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REACTION-DIFFUSION SYSTEMS: DESTABILIZING EFFECT OF CONDITIONS GIVEN BY INCLUSIONS

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Abstract. Sufficient conditions for destabilizing effects of certain unilateral boundary conditions and for the existence of bifurcation points for spatial patterns to reaction-diffusion systems of the activator-inhibitor type are proved. The conditions are related with the mollification method employed to overcome difficulties connected with empty interiors of appropriate convex cones.

Keywords: bifurcation, spatial patterns, reaction-diffusion system, mollification, inclusions

MSC 2000: 35B32, 35K57, 35K58, 47H04

0. Introduction

Systems of reaction-diffusion and the effect of diffusion driven instability, the growth of spatial patterns (stationary but spatially nonconstant solutions) and related eigenvalue and bifurcation problems have been studied for a long time by many authors. The motivation for the study of such problems comes from biology and ecology where the behaviour of two or more species is modeled ([11], [21], [22]); the effect of diffusion driven instability was described for the first time in [27]. Multivalued boundary conditions can describe e.g. a certain control process, a semipermeable or another type of the membrane on a part of the boundary. The system with various types of unilateral boundary conditions was studied by M. Kučera, P. Quittner, M. Bosák, P. Drábek in [2], [3], [4], [6], [12], [15], [16], [19], [26] (the destabilizing effect—the bifurcation for the unilateral problem occurs in a domain of stability of

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the system with classical Dirichlet and/or Neumann boundary conditions) and in [13], [17], [18] (stabilizing effect). For a detailed survey see e.g. [8], [6].

In this paper, the results of [16], i.e. the existence of a bifurcation point for system with multivalued boundary conditions proved for an interval, are generalized to domains with higher dimension and the localization of bifurcation points is specified. In [16] the fact that the Sobolev space $W^{1,2}(\Omega)$ is embedded into the space of continuous functions was used. Therefore, the cone $K := \{v \in W^{1,2}(0,1);\ v(0) = 0, v(1) \geqslant 0\}$ has a nonempty interior. An analogue of this cannot hold for higher dimension. In order to prove the existence of a bifurcation point by a similar process as in [16], we can either define a pseudointerior of K like in [26], [4] or [6] and use a technique similar to [3]—this requires an additional condition for the reaction terms (see Remark 8.1 in Appendix)—or approximate our problem (see Section 3) where the corresponding approximate cone K^δ, defined with help of mollification, has a nonempty interior. Similarly to [16] we show the existence of a bifurcation point for the approximate problem and obtain a bifurcation point for the original one by the limiting process for $\delta \to 0$.

1. Problem formulation

Let Ω be a bounded domain in $\mathbb{R}^n$ with a Lipschitzian boundary, let Γ_D, Γ_N, Γ_U be open (in $\partial\Omega$) disjoint subsets of $\partial\Omega$. Let $\partial\Gamma_U$ be Lipschitz with respect to $\partial\Omega$, $\operatorname{meas}(\partial\Omega \setminus (\Gamma_D \cup \Gamma_N \cup \Gamma_U)) = 0$ and

$$\operatorname{meas}\Gamma_D > 0,\ \operatorname{dist}(\Gamma_D, \Gamma_U) > \delta_0 \text{ with } \delta_0 > 0 \text{ small.} \tag{1.1}$$

Let us consider a reaction-diffusion system

$$\begin{aligned} u_t &= d_1\Delta u + f(u,v), \\ v_t &= d_2\Delta v + g(u,v) \end{aligned} \qquad \text{in } [0,+\infty)\times\Omega \tag{RD}$$

with multivalued boundary conditions

$$\begin{aligned} &u = \tilde{u},\ v = \tilde{v} \ \text{ on } [0,+\infty)\times\Gamma_D, \\ &\frac{\partial u}{\partial n} = 0,\ \frac{\partial v}{\partial n} \in -\frac{m(v-\tilde{v})}{d_2} \ \text{ on } [0,+\infty)\times\Gamma_U, \\ &\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \ \text{ on } [0,+\infty)\times\Gamma_N, \end{aligned} \tag{MC}$$

where d_1, d_2 are positive diffusion parameters, $f, g\colon \mathbb{R}^2 \to \mathbb{R}$ are differentiable functions such that $f(\tilde{u},\tilde{v}) = g(\tilde{u},\tilde{v}) = 0$, $\tilde{u}, \tilde{v} \in \mathbb{R}$ are constants, $m\colon \mathbb{R} \to 2^{\overline{\mathbb{R}}}$ is a suitable

multivalued function (e.g. $m(\xi) = 0$ for $\xi > 0$, $m(0) = [m^0, 0]$, $m(\xi)$ is singlevalued, negative for $\xi < 0$).

We will prove that there is a bifurcation point $d_I = [d_1^I, d_2^I]$ at which stationary spatially nonconstant solutions ("spatial patterns") for the system (RD) with (MC) bifurcate from a branch of the trivial solution $[\tilde{u}, \tilde{v}]$. Moreover, this bifurcation point can lie in the region of stability of $[\tilde{u}, \tilde{v}]$ as a solution of (RD) with classical boundary conditions

$$\text{(CC)} \qquad \begin{aligned} &u = \tilde{u},\ v = \tilde{v} \ \text{ on } [0, +\infty) \times \Gamma_D, \\ &\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \ \text{ on } [0, +\infty) \times (\Gamma_N \cup \Gamma_U), \end{aligned}$$

where the bifurcation for (RD), (CC) is excluded.

Set $b_{11} = \frac{\partial f}{\partial u}(\tilde{u}, \tilde{v})$, $b_{12} = \frac{\partial f}{\partial v}(\tilde{u}, \tilde{v})$, $b_{21} = \frac{\partial g}{\partial u}(\tilde{u}, \tilde{v})$, $b_{22} = \frac{\partial g}{\partial v}(\tilde{u}, \tilde{v})$. It is known that under the assumption

$$\text{(SIGN)} \qquad \begin{aligned} &b_{11} > 0, \qquad b_{12} < 0, \qquad b_{21} > 0, \qquad b_{22} < 0, \\ &b_{11} + b_{22} < 0, \qquad b_{11} b_{22} - b_{12} b_{21} > 0, \end{aligned}$$

the effect of diffusion driven instability occurs: the constant solution $[\tilde{u}, \tilde{v}]$ is stable as a solution of ODE's

$$u_t = f(u, v), \quad v_t = g(u, v) \qquad \text{on } [0, +\infty)$$

but it is stable as a solution of (RD), (CC) only for some $d = [d_1, d_2] \in \mathbb{R}^2_+$ lying in the domain of stability D_S and unstable for the other ones (lying in the domain of instability D_U)—for the notation see Fig. 1, Notation 2.1 and Section 3. Further, spatial patterns of (RD), (CC) bifurcate from $[\tilde{u}, \tilde{v}]$ on the boundary C between D_S and D_U (see Fig. 1 and e.g. [20], [25]).

For the sake of simplicity we assume $\tilde{u} = \tilde{v} = 0$ in the sequel. We study only stationary solutions. Hence we solve the system

$$\text{(SRD)} \qquad \begin{aligned} d_1 \Delta u + f(u, v) = 0 \\ d_2 \Delta v + g(u, v) = 0 \end{aligned} \qquad \text{in } \Omega$$

with boundary conditions (MC) and (CC) in the form

$$\text{(1.2)} \qquad \begin{aligned} &u = v = 0 \ \text{ on } \Gamma_D, \\ &\frac{\partial u}{\partial n} = 0,\ \frac{\partial v}{\partial n} \in -\frac{m(v)}{d_2} \ \text{ on } \Gamma_U, \\ &\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \ \text{ on } \Gamma_N \end{aligned}$$

and

$$u = v = 0 \ \text{ on } \Gamma_D, \qquad \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \ \text{ on } \Gamma_N \cup \Gamma_U. \tag{1.3}$$

2. Weak formulation, general assumptions, model example

N o t a t i o n 2.1. $\mathbb{R}_+$—the set of all positive reals, $\mathbb{R}^2_+ = \mathbb{R}_+ \times \mathbb{R}_+$, $\overline{\mathbb{R}} = \{-\infty\} \cup \mathbb{R}$

$d^A \preceq d^B$ for any $d^A = [d^A_1, d^A_2], d^B = [d^B_1, d^B_2] \in \mathbb{R}^2_+$ if and only if $d^A_1 \leqslant d^B_1$ and $d^A_2 \leqslant d^B_2$

κ_j, e_j $(j = 1, 2, 3, \ldots)$—the eigenvalues and eigenvectors of $-\Delta$ with condition (1.3)

$C_j := \left\{ d = [d_1, d_2] \in \mathbb{R}^2_+ \,;\ d_2 = \frac{b_{12}b_{21}/\kappa_j^2}{d_1 - b_{11}/\kappa_j} + \frac{b_{22}}{\kappa_j} \right\}$, $j = 1, 2, 3, \ldots$ (see Fig. 1)

C—the envelope of the hyperbolas C_j, $j = 1, 2, 3, \ldots$ (see Fig. 1)

$D_U := \{ d = [d_1, d_2] \in \mathbb{R}^2_+ \,;\ d_2 > \frac{b_{12}b_{21}/\kappa_j^2}{d_1 - b_{11}/\kappa_j} + \frac{b_{22}}{\kappa_j}$ for at least one $j = 1, 2, 3, \ldots \}$—the set of all $d \in \mathbb{R}^2_+$ lying to the left from C (domain of instability) (see Fig. 1)

$D_S := \mathbb{R}^2_+ \setminus (C \cup D_U)$—the set of all $d \in \mathbb{R}^2_+$ lying to the right from C (domain of stability) (see Fig. 1)

$\mathcal{T}$—the common tangent to all C_j, $j = 1, 2, 3, \ldots$ (see Fig. 1)

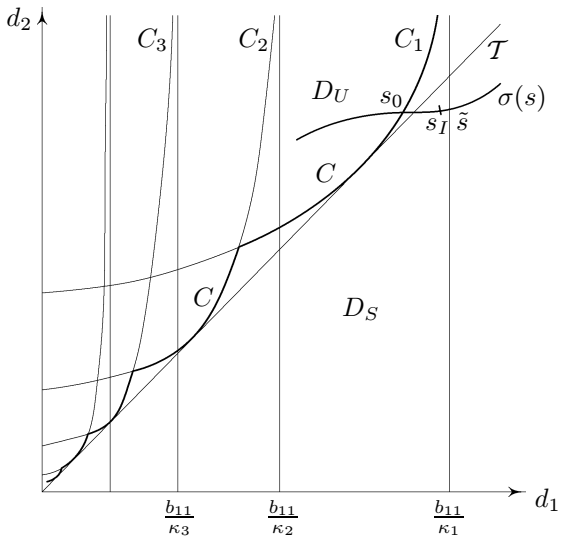

Fig. 1

$C^0(\operatorname{cl}\Omega)$—the space of continuous functions on $\operatorname{cl}\Omega$ equipped with the usual Chebyshev norm

$\mathbb{V}$ a real Hilbert space, $\mathbb{V}^2 = \mathbb{V} \times \mathbb{V}$ endowed with the inner product $\langle U, W \rangle = \langle u, w \rangle + \langle v, z \rangle$, $U = [u, v], W = [w, z] \in \mathbb{V}^2$

A, N_1, N_2—operators satisfying (2.4), (2.5)
$M = [\{0\}, M_2]$, $M_0 = [\{0\}, M_{02}]$—multivalued mappings of $\mathbb{V}^2$ into $2^{\mathbb{V}^2}$ defined in Model Example
$U = [u, v]$ elements of $\mathbb{V}^2$, $AU = [Au, Av]$, $N(U) = [N_1(U), N_2(U)]$ for $U = [u, v] \in \mathbb{V}^2$
$U^* = [\frac{b_{21}}{b_{12}}u, v]$ for $U = [u, v] \in \mathbb{V}^2$
$K := \{U \in \mathbb{V}^2;\ 0 \in M_0(U)\}$—closed convex cone with the vertex at the origin
We denote by $\to, \rightharpoonup$ the strong and weak convergence, respectively.

$$D(d) = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix},\ D^{-1}(d) = \begin{bmatrix} 1/d_1 & 0 \\ 0 & 1/d_2 \end{bmatrix},\ B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},\ B^* = \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix}$$

$E_B(d) := \{U \in \mathbb{V}^2;\ D(d)U - BAU = 0\}$
$E_{B^*}(d) := \{U \in \mathbb{V}^2;\ D(d)U - B^*AU = 0\}$
$E_I(d) := \{U \in \mathbb{V}^2;\ D(d)U - BAU \in -M_0(U)\}$
critical point of a problem (P) (where (P) stands e.g. for (2.7) or (2.11))—a parameter $d \in \mathbb{R}^2_+$ for which (P) has a nontrivial solution
bifurcation point of a problem (P) (where (P) stands e.g. for (2.6) or (2.10))—a parameter $d^0 \in \mathbb{R}^2_+$ such that for any neighbourhood of $[d^0, 0, 0] \in \mathbb{R}^2_+ \times \mathbb{V}^2$ there exists $[d, U] = [d, u, v]$, $\|U\| \neq 0$ satisfying (P).

N o t a t i o n 2.2. Set $\mathbb{V} := \{u \in W^{1,2}(\Omega);\ u = 0$ on Γ_D in the sense of traces$\}$, $\mathbb{V}^2 := \mathbb{V} \times \mathbb{V}$,

$$\langle u, \varphi \rangle := \int_\Omega \sum_{j=1}^{n} u_{x_j} \varphi_{x_j}\, \mathrm{d}x \quad \text{for all } u, \varphi \in \mathbb{V}. \tag{2.1}$$

Then $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{V}$ and the corresponding norm $\| \cdot \|$ is equivalent to the usual Sobolev norm on the space $\mathbb{V}$ under the assumption (1.1) and the embeddings

$$\mathbb{V} \hookrightarrow L^2(\Omega), \mathbb{V} \hookrightarrow L^2(\partial\Omega) \tag{2.2}$$

are compact—see e.g. [10].

Set $n_1(u, v) = f(u, v) - b_{11}u - b_{12}v$, $n_2(u, v) = g(u, v) - b_{21}u - b_{22}v$ and define operators $A\colon\ \mathbb{V} \to \mathbb{V}$, $N_j\colon\ \mathbb{V}^2 \to \mathbb{V}$ $(j = 1, 2)$ by

$$\begin{aligned} \langle Au, \varphi \rangle &= \int_\Omega u\varphi\, \mathrm{d}x \quad \text{for all } u, \varphi \in \mathbb{V} \\ \langle N_j(U), \varphi \rangle &= \int_\Omega n_j(u, v)\varphi\, \mathrm{d}x \quad \text{for all } U = [u, v] \in \mathbb{V}^2,\ \varphi \in \mathbb{V}. \end{aligned} \tag{2.3}$$

It follows from embedding theorems (see e.g. [10]) that

(2.4) A is a linear, symmetric, positive and completely continuous operator.

Further, if $u, v \in W^{1,2}(\Omega)$ then it follows from the embedding theorem that $u, v \in L^q(\Omega)$ with any real $q \geqslant 1$ for $\mathfrak{n} \leqslant 2$ and $1 \leqslant q \leqslant \frac{2\mathfrak{n}}{\mathfrak{n}-2}$ for $\mathfrak{n} > 2$. If n_j satisfy a growth condition $n_j(\xi, \eta) \leqslant C(1 + |\xi|^{q-1} + |\eta|^{q-1})$ for any $\xi, \eta \in \mathbb{R}$ then $n_j(u, v) \in L^{q^*}(\Omega)$ with $q^* = \frac{q}{q-1}$ by the Nemytskii theorem (see e.g. [10]) and this together with the compactness of the embedding mentioned implies that

(2.5)
$$N_1, N_2 \text{ are nonlinear, completely continuous operators from } \mathbb{V}^2 \text{ to } \mathbb{V}$$
$$\lim_{\|U\| \to 0} \frac{\|N_j(U)\|}{\|U\|} = 0 \ (j = 1, 2)$$

(for the last condition, see [18], Lemma 1.A in Appendix).
Now, a weak solution of the problem (SRD), (1.3) is a solution of the operator equations

(2.6)
$$\begin{aligned} d_1 u - b_{11} Au - b_{12} Av - N_1(u, v) &= 0 \\ d_2 v - b_{21} Au - b_{22} Av - N_2(u, v) &= 0. \end{aligned}$$

We also consider the linear problem corresponding to (2.6), i.e.

