $u \in L^{2,\infty}(Q) \cap \widetilde{H}^{1,0}(\nu\psi, Q)$. The set A obviously contains[3] the pair $(2, +\infty)$.

Let us observe that, under the hypotheses on Ω, we have[4]

$$|u|_{\frac{2(m-1)}{m-2+\frac{m}{g}},\partial\Omega_1} \leqslant \gamma\|u\|_1 \quad \text{for any } u \in \widetilde{H}^1(\nu, \Omega). \tag{3.2}$$

Consequently, we obtain:

$$\left(\int_0^T \psi(t)\, |u|^2_{\frac{2(m-1)}{m-2+\frac{m}{g}},\partial\Omega_1}\, \mathrm{d}t\right)^{\frac{1}{2}} \leqslant \gamma\big(\|u\|_{2,\infty} + ||u||_{1,0}\big) \tag{3.3}$$

for any $u \in L^{2,\infty}(Q) \cap \widetilde{H}^{1,0}(\nu\psi, Q)$.

The constant in (3.2) and (3.3) depends on Ω_1.

H y p o t h e s i s 3.1. The functions $a_{ij}(x,t)$, $b_i(x,t)$, $c(x,t)$, $d_i(x,t)$ $(1 \leqslant i, j \leqslant m)$ are defined and measurable in Q;

$$\frac{a_{ij}}{\nu\psi} \in L^\infty(Q), \quad \frac{b_i}{\sqrt{\nu\psi}} \in L^{p^*,p}(Q), \quad c \in L^{q^*,q}(Q), \quad \frac{d_i}{\sqrt{\nu\psi}} \in L^{r^*,r}(Q),$$

where

$$\frac{1}{p^*} + \frac{1}{\alpha_1^*} = \frac{1}{2}, \qquad \frac{1}{p} + \frac{1}{\alpha_1} = \frac{1}{2}, \qquad \frac{1}{q^*} + \frac{2}{\alpha_2^*} = 1,$$
$$\frac{1}{q} + \frac{2}{\alpha_2} = 1, \qquad \frac{1}{r^*} + \frac{1}{\alpha_3^*} = \frac{1}{2}, \qquad \frac{1}{r} + \frac{1}{\alpha_3} = \frac{1}{2}$$

with (α_1^*, α_1), (α_2^*, α_2) and (α_3^*, α_3) belonging to A.

Moreover, if $p = +\infty$ $[q = +\infty,\ r = +\infty]$ and $p^* < +\infty$ $[q^* < +\infty,\ r^* < +\infty]$, then there exists a function $\eta_1(\sigma)$ $[\eta_2(\sigma),\ \eta_3(\sigma)]$, defined for $\sigma \geqslant 0$, non decreasing,

[3] If $\frac{1}{\psi(t)} \in L^t(0,T)$ $(0 < t \leqslant +\infty)$, the set A contains the pair $\big(\frac{2mg}{mg+m\theta-2\theta g}, \frac{2t}{\theta(t+1)}\big)$ for any $\theta \in \big[0, \frac{t}{t+1}\big[$, see, for instance, [13].

[4] See, for instance, [11] Theorem 3.9.

vanishing for σ approaching zero and satisfying for almost all t in the interval $]0,T[$ the inequalities

$$\sum_{i=1}^{m}\left(\int_E\left(\frac{|b_i(x,t)|}{\sqrt{\nu(x)}}\right)^{p^*}\mathrm{d}x\right)^{\frac{1}{p}}\leqslant\eta_1(\sigma)\sqrt{\psi(t)},$$

$$\left[\left(\int_E(|c(x,t)|-c(x,t))^{q^*}\,\mathrm{d}x\right)^{\frac{1}{q}}\leqslant\eta_2(\sigma),\right.$$

$$\left.\sum_{i=1}^{m}\left(\int_E\left(\frac{|d_i(x,t)|}{\sqrt{\nu(x)}}\right)^{r^*}\mathrm{d}x\right)^{\frac{1}{r}}\leqslant\eta_3(\sigma)\sqrt{\psi(t)}\right]$$

for all measurable subsets E of Ω such that $\mathrm{meas}_x\,E\leqslant\sigma$.

The function $\alpha(x,t)$ is defined and measurable on Γ_1 and

$$\frac{\alpha}{\psi}\in L^\infty\Big(0,T;L^{\frac{m-1}{1-\frac{m}{g}}}(\partial\Omega_1)\Big).$$

H y p o t h e s i s 3.2. The functions $f(x,t)$, $g(x,t)$ are defined and measurable respectively in Q and in Γ_1, moreover

$$f\in L^2(Q),\quad \frac{g_1}{\sqrt{\psi}}\in L^2\Big(0,T;L^{\frac{2(m-1)}{m-\frac{m}{g}}}(\partial\Omega_1)\Big).$$

The function $g_2(x,t)$ is defined and measurable in Q and

$$g_2\in\widetilde{H}^{1,0}(\nu\psi,Q),\quad \frac{\partial g_2}{\partial t}\in L^2(Q),\quad g_2(x,t)\leqslant 0 \text{ on } \Gamma_1.$$

Finally, the functions $f^*(x,t)$, $g_1^*(x,t)$ are defined and measurable respectively in Q and in Γ_1, moreover

$$f^*\in L^2(Q),\quad \frac{g_1^*}{\sqrt{\psi}}\in L^2\Big(0,T;L^{\frac{2(m-1)}{m-\frac{m}{g}}}(\partial\Omega_1)\Big).$$