(2.7)
$$\begin{aligned} d_1 u - b_{11} Au - b_{12} Av &= 0 \\ d_2 v - b_{21} Au - b_{22} Av &= 0. \end{aligned}$$

M o d e l E x a m p l e. (Cf. [16].) Let us consider a multivalued mapping $m\colon \mathbb{R} \to 2^{\overline{\mathbb{R}}}$ which is a singlevalued real continuous function on $\mathbb{R} \setminus \{0\}$ and a multivalued one at $\xi = 0$ such that

$$m(\xi) = 0 \text{ for } \xi > 0, \ m(\xi) \leqslant 0 \text{ for } \xi < 0,$$
$$\lim_{\xi \to 0_-} m(\xi) = m^0 \ \text{ with some } m^0 \in (-\infty, 0), \quad m(0) = [m^0, 0].$$

Set

$$\underline{m}(\xi) = \overline{m}(\xi) = m(\xi) \quad \text{for } \xi \neq 0,$$
$$\underline{m}(0) = m^0, \quad \overline{m}(0) = 0$$

and let us assume that

(2.8) $$|\underline{m}(\xi)|, |\overline{m}(\xi)| \leqslant k \cdot (1 + |\xi|) \text{ with some } k > 0.$$

Consider the situation from Notation 2.2 and define a multivalued mapping $M_2\colon \mathbb{V} \to 2^{\mathbb{V}}$ by

$$M_2(v) := \Big\{ z \in \mathbb{V};\ \int_{\Gamma_U} \underline{m}(v)\varphi \ \mathrm{d}\Gamma \leqslant \langle z, \varphi \rangle \leqslant \int_{\Gamma_U} \overline{m}(v)\varphi \ \mathrm{d}\Gamma \quad \text{for all } \varphi \in \mathbb{V}, \varphi \geqslant 0 \text{ on } \Gamma_U \Big\}. \tag{2.9}$$

(The inequalities on Γ_U are understood in the sense of traces.) Then a solution of

$$\begin{aligned} &d_1 u - b_{11} Au - b_{12} Av - N_1(u,v) = 0 \\ &d_2 v - b_{21} Au - b_{22} Av - N_2(u,v) \in -M_2(v) \end{aligned} \tag{2.10}$$

is a weak solution of the problem (SRD), (1.2)—see [9] for details. Further, introduce a positively homogeneous mapping $M_0\colon \mathbb{V}^2 \to 2^{\mathbb{V}^2}$ corresponding to $M(U) = [\{0\}, M_2(v)]$, $U = [u, v]$, which is defined by $M_0(U) = [\{0\}, M_{02}(v)]$ with

$$\begin{aligned} M_{02}(v) &:= \{ z \in \mathbb{V};\ \langle z, v \rangle = 0,\ \langle z, \varphi \rangle \leqslant 0 \text{ for all } \varphi \in \mathbb{V}, \varphi \geqslant 0 \text{ a.e. on } \Gamma_U \} \\ &\qquad \text{if } v \geqslant 0 \text{ a.e. on } \Gamma_U \\ M_{02}(v) &:= \emptyset \text{ if } v < 0 \text{ on a subset of } \Gamma_U \text{ of a positive measure.} \end{aligned}$$

Then a solution of

$$\begin{aligned} &d_1 u - b_{11} Au - b_{12} Av = 0 \\ &d_2 v - b_{21} Au - b_{22} Av \in -M_{02}(v) \end{aligned} \tag{2.11}$$

is a weak solution of

$$\begin{aligned} &d_1 \Delta u + b_{11} u + b_{12} v = 0 \\ &d_2 \Delta v + b_{21} u + b_{22} v = 0 \end{aligned} \qquad \text{in } \Omega$$

with the boundary conditions

$$\begin{aligned} &u = v = 0 \ \text{ on } \Gamma_D, \\ &\frac{\partial u}{\partial n} = 0,\ v \geqslant 0,\ \frac{\partial v}{\partial n} \geqslant 0,\ \frac{\partial v}{\partial n} \cdot v = 0 \text{ on } \Gamma_U, \\ &\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \ \text{ on } \Gamma_N. \end{aligned} \tag{2.12}$$

Note that the problem (2.11) is still nonlinear because M_0 is cone-valued and nonlinear. Hence we cannot use the standard technique (as e.g. the degree theory for linear mappings) to obtain the bifurcation points.

R e m a r k 2.1. We can also consider $m^0 = -\infty$ in Model Example. Then we define M in the same way as M_0 and do not assume (2.8).

R e m a r k 2.2. It is easy to see from the definition of M_0 that the inclusion problem (2.11) is equivalent to the variational inequality

$$\begin{aligned} &U \in K; \\ &\langle D(d)U - BAU, V - U\rangle \geqslant 0 \text{ for any } V \in K \end{aligned} \tag{2.13}$$

with

$$K := \{U \in \mathbb{V}^2;\ 0 \in M_0(U)\} = \mathbb{V} \times \{\varphi \in \mathbb{V};\ \varphi \geqslant 0 \text{ on } \Gamma_U\}. \tag{2.14}$$

Therefore, the inclusion (2.10) is a generalization of such problems (2.13) and also of variational inequalities

$$\langle D(d)U - BAU - N(U), V - U\rangle + \Psi(V) - \Psi(U) \geqslant 0 \ \text{ for any } \ V \in K$$

with a positive convex lower semicontinuous functional $\Psi\colon\ \mathbb{V}^2 \to (-\infty, +\infty]$, $\Psi \not\equiv +\infty$, where $M = \partial\Psi$—the subdifferential of Ψ (cf. e.g. [5]).

3. Properties of the linear equation

In the sequel, we consider a general real Hilbert space $\mathbb{V}$ and operators $A\colon\ \mathbb{V} \to \mathbb{V}$, $N\colon\ \mathbb{V}^2 \to \mathbb{V}$ satisfying (2.4), (2.5).

O b s e r v a t i o n 3.1. (Cf. [4], Section 2, [6], Section 4.) It follows from (2.4) that the characteristic values of A (i.e. the eigenvalues of the Laplacian with (1.3) for A from Notation 2.2) form a sequence $\{\kappa_i\}_{i=1}^{\infty}$, ($\kappa_i \to +\infty$ for $i \to +\infty$) of positive numbers. The set of all corresponding eigenvectors $\{e_i\}_{i=1}^{\infty}$ forms a complete orthonormal system in $\mathbb{V}$.

Proposition 3.1. *The eigenvalue problem*

$$D(d)U - BAU + \mu AU = 0 \tag{3.1}$$

has a system of eigenvalues

$$\mu_i^{(r)} = \tfrac{1}{2}[b_{11} + b_{22} - (d_1 + d_2)\kappa_i \pm \sqrt{\mathcal{D}}],\ r = 1, 2 \tag{3.2}$$

with $\mathcal{D} := [b_{11} + b_{22} - (d_1 + d_2)\kappa_i]^2 - 4 \cdot [(d_1\kappa_i - b_{11})(d_2\kappa_i - b_{22}) - b_{12}b_{21}]$, $i = 1, 2, \ldots$, *which are roots of*

$$\mu^2 - \mu[b_{11} + b_{22} - (d_1 + d_2)\kappa_i] + (d_1\kappa_i - b_{11})(d_2\kappa_i - b_{22}) - b_{12}b_{21} = 0. \tag{3.3}$$

In particular, $d = [d_1, d_2]$ *is a critical point of* (2.7) *if and only if* $\mu = 0$ *is a solution of* (3.3)*, i.e. if and only if*

$$(d_1\kappa_i - b_{11})(d_2\kappa_i - b_{22}) - b_{12}b_{21} = 0, \tag{3.4}$$

i.e. if d *lies on a hyperbola* $C_i = \left\{ d = [d_1, d_2] \in \mathbb{R}^2_+ \,;\; d_2 = \dfrac{b_{12}b_{21}/\kappa_i^2}{d_1 - b_{11}/\kappa_i} + \dfrac{b_{22}}{\kappa_i} \right\}$ *for some* $i = 1, 2, \ldots$.

For the proof see e.g. [4], Section 2.

O b s e r v a t i o n 3.2. (See [20] and [4] for the proof of the following statement.) Under the assumption (SIGN), for a given i there are two real roots $\mu_i^{(1)}(d)$, $\mu_i^{(2)}(d)$ of (3.3) for any d lying to the left from C_i or in the right neighbourhood of C_i (including C_i). The smaller one (say $\mu_i^{(2)}(d)$) is always negative.

It follows from the definition of C_i that $\mu_i^{(1)}(d) < 0$ or $\mu_i^{(1)}(d) > 0$ for d to the right or to the left, respectively, from C_i and in a neighbourhood of C_i. For d lying to the right and sufficiently far from C_i, we have $\mu_i^{(r)}(d) \in \mathbb{C}\backslash\mathbb{R}$ with $\operatorname{Re}\mu_i^{(r)}(d) < 0$, $r = 1, 2$. Further, for any $d \in \mathbb{R}^2_+$, let us set $\mu(d) := \max\{\mu_i^{(1)}(d);\ \mu_i^{(1)}(d) \in \mathbb{R}\}$. Hence, $\mu(d)$ is the greatest eigenvalue of (3.1). Then the envelope C of all C_i, $i = 1, 2, \ldots$ is equal to $\{d \in \mathbb{R}^2_+ \,;\ \mu(d) = 0\}$ and $\mu(d) < 0$ or $\mu(d) > 0$ for d from a neighbourhood of C and to the right or to the left from C, respectively.

O b s e r v a t i o n 3.3. It follows from Proposition 3.1 and Observation 3.2 that $E_B(d) \neq \{0\}$ if and only if $d \in \bigcup_{j=1}^{\infty} C_j$. Moreover, let p be an index such that the characteristic value κ_p of A (i.e. the eigenvalue of the Laplacian with (1.3) for A from Notation 2.2) has a multiplicity k, $\kappa_p = \ldots = \kappa_{p+k-1}$. Then for any $d \in C_p = \ldots = C_{p+k-1}$, $d \notin C_q$ for $C_q \neq C_p$ we have

$$E_B(d) = \operatorname{Lin}\{U_i(d)\}_{i=p}^{p+k-1} \text{ with } U_i(d) = [\alpha_i(d)e_i, e_i], \tag{3.5}$$

where $\alpha_i(d) = \frac{d_2\kappa_p - b_{22}}{b_{21}} > 0$. Further, if $d \in C_p \cap C_q$ for some $C_q \neq C_p$, $\kappa_p \neq \kappa_q = \ldots = \kappa_{q+\ell-1}$ (κ_q has the multiplicity ℓ) then

$$E_B(d) = \operatorname{Lin}\{U_i(d)\}_{i=p,\ldots,p+k-1,q,\ldots,q+\ell-1}. \tag{3.6}$$

For the proof see [4], Section 2.

4. The main result

We will show in Theorem 4.1 the existence of a bifurcation point to (2.10). The method of the proof of this fact will be the same as in [16]. One of the assumptions in [16] was $\operatorname{int} K \neq \emptyset$. Here, we have $\mathfrak{n} > 1$ and therefore $\operatorname{int} K = \emptyset$ in general. We consider an auxiliary problem with an additional parameter δ (see below) which has the property $\operatorname{int} K^\delta \neq \emptyset$ and which approximates our original problem for $\delta \to 0$.

N o t a t i o n 4.1. Let $\delta > 0$ be fixed. Let G be a bounded domain in $\mathbb{R}^{\mathfrak{n}}$ with a Lipschitz boundary such that $\operatorname{cl}\Omega \subset G$. We define a mollification mapping $\Phi^\delta\colon\ \mathbb{V} \to W^{1,2}(G)$ in the following way: Let $\varphi^\delta\colon\ \mathbb{R}^{\mathfrak{n}} \to [0,+\infty)$ be a C^∞-smooth function such that $\varphi^\delta(0) > 0$, $\varphi^\delta(x) \leqslant \varphi^\delta(0)$ for any $x \in \mathbb{R}^{\mathfrak{n}}$, $\varphi^\delta(x) = 0$ for all $x \notin \mathcal{B}_\delta(0)$ (the ball with a radius δ centered at the origin) and $\int_{\mathbb{R}^{\mathfrak{n}}} \varphi^\delta(x)\,\mathrm{d}x = 1$. Then φ^δ is bounded on $\mathbb{R}^{\mathfrak{n}}$ and φ^δ converges in the sense of distributions to the Dirac measure centered at the origin for $\delta \to 0_+$. For an example of such a function see [23]. There exists a continuous "extension" mapping $E\colon\ W^{1,2}(\Omega) \to W_0^{1,2}(G)$ (see [23]). Let us define a mapping

$$\Phi^\delta(v,x) := \int_G \varphi^\delta(x-y)Ev(y)\ \mathrm{d}y \quad \text{for any } v \in \mathbb{V},\ x \in G.$$

Hence, $\Phi^\delta(v,\cdot)$ is a continuous function on $\operatorname{cl}\Omega$ and it is easy to see that if v_n, $v \in \mathbb{V}$, $v_n \to v$ in $\mathbb{V}$ then $\Phi^\delta(v_n,\cdot) \to \Phi^\delta(v,\cdot)$ in $C^0(\operatorname{cl}\Omega)$. Further, define M^δ, M_0^δ, K^δ by $M^\delta(U) = [\{0\}, M_2^\delta(v)]$, $M_0^\delta(U) = [\{0\}, M_{02}^\delta(v)]$, $K^\delta = \mathbb{V} \times K_2^\delta$ with

$$\begin{aligned} M_2^\delta(v) := \Big\{ z \in \mathbb{V};\ & \int_{\Gamma_U} \underline{m}(\Phi^\delta(v,x))[\Phi^\delta(\varphi,x)]^+\,\mathrm{d}\Gamma - \int_{\Gamma_U} \overline{m}(\Phi^\delta(v,x))[\Phi^\delta(\varphi,x)]^-\,\mathrm{d}\Gamma \\ & \leqslant \langle z,\varphi\rangle \leqslant \int_{\Gamma_U} \overline{m}(\Phi^\delta(v,x))[\Phi^\delta(\varphi,x)]^+\,\mathrm{d}\Gamma - \int_{\Gamma_U} \underline{m}(\Phi^\delta(v,x))[\Phi^\delta(\varphi,x)]^-\,\mathrm{d}\Gamma \\ & \qquad \text{for all } \varphi \in \mathbb{V}\Big\}; \end{aligned}$$

$$M_{02}^\delta(v) := \big\{ z \in \mathbb{V};\ \langle z,v\rangle = 0,\ \langle z,\varphi\rangle \leqslant 0 \text{ for all } \varphi \in \mathbb{V}, \Phi^\delta(\varphi,\cdot) \geqslant 0 \text{ on } \Gamma_U \big\} \quad \text{if } \Phi^\delta(v,\cdot) \geqslant 0 \text{ on } \Gamma_U;$$

$$M_{02}^\delta(v) := \emptyset \ \text{ if } \Phi^\delta(v,x_0) < 0 \text{ for some } x_0 \in \Gamma_U;$$

$$K_2^\delta := \{\varphi \in \mathbb{V};\ 0 \in M_{02}^\delta(\varphi)\}.$$

Here, φ^+, φ^- denote the positive and negative parts of φ, respectively, $\varphi = \varphi^+ - \varphi^-$. Note that we have $K_2^\delta = \{\varphi \in \mathbb{V};\ \Phi^\delta(\varphi,\cdot) \geqslant 0 \text{ on } \Gamma_U\}$ and $\operatorname{int} K_2^\delta \supseteq \{\varphi \in \mathbb{V};\ \Phi^\delta(\varphi,\cdot) > 0 \text{ on } \operatorname{cl}\Gamma_U\} \neq \emptyset$ because Φ^δ is $(\mathbb{V} \to C^0(\operatorname{cl}\Omega))$-continuous and the interior of K^δ is the preimage of an open set.

O b s e r v a t i o n 4.1. The mappings M^δ and M_0^δ obviously satisfy the following conditions:

(4.1) $0 \in M^\delta(0)$;

(4.2) K^δ is a closedconvex cone with the vertex at the origin, $\{0\} \neq K^\delta \neq \mathbb{V}^2$;

(4.3) if $U \in K^\delta$ then $U^* \in K^\delta$;

(4.4) $M_0^\delta(tV) = tM_0^\delta(V)$ for all $t > 0,\ V \in \mathbb{V}^2$;

(4.5) if $U \in \mathbb{V}^2$ then $\langle Z, U\rangle = 0$ for all $Z \in M_0^\delta(U)$;

(4.6) if $U \in \mathbb{V}^2$ then $\langle Z, \Psi\rangle \geqslant 0$ for all $\Psi \in K^\delta,\ Z \in -M_0^\delta(U)$.

Proposition 4.1. *Let $U_n \to 0,\ W_n = \frac{U_n}{\|U_n\|} \rightharpoonup W,\ Z_n \to Z$ in $\mathbb{V}^2$ and $d_n \to d$ in $\mathbb{R}^2_+$ such that $D(d_n)W_n + Z_n \in -\frac{M^\delta(U_n)}{\|U_n\|}$. Then $W_n \to W,\ D(d)W + Z \in -M_0^\delta(W)$.*

The proof is given in [9].

There exists a system of continuous functions $p_\tau\colon\ \mathbb{R} \to \mathbb{R}$ with a real parameter $\tau \in [0, +\infty)$ such that

$$p_0 \equiv 0,\ p_\tau(\xi) = 0 \text{ for } \xi \geqslant 0,\ p_\tau(\xi) \in (m(\xi), 0] \text{ for } \xi < 0 \tag{4.7}$$

satisfying the following conditions:

$$\begin{aligned}
&\text{if } \tau_n \to \tau \in [0, +\infty),\ \xi_n \to \xi \ \text{ then } p_{\tau_n}(\xi_n) \to p_\tau(\xi);\\
&\text{if } \tau_n \to \tau \in (0, +\infty),\ \xi_n \to 0_-\ \text{ then } \tilde{p}_\tau := \liminf_{\mathfrak{n}\to+\infty} \frac{p_{\tau_n}(\xi_n)}{\xi_n} > 0;\\
&\text{if } \tau_n \to 0_+,\ \xi_n \to 0_-\ \text{ then } \frac{p_{\tau_n}(\xi_n)}{\xi_n} \to 0,\ \liminf_{\mathfrak{n}\to+\infty} \frac{p_{\tau_n}(\xi_n)}{\tau_n\xi_n} > 0;\\
&\text{if } \tau_n \to +\infty,\ \xi_n \to \xi,\ p_{\tau_n}(\xi_n) \to p\\
&\qquad\qquad \text{then } p \in m(\xi) \text{ or } p = m(\xi) \text{ for } \ \xi = 0 \text{ or } \xi \neq 0, \text{ respectively.}
\end{aligned} \tag{4.8}$$

Let us define for any $\tau \in [0, +\infty)$ a function $p_{0,\tau}\colon\ \mathbb{R} \to \mathbb{R}$ such that $p_{0,\tau}(\xi) = 0$ for all $\xi \geqslant 0$ and $p_{0,\tau}(\xi) = \tilde{p}_\tau \cdot \xi$ for all $\xi < 0$. Moreover, a system of operators $P_\tau^\delta, P_{0,\tau}^\delta\colon\ \mathbb{V}^2 \to \mathbb{V}^2$ with a parameter $\tau \in [0, +\infty)$ is defined by $P_\tau^\delta(U) = [0, P_{\tau,2}^\delta(v)]$, $P_{0,\tau}^\delta(U) = [0, P_{0,\tau,2}^\delta(v)]$ for $U = [u, v]$, where

$$\left.\begin{aligned}
\langle P_{\tau,2}^\delta(v), \psi\rangle &= \int_{\Gamma_U} p_\tau(\Phi^\delta(v,x))\Phi^\delta(\psi,x)\ \mathrm{d}\Gamma\\
\langle P_{0,\tau,2}^\delta(v), \psi\rangle &= \int_{\Gamma_U} p_{0,\tau}(\Phi^\delta(v,x))\Phi^\delta(\psi,x)\ \mathrm{d}\Gamma
\end{aligned}\right\} \text{ for all } v,\ \psi \in \mathbb{V}.$$

O b s e r v a t i o n 4.2. For such a system of operators and a fixed $\delta \in (0, \delta_0)$ the following conditions are clearly fulfilled:

$$\begin{cases} P^\delta_\tau(U) = 0 \ \text{ for all } U \in K^\delta, \\ \langle P^\delta_\tau(U), V\rangle \leqslant 0 \ \text{ for all } U \in \mathbb{V}\times\mathbb{V}, V \in K^\delta,\ \tau \in [0,+\infty); \end{cases} \tag{4.9}$$

$$\langle P^\delta_\tau(U), U\rangle \geqslant 0,\ \langle P^\delta_{0,\tau}(U), U\rangle \geqslant 0 \ \text{ for all } U \in \mathbb{V}\times\mathbb{V},\ \tau \in [0,+\infty). \tag{4.10}$$

The proofs of the following propositions for Model Example will be given in [9].

Proposition 4.2. *Let $U_n \rightharpoonup U$ in $\mathbb{V}^2$, $\tau_n \geqslant 0$, $d_n \to d \in \mathbb{R}^2_+$. Then*

$$\liminf_{n\to+\infty} \langle D^{-1}(d_n) P^\delta_{\tau_n}(U_n), U_n - U\rangle \geqslant 0.$$

If, moreover, $U = 0$, $\frac{P^\delta_{\tau_n}(U_n)}{\|U_n\|}$ are bounded and $W_n = \frac{U_n}{\|U_n\|} \overset{\mathbb{V}^2}{\rightharpoonup} W$, then

$$\liminf_{n\to+\infty} \left\langle \frac{D^{-1}(d_n) P^\delta_{\tau_n}(U_n)}{\|U_n\|}, W_n - W \right\rangle \geqslant 0.$$

Proposition 4.3. *Let $U_n \overset{\mathbb{V}^2}{\rightharpoonup} U$, $\tau_n \to \tau \in [0,+\infty)$. Then $P^\delta_{\tau_n}(U_n) \overset{\mathbb{V}^2}{\to} P^\delta_\tau(U)$. For $\tau = +\infty$ and $P^\delta_{\tau_n}(U_n) \overset{\mathbb{V}^2}{\to} Z$ this Z belongs to $M^\delta(U)$. For $U = 0$ and $\tau = 0$ the convergence*

$$\|U_n\|^{-1} P^\delta_{\tau_n}(U_n) \overset{\mathbb{V}^2}{\to} 0$$

holds. Moreover, if $U = 0$, $W_n = \|U_n\|^{-1} U_n \overset{\mathbb{V}^2}{\rightharpoonup} W$ and $\tau_n \to \tau \in [0,+\infty)$, then $\|U_n\|^{-1} P^\delta_{\tau_n}(U_n) \overset{\mathbb{V}^2}{\to} P^\delta_{0,\tau}(W)$. For $\tau = +\infty$ and $\|U_n\|^{-1} P^\delta_{\tau_n}(U_n) \overset{\mathbb{V}^2}{\to} Z$ we have $Z \in M^\delta_0(W)$.

Proposition 4.4. *Let $U_n \overset{\mathbb{V}^2}{\to} 0$, $W_n = \|U_n\|^{-1} U_n \overset{\mathbb{V}^2}{\rightharpoonup} W \notin K^\delta, \tau_n \to \tau_0 > 0$ and $V \in \operatorname{int} K^\delta$. Then $\limsup_{n\to+\infty} \|U_n\|^{-1} \left\langle P^\delta_{\tau_n}(U_n), V\right\rangle < 0$. For $\tau_0 = 0$, moreover,*

$$\limsup_{n\to+\infty} (\tau_n \|U_n\|)^{-1} \left\langle P^\delta_{\tau_n}(U_n), V\right\rangle < 0.$$

Proposition 4.5. *Let us assume that $U_n \overset{\mathbb{V}^2}{\rightharpoonup} U$, $Z_n \overset{\mathbb{V}^2}{\to} Z$, $d_n \to d \in \mathbb{R}^2_+$ and $\delta_n \to 0_+$. Then the following implications hold:*

$$D(d_n)U_n + Z_n \in -M^{\delta_n}(U_n) \Longrightarrow U_n \overset{\mathbb{V}^2}{\to} U,\ D(d)U + Z \in -M(U); \tag{4.11}$$

$$D(d_n)U_n + Z_n \in -M^{\delta_n}_0(U_n) \Longrightarrow U_n \overset{\mathbb{V}^2}{\to} U,\ D(d)U + Z \in -M_0(U). \tag{4.12}$$

Let us remark that (4.11) is essential for the proof of Theorem 4.1 and (4.12) is used for the proof of the destabilizing effect ($s_I > s_0$)—see Remark 4.2.

Let δ_0 be from (1.1) and let $d^0 \in C_p$ be a fixed point such that there is an eigenfunction e corresponding to the eigenvalue κ_p of the Laplacian with (1.3) such that

$$e \leqslant -\varepsilon \text{ on a } \delta_0\text{-neighbourhood of } \Gamma_U \text{ in } \operatorname{cl}\Omega \text{ for some } \varepsilon > 0. \tag{4.13}$$

Then the system $\{e_i\}_{i=1}^{\infty}$ can be chosen in such a way that $\kappa_p = \ldots = \kappa_{p+k-1}$, k is the multiplicity of κ_p and (4.13) holds with $e = e_p$. In particular, it follows from Observation 3.3 and the definition of K^δ that

$$-U_0 \in E_B(d^0) \cap \operatorname{int} K^\delta \text{ for any } \delta \in (0, \delta_0) \tag{4.14}$$

is fulfilled with $U_0 = U_p(d^0)\big(= [\alpha_p(d^0)e_p, e_p]$, see (3.5)$\big)$.

In the sequel we consider a curve σ given by a differentiable mapping $\sigma\colon\ \mathbb{R} \to \mathbb{R}^2_+$ satisfying

$$\begin{cases} \sigma(s) \in D_S \text{ for all } s \in (s_0, +\infty), \\ \sigma \text{ intersects the envelope } C \text{ at the point } \sigma(s_0) = d^0, \\ \sigma \text{ intersects the line } d_1 = \frac{b_{11}}{\kappa_1} \text{ at a point } \sigma(\tilde{s}),\ \tilde{s} > s_0, \\ \sigma_1(s) < \frac{b_{11}}{\kappa_1} \text{ for all } s \in (s_0, \tilde{s}), \\ \sigma_1(s) > \frac{b_{11}}{\kappa_1} \text{ for } s \in (\tilde{s}, \tilde{s} + \zeta_0) \text{ with some } \zeta_0 > 0. \end{cases} \tag{4.15}$$

It is essential that if $d^0 \in C \cap C_p$ and (4.14) holds with $U_0 = U_p(d^0)$ then

$$\text{the curve } \sigma \text{ is transversal to } C_p \text{ at } d^0. \tag{4.16}$$

Note that if, moreover, $d^0 \in C_p \cap C_q$, $C_p \neq C_q$ then σ has to be transversal to C_p but not necessarily to both C_p and C_q.

R e m a r k 4.1. By introducing the curve $\sigma(s)$ we have changed the two-parametric system (2.10) with $[d_1, d_2] \in \mathbb{R}^2_+$ to the system

$$D(\sigma(s))U - BAU - N(U) \in -M(U) \tag{4.17}$$

with a single parameter $s \in \mathbb{R}$. Further, by a *critical point* of

$$D(\sigma(s))U - BAU = 0 \tag{4.18}$$

or (2.11) written with d_1, d_2 replaced by $\sigma_1(s), \sigma_2(s)$ we mean a parameter s_1 such that $E_B(\sigma(s_1)) \neq \{0\}$ or $E_I(\sigma(s_1)) \neq \{0\}$, respectively, and by a *bifurcation point* of (4.17) we mean a parameter $s_2 \in \mathbb{R}$ such that for any neighbourhood of $[s_2, 0, 0] \in \mathbb{R} \times \mathbb{V}^2$ there exists $[s, U] = [s, u, v]$, $\|U\| \neq 0$ satisfying (4.17). Therefore, by the assumption (4.15) s_0 is the largest critical point of (4.18), because a nontrivial solution of (4.18) exists only for $\sigma(s) \in C_j$ for some $j = 1, 2, \ldots$—see Observation 3.3.

Theorem 4.1. *Let* (SIGN), (1.1), (2.2), (2.4) *and* (2.5) *hold, let* $\sigma(s)$ *be a differentiable curve satisfying* (4.15), *let* $d^0 \in C_p$ *and* (4.16) *hold. Let* (4.14) *hold with* $U_0 = U_p(d^0)$ $\big($ $= [\alpha_p(d^0)e_p, e_p]$, *see* (3.5)$\big)$. *Consider a multivalued mapping* M *such that there exists a system of multivalued mappings* M^δ *and the corresponding homogeneous multivalued mappings* M_0 *and* M_0^δ, *the operators* P_τ^δ, $P_{0,\tau}^\delta$ $(\tau \in [0, +\infty)$, $\delta \in (0, \delta_0))$ *satisfying the assumptions* (4.1)–(4.6), (4.9), (4.10) *and for which Propositions* 4.1–4.4 *and* (4.11) *in Proposition* 4.5 *remain valid. Then there exists a bifurcation point* $s_I \in [s_0, \tilde{s}]$ *of the inclusion* (4.17). *Hence, there is* $\varrho_0 > 0$ *such that for any* $\varrho \in (0, \varrho_0)$ *there are* s_ϱ, U_ϱ *satisfying* (4.17), $\|U_\varrho\|^2 = \varrho$, $s_\varrho \in [s_0, \tilde{s}]$ *and such that if* $\varrho_n \to 0_+$, $s_{\varrho_n} \to s_I$ *then* $s_I \in [s_0, \tilde{s}]$.

Proof will be given in Section 7. For $\mathfrak{n} = 1$, cf. [16], Theorem 2.10.

R e m a r k 4.2. If, moreover, either $\operatorname{int} K \neq \emptyset$ or (2.11) is equivalent to (2.13) (this assumption is satisfied in many reasonable situations) and the conditions

$$\text{if } U \in K \text{ then } U^* \in K, \tag{4.19}$$

$$\text{if } U \in \mathbb{V}^2 \text{ then } \langle Z, \Psi \rangle \geqslant 0 \text{ for all } \Psi \in K,\ Z \in -M_0(U) \tag{4.20}$$

and (4.12) hold then we can prove $s_I > s_0$, which implies that s_ϱ, U_ϱ from Theorem 4.1 do not satisfy

$$D(\sigma(s))U - BAU - N(U) = 0 \tag{4.21}$$

—see the proof of destabilizing effect in Appendix.