H y p o t h e s i s 3.3. The following inequality holds for a.e. (x,t) in Q and for all real numbers $\chi_1,\chi_2,\ldots,\chi_m$:

$$\sum_{i,j=1}^{m}a_{ij}(x,t)\chi_i\chi_j\geqslant\nu(x)\psi(t)\sum_{i=1}^{m}\chi_i^2.$$

4. Preliminary lemmas

Lemma 4.1. *Let us assume that hypotheses* (2.1), (2.2), (3.1) *hold and let* $u(x,t)$ *be an* L_{Γ_1}*-subsolution of the problem* (2.1) *bounded from above on* $\partial\Omega_2\times[0,T]$. *Then if* $0\leqslant\tilde{\tau}_1<\tau<T$ *and* $k>\sup^* u$, *we get*

$$\int_{Q(\tilde{\tau}_1,\tau)}\Bigl(\sum_{i,j=1}^{m}a_{ij}\frac{\partial v}{\partial x_j}\frac{\partial v}{\partial x_i}+\sum_{i=1}^{m}b_i\frac{\partial v}{\partial x_i}v+cuv+\sum_{i=1}^{m}d_iu\frac{\partial v}{\partial x_i}\Bigr)\,\mathrm{d}x\,\mathrm{d}t$$
$$+\frac12\int_\Omega v^2(x,\tau)\,\mathrm{d}x+\int_{\tilde{\tau}_1}^{\tau}\int_{\partial\Omega_1}\alpha uv\,\mathrm{d}\sigma\,\mathrm{d}t\leqslant\frac12\int_\Omega v^2(x,\tilde{\tau}_1)\,\mathrm{d}x,$$

where $v=u-\min(u,k)$ *in* Q; *moreover,* $v\in\widetilde{H}^{1,0}_*(\nu\psi,Q)$.[5]

Proof. Let $\tilde{\tau}_1$, τ be such that $0<\tilde{\tau}_1<\tau<T$; setting $\tau_1=\frac{\tau+T}{2}$, $\tau_2=T-\tau_1$, we denote by $C^\infty_\tau(Q)$ the set of nonnegative functions from $C^*(Q)$ equal to zero for $t\geqslant\tau_1$. Let φ be a function from $C^\infty_\tau(Q)$. We extend u, φ and the coefficients of (2.1) to $\Omega\times(-\infty,+\infty)$, assuming that these functions are equal at zero in those points where they are not defined.

We define in $\Omega\times\,]-\infty,+\infty[$ and for any integer ϱ:

$$\begin{aligned}
\Phi_\varrho(x,t)&=\frac{\varrho}{\tau_2}\int_{t-\frac{\tau_2}{\varrho}}^{t}\varphi(x,\lambda)\,\mathrm{d}\lambda,\\
U_\varrho(x,t)&=\frac{\varrho}{\tau_2}\int_{t}^{t+\frac{\tau_2}{\varrho}}u(x,\lambda)\,\mathrm{d}\lambda,\\
B_\varrho(x,t)&=\frac{\varrho}{\tau_2}\int_{t}^{t+\frac{\tau_2}{\varrho}}\sum_{i=1}^{m}b_i(x,\lambda)\frac{\partial u(x,\lambda)}{\partial x_i}\,\mathrm{d}\lambda,\\
C_\varrho(x,t)&=\frac{\varrho}{\tau_2}\int_{t}^{t+\frac{\tau_2}{\varrho}}c(x,\lambda)u(x,\lambda)\,\mathrm{d}\lambda,\\
A_{i,\varrho}(x,t)&=\frac{\varrho}{\tau_2}\int_{t}^{t+\frac{\tau_2}{\varrho}}\sum_{j=1}^{m}a_{ij}(x,\lambda)\frac{\partial u(x,\lambda)}{\partial x_j}\,\mathrm{d}\lambda,\\
D_{i,\varrho}(x,t)&=\frac{\varrho}{\tau_2}\int_{t}^{t+\frac{\tau_2}{\varrho}}d_i(x,\lambda)u(x,\lambda)\,\mathrm{d}\lambda,\\
\alpha_\varrho(x,t)&=\frac{\varrho}{\tau_2}\int_{t}^{t+\frac{\tau_2}{\varrho}}\alpha(x,\lambda)u(x,\lambda)\,\mathrm{d}\lambda.
\end{aligned}$$

[5] Let us observe that, for t a.e. in $]0,T[$, $v(x,t)\in H^*(\nu,\Omega)$.

From (2.2), in virtue of $\varphi = \Phi_\varrho(x,t)$, via an exchange of the order of integration with respect to t and λ,[6] we get

$$\tag{4.1} \int_Q \Big(\sum_{i=1}^m A_{i,\varrho} \frac{\partial \varphi}{\partial x_i} + B_\varrho \varphi + C_\varrho \varphi + \sum_{i=1}^m D_{i,\varrho} \frac{\partial \varphi}{\partial x_i} + \frac{\partial U_\varrho}{\partial t} \varphi \Big) \,\mathrm{d}x\,\mathrm{d}t + \int_{\Gamma_1} \alpha_\varrho \varphi \,\mathrm{d}\sigma\,\mathrm{d}t \leqslant 0$$

for all φ belonging to the functional class $C_\tau^\infty(Q)$.