R e m a r k 4.3. There are two main improvements in comparison to [16], Theorem 2.10. First, the localization of the bifurcation point is specified—we show that $s_I < \tilde{s}$, $\tilde{s}$ is from (4.15), i.e. $d_I = \sigma(s_I) \notin Z_0$ in the sense of [7], i.e. $d_1^I = \sigma_1(s_I) \leqslant \frac{b_{11}}{\kappa_1}$. Second, in [16] the case $\mathfrak{n} = 1$, $\dim E_B(d^0) = 1$ and $\operatorname{int} K \neq \emptyset$ was considered. Here, $\mathfrak{n} > 1$ is admitted and therefore the possible difficulties $\dim E_B(d^0) > 1$ and $\operatorname{int} K = \emptyset$ must be overcome. To get over the former one the operator L_δ is involved (see Notation 5.2), to get over the latter, the approximate problem (5.13)—see below—is considered. Notice that for δ fixed, the existence of a bifurcation point s_I^δ for this

δ-problem can be shown in the same way as in [16], cf. Remark 8.1 in Appendix for another technique overcoming the emptiness of $\operatorname{int} K$ by using the notion of pseudointerior.

Corollary 4.1. *Let* (SIGN) *and* (1.1) *hold, let* $\sigma(s)$ *be a differentiable curve satisfying* (4.15), *let* $d^0 \in C_p$ *and* (4.16) *hold. Let* m *be the multivalued function from Model Example and let us assume that there exists an eigenfunction* e_p *corresponding to an eigenvalue* κ_p *of the Laplacian with* (1.3) *such that* (4.13) *is fulfilled with* $e = e_p$. *Then stationary spatially nonconstant weak solutions (spatial patterns) of* (SRD), (1.2) *bifurcate at some* $s_I \in (s_0, \tilde{s}]$.

This follows from Theorem 4.1, Propositions 4.1–4.5, Remark 4.2 and the fact that no nontrivial constant functions can satisfy (1.3).

5. Reduction of dimension of the space $E_B(d^0)$

In this section we will keep the assumptions of Theorem 4.1. The following proposition holds (cf. [16], Remark 4.5):

Proposition 5.1. *Let* σ *satisfy* (4.15) *and* (4.16). *Then*

$$\frac{(\kappa_p \sigma_2(s_0) - b_{22})^2}{b_{12} b_{21}} \sigma_1'(s_0) + \sigma_2'(s_0) < 0.$$

For the proof see Appendix.

O b s e r v a t i o n 5.1. Similarly as in [6], Section 4 we will consider an eigenvalue problem

$$(D^{-1}(d)BA - I)U = \mu U. \tag{5.1}$$

We will study the behaviour of eigenvalues of (5.1) with respect to the changing d along the curve $\sigma(s)$. The process will be the same as in [6]. Therefore, the detailed calculations are explained in Appendix and here only the main steps are sketched.

All eigenvalues of (5.1) are the roots of

$$\mu^2 d_1 d_2 \kappa_i^2 - \beta_i(d) \kappa_i \mu + \gamma_i(d) = 0, \tag{5.2}$$

i.e. the numbers

$$\mu_i^{(r)}(d) := \frac{\beta_i(d) \pm \sqrt{\omega(d)}}{2 d_1 d_2 \kappa_i}, \ r = 1, 2. \tag{5.3}$$

Here, $\beta_i(d) := d_1 b_{22} + d_2 b_{11} - 2d_1 d_2 \kappa_i$, $\gamma_i(d) := (d_1\kappa_i - b_{11})(d_2\kappa_i - b_{22}) - b_{12}b_{21}$, $\omega(d) := d_1^2 b_{22}^2 + d_2^2 b_{11}^2 - 2d_1 d_2 b_{11} b_{22} + 4d_1 d_2 b_{12} b_{21}$, $i = 1, 2, \ldots$. The set $\{d \in \mathbb{R}^2_+ ; \omega(d) = 0\}$ is a couple of half-lines, one of them is a common tangent $\mathcal{T}$ to all hyperbolas C_j, $j = 1, 2, \ldots$ (see also [20] and Figures 1 and 2). The calculations of the crucial signs of the eigenvalues $\mu_i^{(1)}(d)$, $\mu_i^{(2)}(d)$ from (5.3) in the domains $\mathcal{D}_1, \ldots, \mathcal{D}_6$, are described in Appendix. They lead to the conclusion that for d lying to the left from C_i, there is one positive root of (5.2) and for d lying to the right from C_i, either none or both roots of (5.2) are positive.

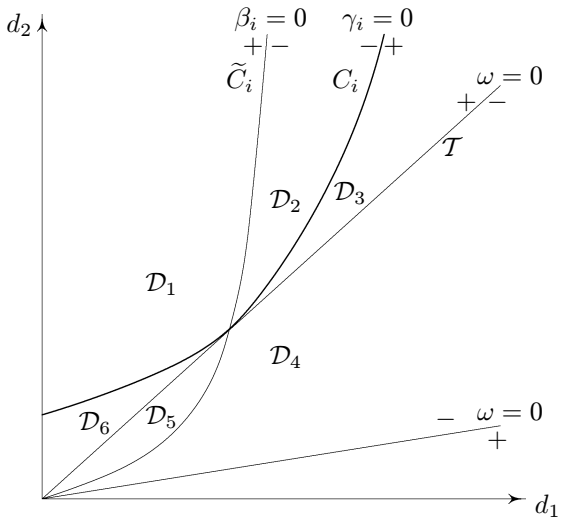

Fig. 2

N o t a t i o n 5.1. (Cf. [8], Notation 4.1.) The vectors

$$U_i^{(r)}(d) = \Big[\frac{d_2\kappa_i - b_{22} + \mu_i^{(r)}(d) d_2 \kappa_i}{b_{21}} e_i, e_i\Big], \ i \in \mathbb{N}, \ r = 1, 2 \tag{5.4}$$

are the eigenvectors of (5.1) corresponding to $\mu_i^{(r)}(d)$.

Let $\eta > 0$ be a small number. Let $d^0 \in C_p = \ldots = C_{p+k-1}$, $d^0 \notin \mathcal{T}$. Then the curve $\sigma(s)$ for $s \in (s_0 - \eta, s_0 + \eta)$ goes either from $\mathcal{D}_2$ into $\mathcal{D}_3$ or from $\mathcal{D}_1$ into $\mathcal{D}_6$ for η small—see Fig. 2. By $\mu_p(s)$ for $s \in (s_0 - \eta, s_0 + \eta)$ we denote the root of (5.2) changing the sign at d^0, i.e.

$$\begin{aligned} \mu_p(s) &= \mu_p^{(1)}(\sigma(s)) \quad \text{if } C_p \cap \mathcal{T} \preceq d^0 \\ &= \mu_p^{(2)}(\sigma(s)) \quad \text{if } d^0 \preceq C_p \cap \mathcal{T} \end{aligned} \tag{5.5}$$

(see Appendix for details).

Let $d^0 \in C_p = \ldots = C_{p+k-1}$, $d^0 \in \mathcal{T}$. Then the curve $\sigma(s)$ goes from the domains $(\mathcal{D}_1 \cup \mathcal{D}_2)$ into $(\mathcal{D}_4 \cup \mathcal{D}_5)$. By $\mu_p(s)$ we denote the positive root of (5.2) on $(s_0 - \eta, s_0)$ (i.e. $\mu_p(s) = \mu_p^{(1)}(\sigma(s))$) and for $[s_0, s_0 + \eta)$ we put $\mu_p(s) = \operatorname{Re} \mu_p^{(r)}(\sigma(s))$, $r = 1, 2$.

Let us denote by

$$U_i(s) = \Big[\frac{\sigma_2(s)\kappa_i - b_{22} + \mu_i(s)\sigma_2(s)\kappa_i}{b_{21}} e_i, e_i\Big],\ i = p, \ldots, p + k - 1 \tag{5.6}$$

the corresponding eigenvectors if $d^0 \notin \mathcal{T}$ or $d^0 \in \mathcal{T}$ and $s \in (s_0 - \eta, s_0]$, or their real parts in the case $d^0 \in \mathcal{T}$ and $s \in (s_0, s_0 + \eta)$.

O b s e r v a t i o n 5.2. (Cf. [8], Observation 4.2.) Let $\mu_q(s) \neq \mu_p(s)$ for all q satisfying $\kappa_q \neq \kappa_p$. Then

$$\operatorname{Ker}(D^{-1}(\sigma(s))BA - (1 + \mu_p(s))I) = \operatorname{Lin}\{U_i(s)\}_{i=p}^{p+k-1}$$

for all $s \in (s_0 - \eta, s_0 + \eta)$ in the case $d^0 \notin \mathcal{T}$ and for all $s \in (s_0 - \eta, s_0]$ in the case $d^0 \in \mathcal{T}$. In particular, if $d^0 \in C_p$ and $d^0 \notin C_q$ for all $C_q \neq C_p$, then

$$E_B(d^0) = \operatorname{Lin}\{U_i(s_0)\}_{i=p}^{p+k-1}. \tag{5.7}$$

If $\mu_q(s) = \mu_p(s)$ for some q satisfying $\kappa_p \neq \kappa_q = \ldots = \kappa_{q+\ell-1}$, where κ_q has the multiplicity ℓ, then $\operatorname{Ker}(D^{-1}(\sigma(s))BA - (1+\mu_p(s))I) = \operatorname{Lin}\{U_i(s)\}_{i=p,\ldots,p+k-1,q,\ldots,q+\ell-1}$ for all $s \in (s_0 - \eta, s_0 + \eta)$. In particular, if $d \in C_p \cap C_q$ for some $C_q \neq C_p$, then

$$E_B(d^0) = \operatorname{Lin}\{U_i(s_0)\}_{i=p,\ldots,p+k-1,q,\ldots,q+\ell-1}. \tag{5.8}$$

N o t a t i o n 5.2. (Cf. [8], Notation 4.2.) Set $I(d^0) = \{i \in \mathbb{N} \setminus \{p\};\ d^0 \in C_i\}$. Set $I_p(d^0) = \{i \in I(d^0);\ C_i = C_p\}$ and $I_q(d^0) = I(d^0) \setminus I_p(d^0)$. Choose $\eta > 0$ such that $\mu_p(s)$ is well defined for any $s \in (s_0 - \eta, s_0 + \eta)$. Moreover, for $i \in I(d^0)$ set

$$\begin{aligned} &\nu_i(d^0) = 1 \quad \text{if } \mu_i(s_0) = \mu_i^{(1)}(d^0) \text{ or } \mu_i^{(1)}(d^0) = 0, \\ &\nu_i(d^0) = -1 \text{ if } \mu_i(s_0) = \mu_i^{(2)}(d^0) \text{ and } \mu_i^{(1)}(d^0) \neq 0, \end{aligned}$$

introduce a continuous cut-off function χ with a support in $(s_0 - \eta, s_0 + \eta)$ such that $\chi(s_0) = 1$, $\chi(\mathbb{R}) \subset [0, 1]$ and for any $\delta > 0$, the linear completely continuous operator $L_\delta(s)$ in $\mathbb{V}^2$ (for any s fixed) by

$$L_\delta(s)\colon\ U \mapsto \delta\chi(s) \cdot \sum_{i \in I(d^0)} \nu_i(d^0) \frac{\langle U_i(s), U\rangle}{\|U_i(s)\|^2} \cdot U_i(s). \tag{5.9}$$

Let us notice that in [4] and [6] a simpler definition of L was taken without a sign term. Here we need also a proper sign in (5.11) below for the proof of the fact that $s_I > s_0$ in Theorem 4.1. Hence, the proof of Lemma 5.1 below is slightly more complicated but its assertion is the same as that in [6].

O b s e r v a t i o n 5.3. (Cf. [8], Remark 4.2.) (5.9) yields that $L_\delta(s) \equiv 0$ for $s \in \mathbb{R}$ if $I(d^0) = \emptyset$, i.e. if $\dim E_B(d^0) = 1$. From (5.4), (5.6), the form of U^* and the fact that $\langle e_i, e_j\rangle = 0$ for $i \neq j$ we deduce that for any $s \in (s_0 - \eta, s_0 + \eta)$ the identities $\langle U_i^{(r)}(\sigma(s)), U_j^{(r)}(\sigma(s))\rangle = \langle U_i(s), U_j(s)\rangle = \langle U_i^*(s), U_j(s)\rangle = 0$ hold for all $j \neq i$, $r = 1, 2$ and

$$
\begin{aligned}
&L_\delta(s)U_p(s) = L_\delta(s)U_p^*(s) = 0,\ L_\delta(s)U_i^{(r)}(\sigma(s)) = 0 \text{ for } i \notin I(d^0),\ r = 1, 2,\\
&L_\delta(s)U_i(s) = \delta\chi(s)U_i(s) \text{ for } i \in I(d^0), \quad \text{if } d^0 \text{ lies above or in } C_i \cap \mathcal{T},\\
&L_\delta(s)U_i(s) = -\delta\chi(s)U_i(s) \text{ for } i \in I(d^0), \quad \text{if } d^0 \text{ lies below } C_i \cap \mathcal{T}.
\end{aligned} \tag{5.10}
$$

Moreover, we have for $\sigma(s)$ lying in the neighbourhood of C_p, $\sigma(s) \notin \mathcal{T}$ that

$$
\begin{aligned}
&\langle D(\sigma(s))L_\delta(s)U, U_p(s)\rangle = \langle D(\sigma(s))L_\delta(s)U, U_p^*(s)\rangle = 0 \text{ for any } U \in \mathbb{V}^2,\\
&\langle D(\sigma(s))L_\delta(s)U_i(s), U_i^*(s)\rangle < 0 \quad \text{for } i \in I(d^0).
\end{aligned} \tag{5.11}
$$

See Appendix for the proof of the last assertion. Note that if $d^0 \in C_p \cap C_q$, $C_p \neq C_q$ and $p < q$, then $d^0 \notin \mathcal{T}$ and d^0 lies below $C_p \cap \mathcal{T}$ and above $C_q \cap \mathcal{T}$.

Lemma 5.1. (Cf. [6], Lemma 4.1.) *There exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ there is $\eta > 0$ such that the following assertions hold.*

(a) *Let $d^0 \in C_p \setminus \mathcal{T}$. Then for all $s \in (s_0 - \eta, s_0 + \eta)$, the eigenvalue $\mu_p(s)$ from (5.5) is simultaneously an algebraically simple eigenvalue of the operator $D^{-1}(\sigma(s))BA - L_\delta(s) - I$ with the corresponding eigenvector $U_p(s)$. It changes the sign as s crosses s_0. The other eigenvalues have constant signs and constant multiplicities on $(s_0 - \eta, s_0 + \eta)$.*

(b) *Let $d^0 \in C_p \cap \mathcal{T}$. Then for $s \in (s_0 - \eta, s_0]$, $\mu_p(s)$ is an eigenvalue of $D^{-1}(\sigma(s))BA - L_\delta(s) - I$ with the only normed eigenvectors $\pm\frac{U_p(s)}{\|U_p(s)\|}$. For $s \in (s_0 - \eta, s_0)$, $\mu_p(s)$ is positive and algebraically simple, $\mu_p(s_0) = 0$ is not algebraically simple. The sum of algebraic multiplicities of the other positive eigenvalues of $D^{-1}(\sigma(s))BA - L_\delta(s) - I$ is even for all $s \in (s_0 - \eta, s_0)$. For $s \in (s_0, s_0 + \eta)$, all eigenvalues of this operator are complex.*

In both cases (a), (b), $\mathrm{Ker}(D^{-1}(\sigma(s_0))BA - L_\delta(s_0) - I) = \mathrm{Lin}\{U_p(s_0)\}$ *and the number $\Theta(s_0 - \varepsilon) - \Theta(s_0 + \varepsilon)$ is odd for all $\varepsilon \in (0, \eta)$ where $\Theta(s)$ is the sum of algebraic multiplicities of all positive eigenvalues of the operator $D^{-1}(\sigma(s))BA - L_\delta(s) - I$.*

The proof, similar to that of [6], Lemma 4.1, is given in Appendix.

Let the parameter $\delta > 0$ be admissible for Lemma 5.1. For $\eta > 0$ as in Lemma 5.1 and such that

$$
s_0 + \eta < \tilde{s} \tag{5.12}
$$

where $\tilde{s}$ is from the assumption (4.15), we arrive at the following inclusions:

$$D(\sigma(s))U - BAU - N(U) + D(\sigma(s))L_\delta(s)U \in -M^\delta(U), \tag{5.13}$$

$$D(\sigma(s))U - BAU + D(\sigma(s))L_\delta(s)U \in -M_0^\delta(U) \tag{5.14}$$

and the corresponding linear equation

$$D(\sigma(s))U - BAU + D(\sigma(s))L_\delta(s)U = 0, \tag{5.15}$$

which is the aim of this section.

6. Properties of solutions to the penalty equation

We will consider the system of penalty equations

$$D(\sigma(s))U - BAU - \frac{\tau}{1+\tau}N(U) + D(\sigma(s))L_\delta(s)U + P_\tau^\delta(U) = 0 \tag{6.1}$$

with the norm condition

$$\|U\|^2 = \frac{\varrho\tau}{1+\tau}. \tag{6.2}$$

Throughout this section $\delta > 0$ is a fixed parameter admissible for Lemma 5.1, hence we can use $\operatorname{int} K^\delta \neq \emptyset$. Moreover, $\varrho > 0$ is fixed and $\tau \in [0, +\infty)$ is a penalty parameter. The penalty equation (6.1) is a linear equation (5.15) for $\tau = 0$ while for $\tau \to +\infty$ we get the inclusion (5.13) (for the proof see Lemma 6.2).

Lemma 6.1. *If* $[s_n, U_n, \tau_n] \in \mathbb{R} \times \mathbb{V}^2 \times \mathbb{R}^+$, $s_n \to s$, $U_n \rightharpoonup U$, $\tau_n \to \tau \in [0, +\infty]$,

$$D(\sigma(s_n))U_n - BAU_n - \frac{\tau_n}{1+\tau_n}N(U_n) + D(\sigma(s_n))L_\delta(s_n)U_n + P_{\tau_n}^\delta(U_n) = 0 \tag{6.3}$$

then $U_n \to U$. *If, moreover,* $\|U\| = 0$, $W_n = \frac{U_n}{\|U_n\|} \rightharpoonup W$ *then* $W_n \to W$.

Since the operator $L_\delta(s)$ is completely continuous, the proof is identical to that of [16], Remark 3.1.

Lemma 6.2. *(Cf.* [16], *Lemma* 3.2.*) Let* $[s_n, U_n, \tau_n] \in \mathbb{R} \times \mathbb{V}^2 \times \mathbb{R}^+$, $s_n \to s$, $U_n \to U$, $\tau_n \to +\infty$ *and let* (6.3) *hold. Then*

$$D(\sigma(s))U - BAU - N(U) + D(\sigma(s))L_\delta(s)U \in -M^\delta(U).$$

P r o o f. From the continuity of L_δ, Proposition 4.3 and (6.3) it follows that

$$\begin{aligned} -Z_n := -P_{\tau_n}^\delta(U_n) &= D(\sigma(s_n))U_n - BAU_n - \frac{\tau_n}{1+\tau_n}N(U_n) + D(\sigma(s_n))L_\delta(s_n)U_n \\ &\to D(\sigma(s))U - BAU - N(U) + D(\sigma(s))L_\delta(s)U = -Z \in -M^\delta(U). \end{aligned}$$

□

Lemma 6.3. *(Cf.* [17]*, Lemma* 1.1.*) Any bifurcation point* $s \in \mathbb{R}$ *of* (5.13) *is simultaneously a critical point of* (5.14).

P r o o f. If s is a bifurcation point of (5.13) then there exist $s_n \to s$ and a sequence $\{U_n\}$ such that $\|U_n\| \to 0,\ \|U_n\| \neq 0,\ W_n = \frac{U_n}{\|U_n\|} \rightharpoonup W$ and

$$D(\sigma(s_n))W_n - BAW_n - \frac{N(U_n)}{\|U_n\|} + D(\sigma(s_n))L_\delta(s_n)W_n \in -\frac{M^\delta(U_n)}{\|U_n\|}. \tag{6.4}$$

Using the compactness of A and L_δ, the assumption (2.5) and Proposition 4.1 we obtain $W_n \to W$ and

$$D(\sigma(s))W - BAW + D(\sigma(s))L_\delta(s)W \in -M_0^\delta(W). \tag{6.5}$$

□

Lemma 6.4. *If* $\operatorname{Ker}(D(\sigma(s_0)) - B^*A + L_\delta(s_0)) \cap \operatorname{int} K^\delta \neq \emptyset$ *then* $\{U \in \mathbb{V}^2;\ D(\sigma(s_0))U - BAU + L_\delta(s_0)U \in -M_0^\delta(U)\} = \operatorname{Ker}(D(\sigma(s_0)) - BA + L_\delta(s_0)) \cap K^\delta$.

The proof is identical to the proof of [16], Lemma 3.3, if we put $U_0 = -U_p(s_0)$.

Lemma 6.5. *If* $\sigma_1(s) > \frac{b_{11}}{\kappa_1}$ *(i.e.* $\sigma(s) \in Z_0$ *in the notation of* [7]*) then the only solution of* (5.14) *is trivial. (The line* $d_1 = \frac{b_{11}}{\kappa_1}$ *is the asymptote to* C_1*—see Fig.* 1.*)*

Proof is done in a similar way as in [7], proof of Theorem 2.1. Note that the condition (M0) in the notation of [7] holds for any $\delta > 0$ small enough due to the assumption (4.5). Moreover, it follows from (5.12) that $L_\delta(s) \equiv 0$ for $s > \tilde{s}$.

Lemma 6.6. *If* $d = [d_1, d_2] \in \mathbb{R}^2_+$, $d_1 > \frac{b_{11}}{\kappa_1}$ *and* $\tau \in [0, +\infty)$ *then the equation*

$$D(d)U - BAU + P^\delta_{0,\tau}(U) = 0$$

has only the trivial solution.

The proof is identical to that of Lemma 3.4 in [7].

Lemma 6.7. *Let* ζ_0 *be from the assumption* (4.15)*. For any* $\zeta \in (0, \zeta_0)$ *there exists* $\varrho_0 > 0$ *such that there is no nontrivial solution* U *of* (6.1) *with* $s = \tilde{s} + \zeta$, $\tau \in [0, +\infty)$ *and* $\|U\|^2 < \varrho_0$.

Proof follows from Lemmas 6.5 and 6.6 and can be done in the same way as that of [8], Lemma 3.9.

Lemma 6.8. *If* $[s_n, U_n, \tau_n] \to [s_0, 0, 0]$, $W_n = \frac{U_n}{\|U_n\|} \to \frac{U_p}{\|U_p\|}$ *and* (6.3) *holds then*

$$\liminf_{n\to+\infty} \frac{s_n - s_0}{\tau_n} > 0.$$

Proof is done in the same way as that of [16], Lemma 3.6 if we put $U_0 = -\frac{U_p(s_0)}{\|U_p(s_0)\|}$.

O b s e r v a t i o n 6.1. (Cf. [16], Remark 3.8.) The assumption (4.9) implies: If $[U_n, \tau_n] \in \mathbb{V}^2 \times \mathbb{R}^+$ and $\frac{P^\delta_{\tau_n}(U_n)}{\|U_n\|} \rightharpoonup F$ then

$$\langle F, W\rangle = \lim_{n\to+\infty} \frac{\langle P^\delta_{\tau_n}(U_n), W\rangle}{\|U_n\|} \leqslant 0 \text{ for any } W \in K^\delta. \tag{6.6}$$

Moreover, let $F \neq 0$ and $V \in \mathbb{V}^2$, $W \in \operatorname{int} K^\delta$ be such that $\langle F, V\rangle > 0$, $\langle F, W\rangle = 0$. Then $\langle F, W + tV\rangle > 0$ for $t > 0$ and simultaneously $W + tV \in K^\delta$ for $t > 0$ small enough. Therefore $\langle F, W\rangle < 0$ for all $W \in \operatorname{int} K^\delta$ and any $F \neq 0$ satisfying (6.6).

Lemma 6.9. *There exists* $\varrho_0 > 0$ *such that if* $\varrho \in (0, \varrho_0)$, s_n, U_n, τ_n *satisfy* (6.1), (6.2), $U_n \notin K^\delta$, $[s_n, U_n, \tau_n] \to [s_0, U, \tau]$, $W_n = \frac{U_n}{\|U_n\|} \to W$, $s_n \geqslant s_0$ *and* $\tau \in [0, +\infty]$ *then* $W \notin K^\delta$.

Proof is similar to that of Lemma 3.7 in [16]. For the sake of completeness, it can be found in Appendix.

Lemma 6.10. *There exists* $\varrho_0 > 0$ *such that if* s, U, τ *satisfy* (6.1), $U \notin K^\delta$, $\|U\| < \varrho_0$ *then* $s \neq s_0$.

Proof can be done in the same way as the proof of [16], Lemma 3.9 where we take $U_0 = -\frac{U_p(s_0)}{\|U_p(s_0)\|}$ again, which is the only normed solution to (5.15) for $s = s_0$ belonging to K^δ.

Lemma 6.11. *There exists* $\varrho_0 > 0$ *such that if* s, U, τ *satisfy* (6.1), $s > s_0$, $0 \neq \|U\| < \varrho_0$ *then* $U \notin \partial K^\delta$.

The proof is identical to that of [16], Lemma 3.10. Again, we take $U_0 = -\frac{U_p(s_0)}{\|U_p(s_0)\|}$ and use the fact that it is the only normed solution to (5.15) for $s = s_0$ belonging to K^δ.

7. Proof of the main result

Let $\delta > 0$ be fixed and such that Lemma 5.1 is satisfied. We rewrite the system (6.1) into the form

$$U - T(s)U + H_\tau(s,U) = 0, \tag{7.1}$$

where

$$\begin{aligned} T(s)U &= D^{-1}(\sigma(s))BAU - \delta L_\delta(s)U, \\ H_\tau(s,U) &= D^{-1}(\sigma(s))\Big[-\frac{\tau}{1+\tau}N(U) + P_\tau^\delta(U)\Big]. \end{aligned} \tag{7.2}$$

If we define $P_\tau^\delta(U) = P_{-\tau}^\delta(U)$ for $\tau < 0$ then

$$\begin{cases} \text{for any } s \in \mathbb{R},\ T(s)\colon\ \mathbb{V}^2 \to \mathbb{V}^2 \text{ is linear completely continuous,} \\ \text{the mapping } s \mapsto T(s) \text{ of } \mathbb{R} \text{ into the space of linear continuous} \\ \text{mappings in } \mathbb{V}^2 \text{ (equipped with the operator norm) is continuous,} \\ \text{the mapping } Q\colon\ \mathbb{R}\times\mathbb{V}^2\times\mathbb{R} \to \mathbb{V}^2 \text{ defined by} \\ Q(s,U,\tau) = T(s)U - H_\tau(s,U) \text{ is completely continuous;} \end{cases} \tag{7.3}$$

$$\begin{cases} \lim\limits_{\|U\|+|\tau|\to 0} \frac{\|H_\tau(s,U)\|}{\|U\|+|\tau|} = 0 \\ \text{uniformly with respect to } s \in [s_0-\gamma, s_0+\gamma],\ \gamma \in (0,+\infty) \end{cases} \tag{7.4}$$

are satisfied under the assumptions from Sections 1 and 4.

The proof of Theorem 4.1 is based on the following theorem (where by a *critical point of* T we mean the parameter $s \in \mathbb{R}$ such that there exists a nontrivial solution of $U - T(s)U = 0$ and by $\Theta_T(s)$ we denote the sum of algebraic multiplicities of all positive eigenvalues of the operator $T(s) - I$):

Theorem 7.1. *Let $\mathcal{K} \neq \mathbb{V}^2$ be a closed convex cone in $\mathbb{V}^2$ with its vertex at the origin and let the mappings T, H satisfy* (7.3) *and* (7.4). *Assume that s_0 is the greatest critical point of T, s_0 is an isolated critical point of T,* $\operatorname{Ker}(I - T(s_0)) = \operatorname{Lin}\{U_0\}$, $-U_0 \in \operatorname{int}\mathcal{K}$ *and*

$$\Theta_T(s_0+\xi) - \Theta_T(s_0-\xi) \text{ is odd for any } \xi \in (0,\xi_0) \tag{7.5}$$

with some $\xi_0 > 0$. *Let the following assumptions hold for any* $\varrho \in (0, \varrho_0)$, $\varrho_0 > 0$ *small,* $[s, U, \tau]$ *and* $[s_n, U_n, \tau_n]$ *satisfying* (7.1), (6.2), $\tau \in [0, +\infty)$:

(7.6) *there exists* $C = C(\varrho_0) > 0$ *such that* $s \leqslant C$;

(7.7) $$\Big(U_n \notin \mathcal{K},\ \tau_n > 0,\ [s_n, U_n, \tau_n] \to [s_0, 0, 0], \frac{U_n}{\|U_n\|} \to U_0\Big) \Longrightarrow \underset{n_0}{\exists}\ \underset{n \geqslant n_0}{\forall}\ s_n > s_0;$$

(7.8) $$\Big(U_n \notin \mathcal{K},\ \tau_n > 0,\ [s_n, U_n, \tau_n] \to [s_0, U, \tau], \frac{U_n}{\|U_n\|} \to W \in \mathcal{K}\Big) \Longrightarrow \underset{n_0}{\exists}\ \underset{n \geqslant n_0}{\forall}\ s_n < s_0;$$

(7.9) *if* $U \notin \mathcal{K}$ *then* $s \neq s_0$;

(7.10) *if* $s > s_0, \|U\| \neq 0$ *then* $U \notin \partial\mathcal{K}$.

Then for any $\varrho \in (0, \varrho_0)$ *there exists a closed connected set* C_ϱ *in* $\mathbb{R} \times \mathbb{V}^2 \times \mathbb{R}$ *containing* $[s_0, 0, 0]$ *such that*

(i) *if* $[s, U, \tau] \in C_\varrho$ *is such that* $[s_0, 0, 0] \neq [s, U, \tau]$ *then* (7.1), (6.2) *are fulfilled,* $s > s_0$, $U \notin \mathcal{K}$;

(ii) *for any* $\tau > 0$ *there exists at least one couple* $[s, U]$ *such that* $[s, U, \tau] \in C_\varrho$.

Proof of this theorem is based on Dancer's global bifurcation theorem ([1], Theorem 2) and on a general continuation theorem proved by Kučera in [14]. The main ideas of the proof are given in [16], proof of Theorem 4.2. Note that the role of the sets C_ϱ^+ and C_ϱ^- is reversed here in comparison with [16].

P r o o f o f T h e o r e m 4.1. We will prove Theorem 4.1 in several steps: In Step 1 we will show for fixed $\delta > 0$ and $\varrho > 0$ small the existence of a solution $\big[s_\varrho^\delta, U_\varrho^\delta\big]$ of (5.13). In Step 2 we obtain by a limiting process $\varrho \to 0_+$ (still with $\delta > 0$ fixed) a bifurcation point $s_I^\delta \in [s_0, \tilde{s} + \zeta_0]$ of (5.13). Finally, we will show in Step 3 the existence of a bifurcation point $s_I \in [s_0, \tilde{s}]$ of (4.17) by a limiting process $\delta \to 0_+$.