Let $\{u_n\}$ be a sequence of functions of $C^1(\overline{Q})$ such that $u_n < \sup^* u$ on Γ_2 and satisfying $(*)$. For all pairs of positive integers ν and n, we define

$$U_{\varrho,n}(x,t) = \frac{\varrho}{\tau_2} \int_t^{t+\frac{\tau_2}{\varrho}} u_n(x,\lambda)\,\mathrm{d}\lambda;$$

the function $U_{\varrho,n}(x,t)$ belongs to $C^1\big(\overline{Q(0,\tau_1)}\big)$.

Let us now introduce the function[7]

$$V_{\varrho,n} = \begin{cases} U_{\varrho,n} - \min(U_{\varrho,n}, k) & \text{in } Q(\tilde\tau_1, \tau), \\ 0 & \text{in } Q \setminus Q(\tilde\tau_1,\tau). \end{cases}$$

Let $\{\Phi_\mu\}_{\mu\in\mathbb{N}}$ be a sequence of nonnegative equibounded functions from $C_\tau^\infty(Q)$ converging to $V_{\varrho,n}$ in $\widetilde{H}^{1,0}\big(\nu\psi, Q(\tilde\tau_1,\tau)\big)$;[8] moreover, let also the functions in the sequence $\big\{\frac{\partial \Phi_\mu}{\partial x_i}\big\}_{\mu\in\mathbb{N}}$ be equibounded.

From (4.1), in virtue of $\varphi = \Phi_\mu(x,t)$, as μ diverges to $+\infty$, we obtain the following relation:

$$\tag{4.2} \int_{Q(\tilde\tau_1,\tau)} \Big(\sum_{i=1}^m A_{i,\varrho} \frac{\partial V_{\varrho,n}}{\partial x_i} + B_\varrho V_{\varrho,n} + C_\varrho V_{\varrho,n} + \sum_{i=1}^m D_{i,\varrho} \frac{\partial V_{\varrho,n}}{\partial x_i} + \frac{\partial U_\varrho}{\partial t} V_{\varrho,n} \Big) \,\mathrm{d}x\,\mathrm{d}t + \int_{\tilde\tau_1}^\tau \int_{\partial\Omega_1} \alpha_\nu V_{\varrho,n} \,\mathrm{d}\sigma\,\mathrm{d}t \leqslant 0.$$

Setting now in Q:

$$V_\varrho = \begin{cases} U_\varrho - \min(U_\varrho, k) & \text{in } Q(\tilde\tau_1, \tau), \\ 0 & \text{in } Q \setminus Q(\tilde\tau_1,\tau); \end{cases}$$

[6] See [10], p. 141.
[7] Since $k > \sup^* u$ there exists a neighbourhood of $\partial\Omega_2$ such that $V_{\varrho,n}(x,t) = 0$ for any $t \in]0,T[$ (see Lemma 4.2 of [3]).
[8] See remark 4.1 of [3].

the sequence $\{V_{\varrho,n}\}$ converges to V_ϱ in $\widetilde{H}^{1,0}\big(\nu\psi, Q(\tilde{\tau}_1,\tau)\big) \cap L^{2,\infty}\big(Q(\tilde{\tau}_1,\tau)\big)$ and satisfies the relation

$$\lim_{n\to\infty} \big\|(V_{\varrho,n} - V_\varrho)\big\|_{\frac{2(m-1)}{m-2+\frac{m}{g}},2,\Gamma_1} = 0.$$

On the other hand, the functions of the sequence $\{V_{\varrho,n}\}$ belong to $\widetilde{H}^{1,0}_{*}\big(\nu\psi, Q(\tilde{\tau}_1,\tau)\big)$ and so also V_ϱ belongs to this space.

From (3.1)–(3.6) we deduce, as n goes to $+\infty$, the following inequality:[9]

$$\begin{aligned}\int_Q \bigg(\sum_{i=1}^m A_{i,\varrho}\frac{\partial V_\varrho}{\partial x_i} + B_\varrho V_\varrho + C_\varrho V_\varrho + \sum_{i=1}^m D_{i,\varrho}\frac{\partial V_\varrho}{\partial x_i} + \frac{\partial U_\varrho}{\partial t}V_{\varrho,n}\bigg)\,\mathrm{d}x\,\mathrm{d}t \\ + \frac{1}{2}\int_{\Omega_\varrho(\tau,k)} \big|U_\varrho(x,\tau)-k\big|^2\,\mathrm{d}x \\ - \frac{1}{2}\int_{\Omega_\varrho(0,k)} \big|U_\varrho(x,0)-k\big|^2\,\mathrm{d}x + \int_{\tilde{\tau}_1}^{\tau}\int_{\partial\Omega_1} \alpha_\varrho V_\varrho\,\mathrm{d}\sigma\,\mathrm{d}t \leqslant 0.\end{aligned} \tag{4.3}$$

Let us remark that the sequence $\{V_\varrho\}$ converges in $\widetilde{H}^{1,0}(\nu\psi,Q)\cap L^{2,\infty}(Q)$ to the function equal to v in $Q(\tilde{\tau}_1,\tau)$ and equal to zero in $Q\setminus Q(\tilde{\tau}_1,\tau)$.

From (4.3), the conclusion follows via another passage to the limit.

For example, we prove that

$$\lim_{n\to\infty}\int_{\tilde{\tau}_1}^{\tau}\int_{\partial\Omega_1}\alpha_\varrho V_\varrho\,\mathrm{d}\sigma\,\mathrm{d}t = \int_{\tilde{\tau}_1}^{\tau}\int_{\partial\Omega_1}\alpha u v\,\mathrm{d}\sigma\,\mathrm{d}t.$$

We get

$$\begin{aligned}&\bigg|\int_{\tilde{\tau}_1}^{\tau}\int_{\partial\Omega_1}\alpha_\varrho V_\varrho - \alpha u v\,\mathrm{d}\sigma\,\mathrm{d}t\bigg| \\ &\qquad\leqslant \gamma\,\frac{\psi(\tau)}{\psi(\tilde{\tau}_1)}\Big\|\frac{\alpha}{\psi}\Big\|_{\frac{m-1}{1-\frac{m}{g}},\infty,\Gamma_1}\big(\|u\|_{2,\infty}+\|u\|_{1,0}\big) \\ &\qquad\quad\times\big(\|V_\varrho - v\|_{2,\infty,(\tilde{\tau}_1,\tau)} + \|V_\varrho - v\|_{1,0,(\tilde{\tau}_1,\tau)}\big) + \Big(\frac{1}{\psi(\tilde{\tau}_1)}\Big)^{\frac{1}{2}} \\ &\qquad\quad\times\big(\|v\|_{2,\infty,(\tilde{\tau}_1,\tau)} + \|v\|_{1,0,(\tilde{\tau}_1,\tau)}\big)\bigg(\int_{\tilde{\tau}_1}^{\tau}\Big(\big|\alpha_\varrho - \alpha u\big|^{\frac{2(m-1)}{m-\frac{m}{g}}}\,\mathrm{d}\sigma\Big)^{\frac{m-\frac{m}{g}}{m-1}}\,\mathrm{d}t\bigg)^{\frac{1}{2}}\end{aligned}$$

for any $\varrho \in \mathbb{N}$.[10]

[9] For a fixed $t \in]0,T[$, we set $\Omega_\varrho(t,k) = \{x\in\Omega\colon\ U_\varrho(x,t) > k\}$.

[10] Let us remark that, by the properties of Steklov averages, it follows that α_ϱ converges to αu in $L^2\Big(\tilde{\tau}_1,\tau; L^{\frac{2(m-1)}{m-\frac{m}{g}}}(\partial\Omega_1)\Big)$.

Next, it is easy to verify that the restriction of the function v to $Q(\tilde\tau_1,\tau)$ belongs to $\widetilde{H}^{1,0}_*(\nu\psi,Q)$ for any $0<\tilde\tau_1<\tau<T$ and, therefore, since v by definition belongs to $\widetilde{H}^{1,0}\big(\nu\psi,Q(\tilde\tau_1,\tau)\big)$, it belongs to $\widetilde{H}^{1,0}_*(\nu\psi,Q)$, too.

Finally, if $\tilde\tau_1=0$, as $\tau>0$ is assumed, it suffices to consider $\tau_n=\frac{\tau}{n+1}$ for $n\in\mathbb{N}$ recalling that the function $v(x,t)$ is continuous in $[0,T]$ with values in $L^2(\Omega)$. $\square$

Lemma 4.2. *Let us assume the hypotheses* (2.1), (2.2), (3.1), (3.3) *hold and let* $u(x,t)$ *be an* L_{Γ_1}*-subsolution of the problem* (2.1) *satisfying the conditions*

$$\operatorname*{ess\,sup}_{\Omega} u(x,0)\leqslant 0,\quad \sup{}^* u\leqslant 0.$$

Then, we have:

$$\operatorname*{ess\,sup}_{Q} u(x,t)\leqslant 0.$$

P r o o f. For any integer n, we consider the functions

$$v=u-\min(u,0),\quad v_n=u-\min\Big(u,\frac{1}{n}\Big).$$

From Lemma 4.1 we deduce that v_n belongs to $\widetilde{H}^{1,0}_*(\nu\psi,Q)$ and that, provided $\tau\in]0,T[$, we have

$$\begin{aligned}(4.4)\qquad &\int_{Q(\tau)}\Big(\sum_{i,j=1}^m a_{ij}\frac{\partial v_n}{\partial x_j}\frac{\partial v_n}{\partial x_i}+\sum_{i=1}^m b_i\frac{\partial v_n}{\partial x_i}v_n+cuv_n+\sum_{i=1}^m d_iu\frac{\partial v_n}{\partial x_i}\Big)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\frac12\int_\Omega v_n^2(x,\tau)\,\mathrm{d}x+\int_0^\tau\int_{\partial\Omega_1}\alpha uv_n\,\mathrm{d}\sigma\,\mathrm{d}t\leqslant 0.\end{aligned}$$

On the other hand, we obtain

$$\lim_{n\to\infty} v_n(x,t)=v(x,t),\quad |v_n(x,t)|\leqslant|u(x,t)|\ \text{ in } Q$$

and

$$\lim_{n\to\infty}\frac{\partial v_n}{\partial x_i}=\frac{\partial v}{\partial x_i},\quad \Big|\frac{\partial v_n}{\partial x_i}\Big|\leqslant\Big|\frac{\partial u}{\partial x_i}\Big|\ \text{a.e. in } Q.$$