S t e p 1. For a fixed $\delta > 0$ we show that the assumptions of Theorem 7.1 are fulfilled with the operators from (7.2), $U_0 = \frac{U_p}{\|U_p\|}$ and with $\mathcal{K} = K^\delta$ from Notation 4.1: It follows from Remark 4.1 and the assumption (4.15) that s_0 from the assumptions of Theorem 4.1 is the greatest critical point of T and Lemma 5.1 gives $\operatorname{Ker}(I - T(s_0)) = \operatorname{Lin}\{U_p\}$, $-U_p \in \operatorname{int} K^\delta$. The assumption (7.5) follows from Lemma 5.1, the assumptions (7.6)–(7.10) follow from Lemmas 6.7–6.11. Hence it follows from Theorem 7.1 that for any $\varrho \in (0, \varrho_0)$ fixed there are $[s_n, U_n, \tau_n]$ satisfying (7.1) and (6.2) (i.e. (6.1) and (6.2)), $U_n \notin K^\delta$, $s_n \to s_\varrho^\delta \geqslant s_0$, $\tau_n \to +\infty$. We can assume $U_n \rightharpoonup U_\varrho^\delta$ and Lemmas 6.1, 6.2 imply that $U_n \to U_\varrho^\delta$ and U_ϱ^δ satisfies

(7.11) $$D(\sigma(s_\varrho^\delta))U - BAU - N(U) + D(\sigma(s_\varrho^\delta))L_\delta(s_\varrho^\delta)U \in -M^\delta(U).$$

Moreover, $U_\varrho^\delta \notin \operatorname{int} K^\delta$ and the limiting process in (6.2) implies $\|U_\varrho^\delta\|^2 = \varrho$. Further, Lemma 6.7 gives $s_\varrho^\delta \in [s_0, \tilde{s} + \zeta_0]$.

S t e p 2. We can construct s_ϱ^δ, U_ϱ^δ for any $\varrho \in (0, \varrho_0)$ and obtain by a limiting process $\varrho \to 0_+$ a bifurcation point $s_I^\delta \in [s_0, \tilde{s} + \zeta_0]$ of (5.13). Lemma 6.3 yields that s_I^δ is a critical point of (5.14). If $s_I^\delta = s_0$ for some $\delta > 0$ then Lemma 6.4 would imply $U_I^\delta \in K^\delta$ and

$$D(\sigma(s_0))U_I^\delta - BAU_I^\delta + D(\sigma(s_0))L_\delta(s_0)U_I^\delta = 0.$$

Therefore $U_I^\delta = -\frac{U_p}{\|U_p\|} \in \operatorname{int} K^\delta$ would hold. On the other hand we had $U_{\varrho_n}^\delta \notin K^\delta$ by Theorem 7.1 and the limiting process $\frac{U_{\varrho_n}^\delta}{\|U_{\varrho_n}^\delta\|} \to U_I^\delta$ gives a contradiction. This implies $s_I^\delta > s_0$ for any $\delta > 0$ small.

If $\operatorname{int} K \neq \emptyset$ and $\dim E_B(\sigma(s_0)) = 1$ then the assertion of Theorem 4.1 is proved, because we can take $\Phi^\delta(v) = v$ for any δ and we have $s_I = s_I^\delta \in (s_0, \tilde{s}]$.

S t e p 3. By the limiting process in (7.11) with $\delta_n \to 0_+$, $s_\varrho^{\delta_n} \to s_\varrho$, $U_\varrho^{\delta_n} \rightharpoonup U_\varrho$ (after choosing subsequences) we obtain by using (4.11) that $U_\varrho^{\delta_n} \to U_\varrho$, $\|U_\varrho\|^2 = \varrho$ and $[s_\varrho, U_\varrho]$ satisfies (4.17). This process can be done for any $\varrho \in (0, \varrho_0)$. Using the fact that ζ_0 can be chosen arbitrarily small we obtain by this procedure a bifurcation point $s_I \in [s_0, \tilde{s}]$ of (4.17).

R e m a r k 7.1. Let us notice that Steps 1 and 2 can be done in the same way as in [16]. Step 3, where δ is not fixed, is new in comparison to [16]. The fact $s_I > s_0$ can be proved under the additional assumptions from Remark 4.2—see the end of Appendix.

8. Appendix

P r o o f o f P r o p o s i t i o n 5.1. If $\sigma_1'(s_0) = 0$ then $\sigma_2'(s_0) < 0$ due to the orientation of the curve $\sigma(s)$ and there is nothing to prove. If $\sigma_1'(s_0) \neq 0$ then we can consider a curve $\sigma(s) = [\sigma_1(s), \sigma_2(s)]$ as $\sigma_2(s) = \tilde{\sigma}(\sigma_1(s))$ on $(s_0 - \eta, s_0 + \eta)$ with some $\eta > 0$ small and the hyperbola C_p as a curve

$$d_2 = h_p(d_1) = \frac{b_{12}b_{21}/\kappa_p^2}{d_1 - b_{11}/\kappa_p} + \frac{b_{22}}{\kappa_p}.$$

Differenting h_p with respect to d_1, we obtain at the point d_1^0 that

$$\frac{\mathrm{d}h_p(d_1^0)}{\mathrm{d}d_1} = -\frac{b_{12}b_{21}/\kappa_p^2}{(d_1^0 - b_{11}/\kappa_p)^2} = -\frac{b_{12}b_{21}}{(d_1^0\kappa_p - b_{11})^2}.$$

It follows that

$$\frac{\mathrm{d}h_p(d_1^0)}{\mathrm{d}d_1} = -\frac{(d_2^0\kappa_p - b_{22})^2}{b_{12}b_{21}}$$

by using (3.4) for d_1^0, d_2^0. Differenting $\tilde{\sigma}$ with respect to σ_1, we obtain $\frac{\mathrm{d}\tilde{\sigma}(\sigma_1(s))}{\mathrm{d}\sigma_1} = \frac{\sigma_2'(s)}{\sigma_1'(s)}$ for any $s \in (s_0 - \eta, s_0 + \eta)$. If the curve $\sigma(s)$ intersects C_p at the point $d^0 = \sigma(s_0)$ transversally then either $\frac{\sigma_2'(s_0)}{\sigma_1'(s_0)} = \frac{\mathrm{d}\tilde{\sigma}(d_1^0)}{\mathrm{d}d_1} < -\frac{(\sigma_2(s_0)\kappa_p - b_{22})^2}{b_{12}b_{21}}$ in the case $\sigma_1'(s_0) > 0$ or $\frac{\sigma_2'(s_0)}{\sigma_1'(s_0)} = \frac{\mathrm{d}\tilde{\sigma}(d_1^0)}{\mathrm{d}d_1} > -\frac{(\sigma_2(s_0)\kappa_p - b_{22})^2}{b_{12}b_{21}}$ in the case $\sigma_1'(s_0) < 0$. In both cases we obtain

$$\sigma_2'(s_0) < -\frac{(\sigma_2(s_0)\kappa_p - b_{22})^2}{b_{12}b_{21}}\sigma_1'(s_0).$$

Our assertion follows.

D e t a i l e d v e r s i o n o f O b s e r v a t i o n 5.1. (Cf. Section 2, [6], Section 4, [4], Section 2.) We can write (5.1) as a system

$$u - \frac{b_{11}}{d_1}Au - \frac{b_{12}}{d_1}Av = -\mu u,$$
$$v - \frac{b_{21}}{d_2}Au - \frac{b_{22}}{d_2}Av = -\mu v$$

and the elements $U = [u, v] \in \mathbb{V}^2$ in the form

$$u = \sum_{j=1}^{\infty}\langle u, e_j\rangle e_j, \qquad v = \sum_{j=1}^{\infty}\langle v, e_j\rangle e_j. \tag{8.1}$$

Using these expansions and the fact that κ_i is a characteristic value of A, multiplying the first equation by $d_1\kappa_i e_i$ and the second by $d_2\kappa_i e_i$, we obtain

$$\langle u, e_i\rangle(d_1\kappa_i - b_{11} + \mu d_1\kappa_i) - \langle v, e_i\rangle b_{12} = 0,$$
$$\langle u, e_i\rangle b_{21} - \langle v, e_i\rangle(d_2\kappa_i - b_{22} + \mu d_2\kappa_i) = 0$$

for $i = 1, 2, \ldots$. A couple $\langle u, e_i\rangle,\ \langle v, e_i\rangle$ can be nontrivial for some i if and only if

$$(d_1\kappa_i - b_{11} + \mu d_1\kappa_i)(d_2\kappa_i - b_{22} + \mu d_2\kappa_i) - b_{12}b_{21} = 0. \tag{8.2}$$

Hence, μ is an eigenvalue of (5.1) if and only if μ is a root of

$$\mu^2 d_1 d_2\kappa_i^2 - \beta_i(d)\kappa_i\mu + \gamma_i(d) = 0$$
$$\text{with } \beta_i(d) = d_1 b_{22} + d_2 b_{11} - 2d_1 d_2\kappa_i, \tag{8.3}$$
$$\gamma_i(d) = (d_1\kappa_i - b_{11})(d_2\kappa_i - b_{22}) - b_{12}b_{21}$$

for at least one i. Now, the coefficient $\beta_i(d)$ can be positive, negative or zero. (Note that the corresponding coefficient in (3.3) in Section 3 was negative by (SIGN) for any d in a neighbourhood of C_i.) The term $\gamma_i(d)$ is negative or positive for d lying to the left or to the right, respectively, from C_i. It is easy to simplify the term

$$\omega_i(d) := \beta_i^2(d) - 4d_1d_2\gamma_i(d) = d_1^2b_{22}^2 + d_2^2b_{11}^2 - 2d_1d_2b_{11}b_{22} + 4d_1d_2b_{12}b_{21}$$

and see that it does not depend on i. Therefore we will write only $\omega(d)$ instead of $\omega_i(d)$. The set $\{d \in \mathbb{R}^2_+ ;\ \omega(d) = 0\}$ is the set of all d satisfying

$$d_1^2b_{22}^2 + d_2^2b_{11}^2 - 2d_1d_2b_{11}b_{22} + 4d_1d_2b_{12}b_{21} = 0.$$

Solving this equation for d_2 with d_1 as a parameter we obtain

$$d_2^{(r)} = \frac{d_1}{b_{11}^2}\big[-b_{12}b_{21} + \det B \pm 2\sqrt{-b_{12}b_{21}}\sqrt{\det B}\big],\ r = 1, 2.$$

Thus the set $\{d \in \mathbb{R}^2_+ ;\ \omega(d) = 0\}$ is a couple of half-lines, one of them is a common tangent $\mathcal{T}$ to all hyperbolas C_j, $j = 1, 2, \ldots$ (see also [20] and Figures 1 and 2). Further, the set $\widetilde{C}_i = \{d \in \mathbb{R}^2_+ ;\ \beta_i(d) = 0\}$ is a hyperbola with the property $\widetilde{C}_i \cap C_i = \mathcal{T} \cap C_i$.

The roots μ of (8.3) are

$$\mu_i^{(r)}(d) := \frac{\beta_i(d) \pm \sqrt{\omega(d)}}{2d_1d_2\kappa_i},\ r = 1, 2.$$

If $\gamma_i(d) < 0$ then $\omega(d) > 0$ and $|\beta_i(d)| < \sqrt{\omega(d)}$. Therefore there are two different real roots $\mu_i^{(1)}(d)$, $\mu_i^{(2)}(d)$, one is negative and the other one is positive. If $\gamma_i(d) > 0$ then $\omega(d)$ can be either negative (and we have a couple of complex roots) or nonnegative but $|\beta_i(d)| > \sqrt{\omega(d)}$ (and we have two real roots, both having the same sign). The possibilities for the signs of $\mu_i^{(1)}(d)$, $\mu_i^{(2)}(d)$ are the following—see Fig. 2:

domain:	$\beta_i(d)$	$\gamma_i(d)$	$\omega(d)$		relation between eigenvalues:
$d \in \mathcal{D}_1$	$+$	$-$	$+$	$\lvert\beta_i(d)\rvert < \sqrt{\omega(d)}$	$\mu_i^{(1)} > 0,\ \mu_i^{(2)} < 0,\ \mu_i^{(1)} \neq \mu_i^{(2)}$
$d \in \widetilde{C}_i$	0	$-$	$+$	$\lvert\beta_i(d)\rvert < \sqrt{\omega(d)}$	$\mu_i^{(1)} > 0,\ \mu_i^{(2)} < 0,\ \mu_i^{(1)} = -\mu_i^{(2)}$
$d \in \mathcal{D}_2$	$-$	$-$	$+$	$\lvert\beta_i(d)\rvert < \sqrt{\omega(d)}$	$\mu_i^{(1)} > 0,\ \mu_i^{(2)} < 0,\ \mu_i^{(1)} \neq \mu_i^{(2)}$
$d \in C_i$	$-$	0	$+$	$\lvert\beta_i(d)\rvert = \sqrt{\omega(d)}$	$\mu_i^{(1)} = 0,\ \mu_i^{(2)} < 0,\ \mu_i^{(1)} \neq \mu_i^{(2)}$
$d \in \mathcal{D}_3$	$-$	$+$	$+$	$\lvert\beta_i(d)\rvert > \sqrt{\omega(d)}$	$\mu_i^{(1)} < 0,\ \mu_i^{(2)} < 0,\ \mu_i^{(1)} \neq \mu_i^{(2)}$
$d \in \mathcal{T}$	$-$	$+$	0	$\lvert\beta_i(d)\rvert > \sqrt{\omega(d)}$	$\mu_i^{(1)} = \mu_i^{(2)} < 0$
$d \in \mathcal{D}_4$	$-$	$+$	$-$		$\mu_i^{(1)} \neq \mu_i^{(2)},\ \mu_i^{(r)} \in \mathbb{C} \setminus \mathbb{R},\ \operatorname{Re}\mu_i^{(r)} < 0$
$d \in \widetilde{C}_i$	0	$+$	$-$		$\mu_i^{(1)} = -\mu_i^{(2)} \in \mathrm{i}\mathbb{R},\ \operatorname{Re}\mu_i^{(r)} = 0$
$d \in \mathcal{D}_5$	$+$	$+$	$-$		$\mu_i^{(1)} \neq \mu_i^{(2)},\ \mu_i^{(r)} \in \mathbb{C} \setminus \mathbb{R},\ \operatorname{Re}\mu_i^{(r)} > 0$
$d \in \mathcal{T}$	$+$	$+$	0	$\lvert\beta_i(d)\rvert > \sqrt{\omega(d)}$	$\mu_i^{(1)} = \mu_i^{(2)} > 0$
$d \in \mathcal{D}_6$	$+$	$+$	$+$	$\lvert\beta_i(d)\rvert > \sqrt{\omega(d)}$	$\mu_i^{(1)} > 0,\ \mu_i^{(2)} > 0,\ \mu_i^{(1)} \neq \mu_i^{(2)}$
$d \in C_i$	$+$	0	$+$	$\lvert\beta_i(d)\rvert = \sqrt{\omega(d)}$	$\mu_i^{(1)} > 0,\ \mu_i^{(2)} = 0,\ \mu_i^{(1)} \neq \mu_i^{(2)}$
$d \in C_i \cap \widetilde{C}_i \cap \mathcal{T}$	0	0	0	$\lvert\beta_i(d)\rvert = \sqrt{\omega(d)}$	$\mu_i^{(1)} = \mu_i^{(2)} = 0.$

These calculations lead to the conclusion that for d lying to the left there is one positive root of (8.3) and for d lying to the right from C_i, either none or both roots of (8.3) are positive.