Furthermore, also v belongs to $\widetilde{H}^{1,0}_*(\nu\psi,Q)$ and so, as n goes to $+\infty$ in (4.4), we get

$$\begin{aligned}(4.5)\qquad &\int_{Q(\tau)}\Big(\sum_{i,j=1}^m a_{ij}\frac{\partial v}{\partial x_j}\frac{\partial v}{\partial x_i}+\sum_{i=1}^m b_i\frac{\partial v}{\partial x_i}v+cv^2+\sum_{i=1}^m d_iv\frac{\partial v}{\partial x_i}\Big)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\frac12\int_\Omega v^2(x,\tau)\,\mathrm{d}x+\int_0^\tau\int_{\partial\Omega_1}\alpha v^2\,\mathrm{d}\sigma\,\mathrm{d}t\leqslant 0.\end{aligned}$$

From (4.5) we deduce that $|v|_{2,\infty}=0$ and the conclusion easily follows.

The proof is similar to that given in Lemma 4.1 of [13]; let us remark that since $v \in \widetilde{H}^{1,0}(\nu\psi, Q) \cap L^{2,\infty}(Q)$ we can apply the relations (3.1) and (3.3) instead of the hypothesis A) of [13]. $\square$

5. A generalized maximum principle

We will prove

Theorem 5.1. *Let us assume the hypotheses* (2.1), (2.2), (3.1), (3.3) *hold and let* $w(x,t)$ *be an* L_{Γ_1}*-supersolution of the problem* (2.1) *satisfying the conditions*

$$w(x,t) > 0 \quad \text{a.e. in } Q,$$
$$w(x,0) > 0 \quad \text{a.e. in } \Omega, \qquad w(x,t) \geqslant 0 \quad \text{on } \Gamma_2.$$

Then

$$\operatorname*{ess\,sup}_Q \frac{u(x,t)}{w(x,t)} \leqslant \max\Big(0, \operatorname*{ess\,sup}_\Omega \frac{u(x,0)}{w(x,0)}, \sup{}^* \frac{u}{w}\Big) \tag{5.1}$$

for any L_{Γ_1}*-subsolution* $u(x,t)$ *of the problem* (2.1).

P r o o f. The conclusion is obvious if the second term of (5.1) is equal to $+\infty$. Let us suppose, now, that this term is finite and let us denote by h some real number greater than its value. Consequently, the function $u(x,t)-hw(x,t)$ is an L_{Γ_1}-subsolution of the problem (2.1) such that $\operatorname{ess\,sup}_\Omega\big[u(x,0)-hw(x,0)\big] \leqslant 0$, $\sup^*(u-hw) \leqslant 0$. From Lemma 4.2 we can see that $u(x,t) - hw(x,t) \leqslant 0$ a.e. in Q. So, we obtain $\operatorname{ess\,sup}_Q \frac{u(x,t)}{w(x,t)} \leqslant h$ and the conclusion easily follows. $\square$

6. A comparison theorem

Let us define the following closed convex sets:

$$K^* = \big\{z \in \widetilde{H}^{1,0}(\nu\psi, Q),\ z \in C\big([0,T]; L^2(\Omega)\big),\ z(x,0) = 0,\ z \geqslant g_2 \text{ on } \Gamma_2\big\},$$
$$\Phi^* = \Big\{\varphi \in \widetilde{H}^{1,0}(\nu\psi, Q),\ \frac{\partial\varphi}{\partial t} \in L^2(Q),\ \varphi(x,T) = 0,\ \varphi \geqslant g_2 \text{ on } \Gamma_2\Big\}$$

and let us suppose that there exists a solution $z \in K^*$ of the variational inequality

$$\begin{aligned}\int_Q \Big(\sum_{i,j=1}^m a_{ij}\frac{\partial z}{\partial x_j}\frac{\partial(\varphi - z)}{\partial x_i} + \sum_{i=1}^m b_i \frac{\partial z}{\partial x_i}(\varphi - z) + cz(\varphi - z) \\ + \sum_{i=1}^m d_i z \frac{\partial(\varphi - z)}{\partial x_i} - z\frac{\partial\varphi}{\partial t}\Big)\,\mathrm{d}x\,\mathrm{d}t + \int_{\Gamma_1} \alpha z(\varphi - z)\,\mathrm{d}\sigma\,\mathrm{d}t \\ \geqslant \int_Q f^*(\varphi - z)\,\mathrm{d}x\,\mathrm{d}t + \int_{\Gamma_1} g_1^*(\varphi - z)\,\mathrm{d}\sigma\,\mathrm{d}t\end{aligned} \tag{6.1}$$

for any $\varphi \in \Phi^*$. The problem (6.1) is formally equivalent to the problem

$$\begin{cases} -\sum_{i=1}^{m} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{m} a_{ij}\frac{\partial z}{\partial x_j} + d_i z\right) + \left(\sum_{i=1}^{m} b_i \frac{\partial z}{\partial x_i} + cz\right) + \frac{\partial z}{\partial t} = f^* & \text{in } Q \\ \frac{\partial z}{\partial \nu} + \alpha z + \sum_{i=1}^{m} d_i z \cos nx_i = g_1^* & \text{on } \Gamma_1 \\ z \geqslant g_2,\ \frac{\partial z}{\partial \nu} + \sum_{i=1}^{m} d_i z \cos nx_i \geqslant 0,\ (z-g_2)\left(\frac{\partial z}{\partial \nu} + \sum_{i=1}^{m} d_i z\cos nx_i\right) = 0 & \text{on } \Gamma_2 \\ z(x,0) = 0 & \text{in } \Omega. \end{cases}$$

We will prove

Theorem 6.1. *Let us assume the hypotheses* (2.1), (2.2), (3.1), (3.2), (3.3) *hold and let* $w(x,t)$ *be an* L_{Γ_1}*-supersolution of the problem* (2.1) *satisfying the conditions*

$$w(x,t) > 0 \quad \text{a.e. in } Q,$$
$$w(x,0) > 0 \quad \text{a.e. in } \Omega, \quad w(x,t) \geqslant 0 \text{ on } \Gamma_2.$$

Let $z(x,t)$ *be a solution of the problem* (6.1) *with* $f^* \geqslant f$ *in* Q, $g_1^* \geqslant g_1$ *on* Γ_1.

Then, we have the inequality

$$u(x,t) \leqslant z(x,t) \quad \text{a.e. in } Q$$

for any solution $u(x,t)$ *of the problem* (2.4).

P r o o f. Let us extend $z(x,t)$ to $\mathbb{R}^{m+1}$ assuming that it vanishes at points not belonging to Q; for a fixed $\tau \in\ ]0,T[$ and for any pairs of integers ϱ, n we introduce the functions

$$\theta_n(t) = \begin{cases} 0 & \text{if } t < \tau - \frac{2}{n}, \\ n\left(t + \frac{2}{n} - \tau\right) & \text{if } \tau - \frac{2}{n} \leqslant t \leqslant \tau - \frac{1}{n}, \\ 1 & \text{if } t > \tau - \frac{1}{n}; \end{cases}$$

$$z_{n,\varrho}(x,t) = \varrho\theta_n(t)\int_{t-\frac{1}{2\varrho}}^{t+\frac{1}{2\varrho}} z(x,y)\theta_n(y)\,\mathrm{d}y.$$

We have

$$\frac{\partial z_{n,\varrho}}{\partial t} = \varrho\theta_n'(t)\int_{t-\frac{1}{2\varrho}}^{t+\frac{1}{2\varrho}} z(x,y)\theta_n(y)\,\mathrm{d}y + \varrho\theta_n(t)\Big(z\Big(x,t+\frac{1}{2\varrho}\Big)\theta_n\Big(t+\frac{1}{2\varrho}\Big) - z\Big(x,t-\frac{1}{2\varrho}\Big)\theta_n\Big(t-\frac{1}{2\varrho}\Big)\Big).$$

Choosing $\varphi = z_{n,\varrho} + \beta$, $0 \leqslant \beta \in C^*(Q)$, $\beta(x,t) = 0$ a.e. in Q, we get from (6.1)

$$\begin{aligned}\int_Q\Big(\sum_{i,j=1}^m a_{ij}\frac{\partial z}{\partial x_j}\frac{\partial(z_{n,\varrho}+\beta-z)}{\partial x_i} + \sum_{i=1}^m b_i\frac{\partial z}{\partial x_i}(z_{n,\varrho}+\beta-z)\\ + cz(z_{n,\varrho}+\beta-z) + \sum_{i=1}^m d_i z\frac{\partial(z_{n,\varrho}+\beta-z)}{\partial x_i}\Big)\,\mathrm{d}x\,\mathrm{d}t\\ - \int_Q z\frac{\partial z_{n,\varrho}}{\partial t}\,\mathrm{d}x\,\mathrm{d}t - \int_Q z\frac{\partial\beta}{\partial t}\,\mathrm{d}x\,\mathrm{d}t + \int_{\Gamma_1}\alpha z(z_{n,\varrho}+\beta-z)\,\mathrm{d}\sigma\,\mathrm{d}t\\ \geqslant \int_Q f^*(z_{n,\varrho}+\beta-z)\,\mathrm{d}x\,\mathrm{d}t + \int_{\Gamma_1} g_1^*(z_{n,\varrho}+\beta-z)\,\mathrm{d}\sigma\,\mathrm{d}t;\end{aligned}\tag{6.2}$$

now, taking into account the relation

$$\begin{aligned}&\int_Q z(x,t)\varrho\theta_n(t)\theta_n\Big(t+\frac{1}{2\varrho}\Big)z\Big(x,t+\frac{1}{2\varrho}\Big)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad= \int_\Omega \mathrm{d}x\int_{-\infty}^{+\infty} z(x,t)\varrho\theta_n(t)\theta_n\Big(t+\frac{1}{2\varrho}\Big)z\Big(x,t+\frac{1}{2\varrho}\Big)\,\mathrm{d}t\\ &\quad= \int_\Omega \mathrm{d}x\int_{-\infty}^{+\infty} z\Big(x,t-\frac{1}{2\varrho}\Big)\varrho\theta_n\Big(t-\frac{1}{2\varrho}\Big)\theta_n(t)z(x,t)\,\mathrm{d}t\\ &\quad= \int_Q z(x,t)\varrho\theta_n(t)\theta_n\Big(t-\frac{1}{2\varrho}\Big)z\Big(x,t-\frac{1}{2\varrho}\Big)\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