P r o o f o f t h e s e c o n d p a r t o f (5.11). Using (5.9), (5.6), (8.2), (SIGN) and (5.3) we obtain

$$\begin{aligned}
\langle D(\sigma(s))L_\delta(s)U_i(s), U_i^*(s)\rangle &= \delta\chi(s)\cdot\sum_{j\in I(d^0)}\nu_j(d^0)\cdot\langle D(\sigma(s))U_j(s), U_i^*(s)\rangle\\
&= \delta\chi(s)\nu_i(d^0)\Big[\frac{\sigma_1(s)(\sigma_2(s)\kappa_i - b_{22} + \mu_i(s)\sigma_2(s)\kappa_i)^2}{b_{12}b_{21}} + \sigma_2(s)\Big]\\
&= -\delta\chi(s)\nu_i(d^0)\frac{\sigma_2(s)\kappa_i - b_{22} + \mu_i(s)\sigma_2(s)\kappa_i}{b_{12}b_{21}}\\
&\qquad\times[\sigma_2(s)b_{11} + \sigma_1(s)b_{22} - 2\sigma_1(s)\sigma_2(s)\kappa_i - 2\sigma_1(s)\sigma_2(s)\kappa_i\mu_i(s)]\\
&= \delta\chi(s)\cdot\frac{[\sigma_2(s)\kappa_i - b_{22} + \mu_i(s)\sigma_2(s)\kappa_i]\sqrt{\omega(\sigma(s))}}{b_{12}b_{21}} < 0 \quad \text{for } i \in I(d^0).
\end{aligned}$$

P r o o f o f L e m m a 5.1. Analogously as in Observation 5.1 we obtain that μ is an eigenvalue of the problem

$$D^{-1}(\sigma(s))BAU - L_\delta(s)U - U = \mu U$$

if and only if μ is a root of the quadratic equation

$$\mu^2 - \beta_i^\delta(s)\mu + \gamma_i^\delta(s) = 0 \tag{8.4}$$

with coefficients $\beta_i^\delta(s)$, $\gamma_i^\delta(s)$ depending continuously on s and δ. For the sake of efficiency, the structure of the proof differs from the structure of the lemma. We shall distinguish the following cases:

A1. Let $i \notin I(d^0)$. It follows from (5.10) that $\mu_i^{(r)}(\sigma(s))$ and $U_i^{(r)}(\sigma(s))$, $r = 1, 2$, from Observation 5.1 and Notation 5.1 are simultaneously eigenvalues and eigenvectors of $D^{-1}(\sigma(s))BA - L_\delta(s) - I$ and (8.4) is equivalent to (5.2) for any $s \in \mathbb{R}$. In particular, this means by the definitions of $\mu_p(s)$, $U_p(s)$ that $\mu_p(s)$ and $U_p(s)$ is an eigenvalue and an eigenvector of $D^{-1}(\sigma(s))BA - L_\delta(s) - I$ for any $s \in (s_0 - \eta, s_0 + \eta)$ or $s \in (s_0 - \eta, s_0]$ in the case $d^0 \in C_p \setminus \mathcal{T}$ or $d^0 \in C_p \cap \mathcal{T}$, respectively.

A2. If $i \notin I(d^0) \cup \{p\}$ then d^0 and also $\sigma(s)$ for any $s \in (s_0 - \eta, s_0 + \eta)$ lie to the right from C_i. (Recall that $d^0 \in C$.) It follows from Observation 5.1 that if $d^0 \in C_p \setminus \mathcal{T}$, $i \notin I(d^0) \cup \{p\}$ then the sign of both $\mu_i^{(1)}(\sigma(s)) \neq \mu_i^{(2)}(\sigma(s))$ is constant on $(s_0 - \eta, s_0 + \eta)$ (more precisely, $\mu_i^{(1)}(\sigma(s)) \neq \mu_i^{(2)}(\sigma(s))$ are both negative or positive on $(s_0 - \eta, s_0 + \eta)$ for $C_i \cap \mathcal{T} \preceq d^0$ or $d^0 \preceq C_i \cap \mathcal{T}$, respectively). If $d^0 \in C_p \cap \mathcal{T}$, $i \notin I(d^0) \cup \{p\}$ then $\mu_i^{(1)}(\sigma(s)) \neq \mu_i^{(2)}(\sigma(s))$ are both negative or positive on $(s_0 - \eta, s_0)$ for $C_i \cap \mathcal{T} \preceq d^0$ or $d^0 \preceq C_i \cap \mathcal{T}$, respectively, and complex on $(s_0, s_0 + \eta)$.

A3. For $i = p$, $\mu_p(s)$ changes its sign at s_0 and the sign of the other root is constant on $(s_0 - \eta, s_0 + \eta)$ in the case $d^0 \in C_p \setminus \mathcal{T}$. More precisely, if $C_i \cap \mathcal{T} \preceq d^0$ then $\mu_p(s) = \mu_p^{(1)}(\sigma(s)) > 0$ on $(s_0 - \eta, s_0)$, $\mu_p(s) = \mu_p^{(1)}(\sigma(s)) < 0$ on $(s_0, s_0 + \eta)$, $\mu_p^{(2)}(\sigma(s)) < 0$ on $(s_0 - \eta, s_0 + \eta)$, and if $d^0 \preceq C_i \cap \mathcal{T}$ then $\mu_p(s) = \mu_p^{(2)}(\sigma(s)) < 0$ on $(s_0 - \eta, s_0)$, $\mu_p(s) = \mu_p^{(2)}(\sigma(s)) > 0$ on $(s_0, s_0 + \eta)$, $\mu_p^{(1)}(\sigma(s)) > 0$ on $(s_0 - \eta, s_0 + \eta)$. In the case $d^0 \in C_p \cap \mathcal{T}$ we have $\mu_p(s) > 0$ and the other root is negative on $(s_0 - \eta, s_0)$, both the roots being complex on $(s_0, s_0 + \eta)$.

B1. Let $i \in I(d^0)$. Let $d^0 \in C_p$, $d^0 \notin C_q$ for $C_q \neq C_p$. Then $i \in I_p(d^0)$, $\mu_i(s) = \mu_p(s)$ and $I_q(d^0) = \emptyset$. Let $C_i \cap \mathcal{T} \preceq d^0$. Notation 5.1, 5.2 and (5.10) yield that $\mu_i(s) - \delta\chi(s)$ is an eigenvalue of $D^{-1}(\sigma(s))BA - L_\delta(s) - I$ and one of the roots of (8.4). It follows from Notation 5.1 and Observation 5.1 that we can choose $\delta_0 > 0$ and $\eta > 0$ such that $\mu_i(s) - \delta\chi(s) < 0$ on $(s_0 - \eta, s_0 + \eta)$ for any $\delta \in (0, \delta_0)$. The roots of (8.4) depend continuously on $s \in \mathbb{R}$, $\delta \geqslant 0$ and therefore the choice of $\delta_0 > 0$ and $\eta > 0$ can be such that the other root is negative on $(s_0 - \eta, s_0 + \eta)$ for any $\delta \in (0, \delta_0)$. Let $d^0 \preceq C_i \cap \mathcal{T}$. Similarly as above, Notation 5.1, 5.2 and (5.10) yield that $\mu_i(s) + \delta\chi(s)$ is an eigenvalue of $D^{-1}(\sigma(s))BA - L_\delta(s) - I$ and one of the roots of (8.4). It follows from Notation 5.1 and Observation 5.1 that we can choose $\delta_0 > 0$ and $\eta > 0$ such that $\mu_i(s) + \delta\chi(s) > 0$ on $(s_0 - \eta, s_0 + \eta)$ for any $\delta \in (0, \delta_0)$ and that

the other root is also positive on $(s_0-\eta, s_0+\eta)$. Let $d^0 \in C_i \cap \mathcal{T}$. Notation 5.1, 5.2 and (5.10) yield that $\mu_i(s) - \delta\chi(s)$ is an eigenvalue of $D^{-1}(\sigma(s))BA - L_\delta(s) - I$ and one of the roots of (8.4) again. It follows from Notation 5.1 and Observation 5.1 that we can choose $\delta_0 > 0$ and $\eta > 0$ such that both $\mu_i(s) - \delta\chi(s)$ and the other root of (8.4) are negative on $(s_0-\eta, s_0]$ and complex on $(s_0, s_0+\eta)$ for any $\delta \in (0, \delta_0)$. (See Observation 5.1.)

B2. Let $i \in I(d^0)$. Let $d^0 \in C_p \cap C_q$, $C_q \neq C_p$. Let $p > q$. Then $C_i \cap \mathcal{T} \preceq d^0 \preceq C_j \cap \mathcal{T}$ for $i \in I_p(d^0) \cup \{p\}$, $j \in I_q(d^0)$. Similarly as above, Notation 5.1, 5.2 and (5.10) yield that $\mu_i(s) - \delta\chi(s) = \mu_p(s) - \delta\chi(s)$ or $\mu_i(s) + \delta\chi(s) = \mu_q(s) + \delta\chi(s)$ for $i \in I_p(d^0)$ or $i \in I_q(d^0)$, respectively, is an eigenvalue of $D^{-1}(\sigma(s))BA - L_\delta(s) - I$ and one of the roots of (8.4). (Let us note that $\mu_i(s_0) = 0$ for any $i \in I(d^0)$.) It follows from Notation 5.1 and Observation 5.1 that we can choose $\delta_0 > 0$ and $\eta > 0$ such that $\mu_i(s) - \delta\chi(s) < 0$ or $\mu_i(s) + \delta\chi(s) > 0$ for $i \in I_p(d^0)$ or $i \in I_q(d^0)$, respectively, on $(s_0-\eta, s_0+\eta)$, $\delta \in (0, \delta_0)$. The roots of (8.4) depend continuously on $s \in \mathbb{R}$, $\delta \geqslant 0$ and therefore the choice of $\delta_0 > 0$ and $\eta > 0$ can be such that the other root is negative or positive for $i \in I_p(d^0)$ or $i \in I_q(d^0)$, respectively, on $(s_0-\eta, s_0+\eta)$, $\delta \in (0, \delta_0)$.

B3. Let $i \in I(d^0)$. Let $d^0 \in C_p \cap C_q$, $C_q \neq C_p$. Let $p < q$. Then $C_i \cap \mathcal{T} \preceq d^0 \preceq C_j \cap \mathcal{T}$ for $j \in I_p(d^0) \cup \{p\}$, $i \in I_q(d^0)$. Similarly as above, Notation 5.1, 5.2 and (5.10) yield that $\mu_i(s) + \delta\chi(s) = \mu_p(s) + \delta\chi(s)$ or $\mu_i(s) - \delta\chi(s) = \mu_q(s) - \delta\chi(s)$ for $i \in I_p(d^0)$ or $i \in I_q(d^0)$, respectively, is an eigenvalue of $D^{-1}(\sigma(s))BA - L_\delta(s) - I$ and one of the roots of (8.4). It follows from Notation 5.1 and Observation 5.1 that we can choose $\delta_0 > 0$ and $\eta > 0$ such that $\mu_i(s) + \delta\chi(s) > 0$ or $\mu_i(s) - \delta\chi(s) < 0$ for $i \in I_p(d^0)$ or $i \in I_q(d^0)$, respectively, on $(s_0-\eta, s_0+\eta)$, $\delta \in (0, \delta_0)$. The roots of (8.4) depend continuously on $s \in \mathbb{R}$, $\delta \geqslant 0$ and therefore the choice of $\delta_0 > 0$ and $\eta > 0$ can be such that the other root is positive or negative for $i \in I_p(d^0)$ or $i \in I_q(d^0)$, respectively, on $(s_0-\eta, s_0+\eta)$, $\delta \in (0, \delta_0)$.

Now, it follows from the relation of the eigenvalues of the operator $D^{-1}(\sigma(s))BA - L_\delta(s) - I$ and the roots of (8.4) mentioned above that there are no further eigenvalues and eigenvectors besides those discussed in A1–B3.

Let us show that for $s \in (s_0-\eta, s_0+\eta)$ or $s \in (s_0-\eta, s_0)$ in the case $d^0 \in C_p \setminus \mathcal{T}$ or $d^0 \in C_p \cap \mathcal{T}$, respectively, the algebraic and geometric multiplicities of any positive eigenvalue of the operator $D^{-1}(\sigma(s))BA - L_\delta(s) - I$ coincide. First, we will show the coincidence of the algebraic and geometric multiplicities of any positive eigenvalue $\mu_i^{(r)}(d)$, $r = 1, 2$, of the operator $D^{-1}(\sigma(s))BA - I$.

The adjoint equation to (5.1) is

$$B^* D^{-1}(d) A U - U = \mu U$$

and similar considerations as in Observation 5.1 imply that the eigenvectors of this equation corresponding to $\mu_i^{(r)}(d)$ are

$$\widetilde{U}_i^{(r)}(d) = \Big[\frac{d_1}{d_2}\frac{d_2\kappa_i - b_{22} + \mu_i^{(r)}(d)d_2\kappa_i}{b_{12}}e_i, e_i\Big] = \Big[\frac{d_1}{d_2}\frac{b_{21}}{b_{12}}\alpha_i^{(r)}(d)e_i, e_i\Big],\ r = 1, 2.$$

(Recall that $U_i^{(r)}(d) = [\alpha_i^{(r)}(d)e_i, e_i],\ r = 1, 2$—see Observation 5.2.) An elementary calculation using (5.3) gives for $\mu_i^{(r)}(d) > 0$ that

$$\begin{aligned}
&|\langle U_i^{(r)}(d), \widetilde{U}_i^{(r)}(d)\rangle| = \Big|\frac{d_1}{d_2}\frac{(d_2\kappa_i - b_{22} + \mu_i^{(r)}(d)d_2\kappa_i)^2}{b_{12}b_{21}} + 1\Big| \\
&= -\frac{d_2\kappa_i - b_{22} + \mu_i^{(r)}(d)d_2\kappa_i}{d_2 b_{12}b_{21}}\big[d_2 b_{11} + d_1 b_{22} - 2d_1 d_2\kappa_i - 2d_1 d_2\kappa_i\mu_i^{(r)}(d)\big] \qquad (8.5)\\
&= -\frac{[d_2\kappa_i - b_{22} + \mu_i^{(r)}(d)d_2\kappa_i]\sqrt{\omega(d)}}{d_2 b_{12}b_{21}} \neq 0 \text{ for } i = 1, 2, \dots,\ r = 1, 2,\ d \notin \mathcal{T}, \\
&\langle U_i^{(r)}(d), \widetilde{U}_j^{(r)}(d)\rangle = 0 \text{ for any } i \neq j,\ r = 1, 2,
\end{aligned}$$

cf. (5.11). Hence,

$$\det(\langle \widetilde{U}_i^{(r)}(d), U_j^{(r)}(d)\rangle)_{i,j\in J} \neq 0 \text{ for any } J \subset \mathbb{N},\ r = 1, 2,\ d \notin \mathcal{T}. \qquad (8.6)$$

This yields that the algebraic and geometric multiplicities of $\mu_i^{(r)}(d)$ coincide for $i \in \mathbb{N},\ r = 1, 2,\ d \notin \mathcal{T}$ (see e.g. [24]). In particular, this holds for $d = \sigma(s)$ with $s \in \mathcal{U}_\eta(s_0)$, where $\mathcal{U}_\eta(s_0) := (s_0 - \eta, s_0 + \eta)$ for $d^0 \in C_p \setminus \mathcal{T}$, $\mathcal{U}_\eta(s_0) := (s_0 - \eta, s_0)$ for $d^0 \in C_p \cap \mathcal{T}$ (let us note that $\sigma(s) \notin \mathcal{T}$ for $s \in \mathcal{U}_\eta(s_0)$).

By a standard treatment of the adjoint operator we obtain

$$L_\delta^*(s)\widetilde{U}_i^{(r)}(\sigma(s)) = 0 \text{ for all } i \notin I(d^0),\ r = 1, 2,\ s \in \mathcal{U}_\eta(s_0).$$

This implies that $\widetilde{U}_i^{(r)}(\sigma(s))$ for $i \notin I(d^0)$, $r = 1, 2$, $s \in \mathcal{U}_\eta(s_0)$, is simultaneously an eigenvector of the adjoint operator $(D^{-1}(\sigma(s))BA)^* - L_\delta^*(s) - I$ corresponding to $\mu_i^{(r)}(\sigma(s))$. The above considerations imply the coincidence of the algebraic and geometric multiplicities of any $\mu_i^{(r)}(d) > 0$ with $i \notin I(d^0)$, $r = 1, 2$, $s \in \mathcal{U}_\eta(s_0)$, as the eigenvalue of the operator $D^{-1}(\sigma(s))BA - L_\delta(s) - I$.

If $d^0 \in C_p \cap \mathcal{T}$ then all eigenvalues of $D^{-1}(\sigma(s))BA - L_\delta(s) - I$ corresponding to $i \in I(d^0)$ are negative on $(s_0 - \eta, s_0)$ and complex on $(s_0, s_0 + \eta)$—see the first part of this proof.