we obtain from (6.2)

$$\begin{aligned}\int_Q\Big(\sum_{i,j=1}^m a_{ij}\frac{\partial z}{\partial x_j}\frac{\partial(z_{n,\varrho}+\beta-z)}{\partial x_i} + \sum_{i=1}^m b_i\frac{\partial z}{\partial x_i}(z_{n,\varrho}+\beta-z)\\ + cz(z_{n,\varrho}+\beta-z) + \sum_{i=1}^m d_i z\frac{\partial(z_{n,\varrho}+\beta-z)}{\partial x_i}\Big)\,\mathrm{d}x\,\mathrm{d}t\\ - \int_Q\Big(z\varrho\theta_n'(t)\int_{t-\frac{1}{2\varrho}}^{t+\frac{1}{2\varrho}} z(x,y)\theta_n(y)\,\mathrm{d}y\Big)\,\mathrm{d}x\,\mathrm{d}t - \int_Q z\frac{\partial\beta}{\partial t}\,\mathrm{d}x\,\mathrm{d}t\\ + \int_{\Gamma_1}\alpha z(z_{n,\varrho}+\beta-z)\,\mathrm{d}\sigma\,\mathrm{d}t\\ \geqslant \int_Q f^*(z_{n,\varrho}+\beta-z)\,\mathrm{d}x\,\mathrm{d}t + \int_{\Gamma_1} g_1^*(z_{n,\varrho}+\beta-z)\,\mathrm{d}\sigma\,\mathrm{d}t,\end{aligned}$$

and therefore, letting ϱ tend to $+\infty$, we find that

$$
\begin{aligned}
\int_Q \Big(\sum_{i,j=1}^m a_{ij} \frac{\partial z}{\partial x_j} \frac{\partial(\theta_n^2(t)z+\beta-z)}{\partial x_i} &+ \sum_{i=1}^m b_i \frac{\partial z}{\partial x_i}\big(\theta_n^2(t)z+\beta-z\big) \\
+ cz\big(\theta_n^2(t)z+\beta-z\big) &+ \sum_{i=1}^m d_i z \frac{\partial(\theta_n^2(t)z+\beta-z)}{\partial x_i}\Big) \,\mathrm{d}x\,\mathrm{d}t \\
- \int_Q z^2\theta_n(t)\theta_n'(t)\,\mathrm{d}x\,\mathrm{d}t &- \int_Q z\frac{\partial \beta}{\partial t}\,\mathrm{d}x\,\mathrm{d}t + \int_{\Gamma_1} \alpha z\big(\theta_n^2(t)z+\beta-z\big)\,\mathrm{d}\sigma\,\mathrm{d}t \\
\geqslant \int_Q f^*\big(\theta_n^2(t)z+\beta-z\big)\,\mathrm{d}x\,\mathrm{d}t &+ \int_{\Gamma_1} g_1^*\big(\theta_n^2(t)z+\beta-z\big)\,\mathrm{d}\sigma\,\mathrm{d}t.
\end{aligned}
\tag{6.3}
$$

Let us observe that $\theta_n(t)\theta_n'(t) \geqslant 0$ a.e. in $]0,T[$ and $\theta_n(t)\theta_n'(t) > \frac{n}{2}$ if $\tau - \frac{3}{2n} < t < \tau - \frac{1}{n}$. Then, from (6.3) we have

$$
\begin{aligned}
\int_Q \Big(\sum_{i,j=1}^m a_{ij} \frac{\partial z}{\partial x_j} \frac{\partial(\theta_n^2(t)z+\beta-z)}{\partial x_i} &+ \sum_{i=1}^m b_i \frac{\partial z}{\partial x_i}\big(\theta_n^2(t)z+\beta-z\big) \\
+ cz\big(\theta_n^2(t)z+\beta-z\big) &+ \sum_{i=1}^m d_i z \frac{\partial(\theta_n^2(t)z+\beta-z)}{\partial x_i}\Big) \,\mathrm{d}x\,\mathrm{d}t \\
- \int_Q z\frac{\partial \beta}{\partial t}\,\mathrm{d}x\,\mathrm{d}t &+ \int_{\Gamma_1} \alpha z\big(\theta_n^2(t)z+\beta-z\big)\,\mathrm{d}\sigma\,\mathrm{d}t \\
\geqslant \int_Q z^2\theta_n(t)\theta_n'(t)\,\mathrm{d}x\,\mathrm{d}t &+ \int_Q f^*\big(\theta_n^2(t)z+\beta-z\big)\,\mathrm{d}x\,\mathrm{d}t \\
+ \int_{\Gamma_1} g_1^*\big(\theta_n^2(t)z+\beta-z\big)\,\mathrm{d}\sigma\,\mathrm{d}t & \\
\geqslant \frac{n}{2}\int_{\tau-\frac{3}{2n}}^{\tau-\frac{1}{n}} \mathrm{d}t \int_\Omega z^2(x,t)\,\mathrm{d}x &+ \int_Q f^*\big(\theta_n^2(t)z+\beta-z\big)\,\mathrm{d}x\,\mathrm{d}t \\
+ \int_{\Gamma_1} g_1^*\big(\theta_n^2(t)z+\beta-z\big)\,\mathrm{d}\sigma\,\mathrm{d}t. &
\end{aligned}
$$