If $d^0 \in C_p \setminus \mathcal{T}$ then, due to the continuous dependence on $s \in \mathbb{R}$, $\delta \geqslant 0$, we can choose $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$ there is $\eta > 0$ for which the determinant

corresponding to (8.6) with the scalar products of the corresponding eigenvectors of the operators $D^{-1}(\sigma(s))BA - L_\delta(s) - I$ and $(D^{-1}(\sigma(s))BA)^* - L_\delta^*(s) - I$, respectively, with $i, j \in J \subset I(d^0)$, remains nonzero on $(s_0 - \eta, s_0 + \eta)$. Therefore, the algebraic and geometric multiplicities of any positive eigenvalue corresponding to $i \in I(d^0)$ coincide again.

Our considerations lead to the following conclusion. If $d^0 \in C_p \setminus \mathcal{T}$ then $\mu_p(s)$ is the only eigenvalue of the operator $D^{-1}(\sigma(s))BA - L_\delta(s) - I$ changing its sign at s_0 and it is algebraically simple. The other eigenvalues have constant signs and multiplicities on $(s_0 - \eta, s_0 + \eta)$. If $d^0 \in C_p \cap \mathcal{T}$ then $\mu_p(s)$ is a real positive algebraically simple eigenvalue on $(s_0 - \eta, s_0)$. The other possible positive eigenvalues (which can correspond only to $i \notin I(d^0)$) form pairs $\mu_i^{(1)}(\sigma(s)) \neq \mu_i^{(2)}(\sigma(s))$ where $\mu_i^{(1)}(\sigma(s))$, $\mu_i^{(2)}(\sigma(s))$ have the same algebraic multiplicity, i.e. the sum of algebraic multiplicities for any such pair is even. All eigenvalues are complex for $(s_0, s_0 + \eta)$. The assertion of Lemma 5.1 follows.

P r o o f o f L e m m a 6.9. Assume that there are $\varrho^m \to 0_+$ such that for any m fixed there exist s_n^m, U_n^m, τ_n^m $(n = 1, 2, \ldots)$ satisfying

$$D(\sigma(s_n^m))U_n^m - BAU_n^m - \frac{\tau_n^m}{1+\tau_n^m} N(U_n^m) + D(\sigma(s_n^m))L_\delta(s_n^m)U_n^m + P_{\tau_n^m}(U_n^m) = 0, \tag{8.7}$$

$$\|U_n^m\|^2 = \frac{\varrho^m \tau_n^m}{1+\tau_n^m} \tag{8.8}$$

with $U_n^m \notin K^\delta$, $[s_n^m, U_n^m, \tau_n^m] \to [s_0, U^m, \tau^m]$, $\frac{U_n^m}{\|U_n^m\|} \to W^m \in K^\delta$ if $n \to +\infty$. We have $\|U_n^m\|^2 = \frac{\varrho^m \tau_n^m}{1+\tau_n^m} \leqslant \varrho^m \to 0$ by (8.8). We can choose a subsequence $\{U_k\}_{k=1}^{+\infty}$ from $\{U_n^m\}_{n,m=1}^{+\infty}$ such that $Z_k = \frac{U_k}{\|U_k\|} \notin K^\delta$, $Z_k \rightharpoonup Z \in K^\delta$ and $\tau_k \to \tau \in [0, +\infty]$. Lemma 6.1 gives $Z_k \to Z$.

First let $\tau = 0$. Dividing (8.7) by $\|U_k\|$, the limiting process gives

$$D(\sigma(s_0))Z - BAZ + D(\sigma(s_0))L_\delta(s_0)Z = 0 \tag{8.9}$$

with help of (2.5) and Proposition 4.3. It means $Z = -\frac{U_p}{\|U_p\|} \in \operatorname{int} K^\delta$ because of $W^m \in K^\delta$ and the fact that $\pm\frac{U_p}{\|U_p\|}$ are the only normed solutions of (8.9). For $\tau \in (0, +\infty]$ the equation (8.7) gives that $\frac{P_{\tau_k}^\delta(U_k)}{\|U_k\|}$ are bounded and therefore we can assume $\frac{P_{\tau_k}^\delta(U_k)}{\|U_k\|} \rightharpoonup F$,

$$D(\sigma(s_0))Z - BAZ + D(\sigma(s_0))L_\delta(s_0)Z + F = 0, \tag{8.10}$$

where we have employed (2.5) again. Multiplying (8.10) by $-U_p^*$, the equation

$$D(\sigma(s_0))U_p^* - B^*AU_p^* = 0$$

by Z and adding them we obtain $\langle F, -U_p^* \rangle = 0$ due to (5.10). Observation 6.1 implies $F = 0$, i.e. we have (8.9) and $Z = -\frac{U_p}{\|U_p\|}$ again. In both cases, this is a contradiction because $Z_k \notin K^\delta$ and $Z_k \to Z = -\frac{U_p}{\|U_p\|} \in \operatorname{int} K^\delta$ and our assertion is proved.

R e m a r k 8.1. The other possibility to avoid the condition $\operatorname{int} K \neq \emptyset$ is to define a pseudointerior

$$K^- := \big\{U \in K\,;\ \underset{\substack{V \notin K \\ \tau > 0}}{\forall} \langle P_\tau V, U\rangle < 0 \ \&\ \underset{\substack{F \in \mathbb{V}^2 \\ F \neq 0}}{\forall} \underset{W \in \mathbb{V}^2}{\exists} \langle F, W\rangle > 0,\ U \pm W \in K\big\}$$

(cf. [26], [6]) and assume $-U_p, -U_p^* \in K^-$ instead of the assumption $-U_p, -U_p^* \in \operatorname{int} K^\delta$ for any $\delta \in (0, \delta_0)$ in (4.14). In order to prove Lemmas in Section 5, one has to add a special assumption about the nonlinearity term N or about the sign of a scalar product of a certain type, respectively:

$$\begin{aligned} &\text{if } U_n \to 0,\ W_n = \frac{U_n}{\|U_n\|} \rightharpoonup \frac{U_p}{\|U_p\|} \\ &\text{then } \left\langle \frac{N(U_n)}{\|U_n\|}, U_p^* \right\rangle \geqslant 0 \text{ for } n \text{ large enough.} \end{aligned} \tag{8.11}$$

The meaning of this condition for (2.6) is the following: Let $s_n,\ U_n$ satisfy (2.6). Let $s_n \to s_0,\ U_n \to 0,\ W_n = \frac{U_n}{\|U_n\|} \rightharpoonup \frac{U_p}{\|U_p\|}$. After some calculation (similar to that in the proof of [16], Lemma 3.6), condition (8.11) leads us to the conclusion that $s_n \leqslant s_0$. This corresponds to the fact that a branch of bifurcating spatial patterns of (2.6) goes to the left from C, i.e. to the domain of instability of the trivial solution.

P r o o f o f t h e d e s t a b i l i z i n g e f f e c t i n T h e o r e m 4.1 under the additional assumption from Remark 4.2.

We will show that there exists $\varepsilon > 0$ such that

$$s_I^\delta > s_0 + \varepsilon \text{ for all } \delta > 0 \text{ small enough.} \tag{8.12}$$

Assume that for $\delta_n \to 0$ we have $s_I^{\delta_n} \to s_0$, $U_n = U_I^{\delta_n} \rightharpoonup U$ satisfying

$$D(\sigma(s_I^{\delta_n}))U_n - BAU_n + D(\sigma(s_I^{\delta_n}))L_{\delta_n}(s_I^{\delta_n})U_n \in -M_0^{\delta_n}(U_n), \tag{8.13}$$

where $U_I^{\delta_n}$ are from Step 2 of the proof of Theorem 4.1. With help of (4.12) the limiting process in (8.13) gives $U_n \to U$ and

$$D(\sigma(s_0))U - BAU \in -M_0(U),$$

i.e. $U \in E_I(d^0)$. Under the assumption of the equivalence of relations (2.11) and (2.13) we can use Lemma 7 together with Remark 5 from [26] to obtain $U \in K$ and $U \in E_B(d^0)$. Hence

$$U = \sum_{i \in I(d^0) \cup \{p\}} a_i U_i(s_0) \tag{8.14}$$

with some $a_i \in \mathbb{R}$ (see (5.7) or (5.8), respectively). Setting

$$F_n := D(\sigma(s_I^{\delta_n}))U_n - BAU_n + D(\sigma(s_I^{\delta_n}))L_{\delta_n}(s_I^{\delta_n})U_n \tag{8.15}$$

we rewrite (8.13) into the form $F_n \in -M_0(U_n)$. Then the assumptions (4.19) and (4.20) imply

$$\langle F_n, U^* \rangle \geqslant 0. \tag{8.16}$$

To get a contradiction we prove that

$$\begin{aligned} \langle F_n, U^* \rangle &= \langle D(\sigma(s_I^{\delta_n}))U_n - BAU_n + D(\sigma(s_I^{\delta_n}))L_{\delta_n}(s_I^{\delta_n})U_n, U^* \rangle \\ &= \langle [D(\sigma(s_I^{\delta_n})) - D(\sigma(s_0))]U_n, U^* \rangle + \langle D(\sigma(s_0))U_n - BAU_n, U^* \rangle \\ &\quad + \langle D(\sigma(s_I^{\delta_n}))L_{\delta_n}(s_I^{\delta_n})U_n, U^* \rangle < 0. \end{aligned} \tag{8.17}$$

Indeed, the first scalar product is negative for $s_I^{\delta_n} > s_0$ because we have

$$\langle [D(\sigma(s_I^{\delta_n})) - D(\sigma(s_0))]U_n, U^* \rangle = (s_I^{\delta_n} - s_0)R_n$$

where $R_n := \sigma_1'(\overline{s}_n)\langle u_n, u^* \rangle + \sigma_2'(\tilde{s}_n)\langle z_n, v^* \rangle$ with some $\overline{s}_n, \tilde{s}_n$ lying between $s_I^{\delta_n}$ and s_0. It follows from (8.14) and Proposition 5.1 that

$$\lim_{n \to +\infty} R_n = \sum_{i \in I(d^0) \cup \{p\}} a_i^2 \Big[\frac{(\kappa_i \sigma_2(s_0) - b_{22})^2}{b_{12} b_{21}} \sigma_1'(s_0) + \sigma_2'(s_0) \Big] < 0. \tag{8.18}$$

Note that (4.16) implies that the term in brackets in (8.18) is negative for all $i \in I_p(d^0) \cup \{p\}$ and nonpositive for $i \in I_q(d^0)$ in the case $d^0 \in C_p \cap C_q$, $C_p \neq C_q$—see Notation 5.2. The second scalar product in (8.17) vanishes because

$$\langle D(\sigma(s_0))U_n - BAU_n, U^* \rangle = \langle U_n, D(\sigma(s_0))U^* - B^*AU^* \rangle = 0.$$

If $U = \pm\frac{U_p}{\|U_p\|}$ then the last term in (8.17) is zero by (5.11). Further, we have

$$
\begin{aligned}
&\frac{1}{\delta_n}\langle D(\sigma(s_I^{\delta_n}))L_{\delta_n}(s_I^{\delta_n})U_n, U^*\rangle \\
&\qquad = \chi(s_I^{\delta_n}) \sum_{i\in I(d^0)} \nu_i(d^0)\frac{\langle U_i(s_I^{\delta_n}), U_n\rangle}{\|U_i(s_I^{\delta_n})\|^2}\langle D(\sigma(s_I^{\delta_n}))U_i(s_I^{\delta_n}), U^*\rangle \\
&\qquad = \chi(s_I^{\delta_n}) \sum_{i\in I(d^0)} \nu_i(d^0)\frac{\langle U_i(s_I^{\delta_n}), U_n\rangle}{\|U_i(s_I^{\delta_n})\|^2}\langle D(\sigma(s_I^{\delta_n}))U_i(s_I^{\delta_n}), a_i U_i^*(s_0)\rangle \\
&\qquad \to \sum_{i\in I(d^0)} \frac{a_i^2[d_2^0\kappa_i - b_{22}]\sqrt{\omega(d^0)}}{b_{12}b_{21}} \text{ for } n \to +\infty
\end{aligned}
\tag{8.19}
$$

(see the proof of the second part of (5.11) with $s = s_0$). If $d^0 \notin \mathcal{T}$ then the limit in (8.19) is negative and therefore the last term in (8.17) is negative for large n.

If $d^0 \in \mathcal{T}$ then $\omega(d^0) = 0$ and therefore the limit in (8.19) is zero. But $I(d^0) = \{p+1, \dots, p+k-1\}$ (k is the multiplicity of κ_p), $\nu_i(d^0) = 1$ and $\langle D(\sigma(s_0))U_i(s_0), U_i^*(s_0)\rangle = \frac{a_i^2[d_2^0\kappa_i - b_{22}]\sqrt{\omega(d^0)}}{b_{12}b_{21}} = 0$ for any $i \in I(d^0)$. Therefore, by the definition of L_δ and by (5.11) we have

$$
\begin{aligned}
&\frac{1}{\delta_n}\langle D(\sigma(s_I^{\delta_n}))L_{\delta_n}(s_I^{\delta_n})U_n, U^*\rangle \\
&\qquad = \frac{1}{\delta_n}\langle D(\sigma(s_I^{\delta_n}))L_{\delta_n}(s_I^{\delta_n})U_n, U^*\rangle \\
&\qquad\qquad - \chi(s_I^{\delta_n}) \sum_{i\in I(d^0)} \frac{\langle U_i(s_I^{\delta_n}), U_n\rangle}{\|U_i(s_I^{\delta_n})\|^2}\langle D(\sigma(s_0))U_i(s_0), a_i U_i^*(s_0)\rangle \\
&\qquad = \chi(s_I^{\delta_n}) \sum_{i\in I(d^0)} \frac{\langle U_i(s_I^{\delta_n}), U_n\rangle}{\|U_i(s_I^{\delta_n})\|^2} \\
&\qquad\qquad \times \langle [D(\sigma(s_I^{\delta_n}))U_i(s_I^{\delta_n}) - D(\sigma(s_0))U_i(s_0)], a_i U_i^*(s_0)\rangle \\
&\qquad = \chi(s_I^{\delta_n}) \sum_{i\in I(d^0)} \frac{\langle U_i(s_I^{\delta_n}), U_n\rangle}{\|U_i(s_I^{\delta_n})\|^2}(s_I^{\delta_n} - s_0)a_i R_n^i
\end{aligned}
\tag{8.20}
$$

with $R_n^i := \sigma_1'(\overline{s}_n^i)\langle u_n, u_i^*\rangle + \sigma_2'(\tilde{s}_n^i)\langle z_n, v_i^*\rangle$ for suitable $\overline{s}_n^i, \tilde{s}_n^i \in (s_0, s_I^{\delta_n})$. Hence the last expression in (8.20) is negative for large n by the same argument used for the first term in the last part of (8.17) $\big($cf. (8.18)$\big)$. The assertion (8.17) follows and we have a contradiction with $s_I^{\delta_n} \to s_0$. Therefore, $s_I^{\delta_n} > s_0 + \varepsilon$ with some $\varepsilon > 0$ for any n and thus $s_I \geqslant s_0 + \varepsilon$.

It remains to show that $[s_\varrho, U_\varrho]$ do not satisfy (4.21): Assume by contradiction that there are $\varrho_n \to 0$, $s_{\varrho_n} \to s_I$, $U_{\varrho_n} \to 0$ satisfying

$$D(\sigma(s_{\varrho_n}))U_{\varrho_n} - BAU_{\varrho_n} - N(U_{\varrho_n}) = 0. \tag{8.21}$$

Dividing this equation by $\|U_{\varrho_n}\|$ we obtain after the limiting process

$$D(\sigma(s_I))U_I - BAU_I = 0$$

where U_I is an accumulating point of $\frac{U_{\varrho_n}}{\|U_{\varrho_n}\|}$. This is impossible because $s_0 < s_I$ is the greatest critical point of (4.18).

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