Finally, as $n \to \infty$ and $\tau \to 0$, we get

$$
\begin{aligned}
\int_Q \Big(\sum_{i,j=1}^m a_{ij} \frac{\partial z}{\partial x_j}\frac{\partial \beta}{\partial x_i} + \sum_{i=1}^m b_i \frac{\partial z}{\partial x_i}\beta + cz\beta + \sum_{i=1}^m d_i z \frac{\partial \beta}{\partial x_i} - z\frac{\partial \beta}{\partial t}\Big)\,\mathrm{d}x\,\mathrm{d}t \\
+ \int_{\Gamma_1} \alpha z\beta\,\mathrm{d}\sigma\,\mathrm{d}t \geqslant \int_Q f^*\beta\,\mathrm{d}x\,\mathrm{d}t + \int_{\Gamma_1} g_1^*\beta\,\mathrm{d}\sigma\,\mathrm{d}t
\end{aligned}
\tag{6.4}
$$

for any $0 \leqslant \beta \in C^*(Q)$, $\beta(x,T) = 0$ a.e. in Q.

By virtue of (6.4) and (2.4) we conclude that

$$\tilde{a}(u-z,\varphi) \leqslant \int_Q (f-f^*)\beta \,\mathrm{d}x\,\mathrm{d}t + \int_{\Gamma_1} (g_1 - g_1^*)\beta \,\mathrm{d}\sigma\,\mathrm{d}t \leqslant 0$$

for any $0 \leqslant \beta \in C^*(Q)$, $\beta(x,T) = 0$ a.e. in Q.

Applying the above maximum principle to the L_{Γ_1}-subsolution $(u-z)$ and to the L_{Γ_1}-supersolution w, we obtain

$$\operatorname*{ess\,sup}_Q \frac{u(x,t)-z(x,t)}{w(x,t)} \leqslant \max\Big(0, \sup{}^* \frac{u-z}{w}\Big) = 0.$$

This completes the proof. □

References

[1] *Adams, R. A.*: Sobolev Spaces. Academic Press, New York, 1975.

[2] *Baiocchi, C.*: Soluzioni deboli dei problemi ai limiti per le equazioni paraboliche del tipo del calore. Ist. Lombardo Accad. Sci. Lett. Rend. Sez. A *103* (1969), 704–726.

[3] *Bonafede, S.*: On maximum principle for weak subsolutions of degenerate parabolic linear equations. Comment. Math. Univ. Carolin. *35* (1994), 417–430.

[4] *Bonafede, S.*: A generalized maximum principle for weak subsolutions of degenerate parabolic equations. Rend. Circ. Mat. Palermo (2) *41* (1992), 81–95.

[5] *Drábek, P., Nicolosi, F.*: Existence of bounded solutions for some degenerated quasilinear elliptic equations. Ann. Mat. Pura Appl. (4) *165* (1993), 217–238.

[6] *Eklund, N. A.*: Generalized super-solution of parabolic operators. Trans. Amer. Math. Soc. *220* (1976), 235–242.

[7] *Guglielmino, F., Nicolosi, F.*: W-solutions of boundary value problems for degenerate elliptic operators. Ricerche Mat. Suppl. *36* (1987), 59–72.

[8] *Guglielmino, F., Nicolosi, F.*: Existence theorems for boundary value problems associated with quasilinear elliptic equations. Ricerche Mat. *37* (1988), 157–176.

[9] *Guglielmino, F., Nicolosi, F.*: Existence results for boundary value problems for a class of quasilinear parabolic equations. Actual problems in analysis and mathematical physics, Proceeding of the international symposium in Taormina, Italy, 1992. Univ. di Roma, Roma, 1993, pp. 95–117.

[10] *Ladyzhenskaja, O. A., Solonnikov, V. A., Ural'tseva, N. N.*: Linear and Quasi-Linear Equations of Parabolic Type. Translation of mathematical monographs, vol. 23, A. M. S. Providence, 1968.

[11] *Murthy, M. K. V., Stampacchia, G.*: Boundary value problems for some degenerate-elliptic operators. Ann. Mat. Pura Appl. (4) *80* (1968), 1–122.

[12] *Nicolosi, F.*: Sottosoluzioni deboli delle equazioni paraboliche lineari del secondo ordine superiormente limitate. Matematiche *28* (1973), 361–378.

[13] *Nicolosi, F.*: Soluzioni dei problemi al contorno per operatori parabolici che possono degenerare. Ann. Mat. Pura Appl., IV. Ser. *125* (1980), 135–155.

[14] *Platone Garroni, M. G.*: A generalized maximum principle for boundary value problems for elliptic operators with discontinuos coefficients. Boll. U. M. I. (4) *9* (1974), 1–9.

[15] *Protter, M. H., Weinberger H. F.*: Regular points for elliptic equations with discontinuous coefficients. Ann. Scuola Sup. Pisa *17* (1963), 45–79.

[16] *Stampacchia, G.*: Le probleme de Dirichlet pour les equations elliptiques du second ordre a coefficients discontinus. Annal. Inst. Fourier *15* (1965), 187–257.

[17] *Troianello, G. M.*: On weak subsolutions for parabolic second-order operators. Comm. Partial Differential Equations *3*, 933–948.

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