# REACTION-DIFFUSION SYSTEMS: DESTABILIZING EFFECT OF CONDITIONS GIVEN BY INCLUSIONS II, EXAMPLES

Jan Eisner, Praha

(Received March 9, 1999; revised September 22, 2000)

*Abstract.* The destabilizing effect of four different types of multivalued conditions describing the influence of semipermeable membranes or of unilateral inner sources to the reaction-diffusion system is investigated. The validity of the assumptions sufficient for the destabilization which were stated in the first part is verified for these cases. Thus the existence of points at which the spatial patterns bifurcate from trivial solutions is proved.

*Keywords*: bifurcation, spatial patterns, reaction-diffusion system, mollification, inclusions

*MSC 2000*: 35B32, 35K57, 35K58, 47H04

## 9. Auxiliary assertions

This paper is a continuation of [1]. We study the bifurcation points $s_I \in \mathbb{R}$ at which nontrivial solutions to

$$\begin{aligned} &\sigma_1(s)u - b_{11}Au - b_{12}Av - N_1(u,v) = 0 \\ &\sigma_2(s)v - b_{21}Au - b_{22}Av - N_2(u,v) \in -M_2(v) \end{aligned} \tag{9.1}$$

bifurcate from the trivial solution. Recall that solutions to (9.1) for $M_2$ defined by (2.9) are weak solutions to

$$\begin{aligned} &\sigma_1(s)u + b_{11}u + b_{12}v + n_1(u,v) = 0 \\ &\sigma_2(s)v + b_{21}u + b_{22}v + n_2(u,v) = 0 \end{aligned} \quad \text{in } \Omega \tag{9.2}$$

The support of this work by the Grant Agency of the Czech Republic under the grant No. 201/98/1453 and by the Ministry of Education of the Czech Republic under the grant VS 97156 is gratefully acknowledged.

with the boundary conditions

$$(9.3)\quad u = v = 0 \ \text{ on } \Gamma_D,\ \frac{\partial u}{\partial n} = 0,\ \frac{\partial v}{\partial n} \in -\frac{m(v)}{\sigma_2(s)} \ \text{ on } \Gamma_U,\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \ \text{ on } \Gamma_N$$

(see Section 12). In the main result of [1]—Theorem 4.1—the existence of such bifurcation points is proved under certain assumptions on the multivalued mapping $M$. Here we shall concentrate on a verification of these assumptions for some important particular examples of the mapping $M$ (related to a multivalued function $m$) to show the existence of bifurcation points of spatial patterns to (9.2).

Let us note that the references to Sections 1–8 correspond to [1].

In the sequel, we need the following assertions and the corresponding

Notation 9.1.
$H^1_L(\Omega) = \{\varphi \in W^1_2(\Omega);\ \Delta\varphi \in L^2(\Omega)\}$,
$\mathbb{H} = H^1_L(\Omega) \cap \mathbb{V}$,
$H^{\frac{1}{2}}(\partial\Omega)$—the space of traces of functions from $W^1_2(\Omega)$,
$H^{-\frac{1}{2}}(\partial\Omega)$—the dual space of $H^{\frac{1}{2}}(\partial\Omega)$,
$\mathcal{D}(\Omega)$—the space of $C^\infty$—smooth functions with compact support in $\Omega$.

Observation 9.1. There is a uniquely defined continuous mapping $\mathfrak{T}$: $H^1_L(\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ such that $\mathfrak{T}u = \frac{\partial u}{\partial n}$ if $u \in C^1(\operatorname{cl}\Omega)$—see [5].

Let $m\colon \mathbb{R} \to 2^{\overline{\mathbb{R}}}$ be a multivalued function. Let $\underline{m}(\xi) := \inf\{m(\xi)\}$, $\overline{m}(\xi) := \sup\{m(\xi)\}$ for $\xi \in \mathbb{R}$. If $u \in \mathbb{V}$ then $u$ on $\partial\Omega$ is understood in the sense of traces and $\frac{\partial u}{\partial n}$ is understood as a functional $\mathfrak{T}$ from $H^{-\frac{1}{2}}(\partial\Omega)$. This means that $\frac{\partial u}{\partial n}(x) \in -m(u(x))$ stands for

$$-\int_{\Gamma_U} \overline{m}(u(x))\psi(x)\,\mathrm{d}\Gamma \leqslant \int_{\Gamma_U} \frac{\partial u}{\partial n}\psi\,\mathrm{d}\Gamma \leqslant -\int_{\Gamma_U} \underline{m}(u(x))\psi(x)\,\mathrm{d}\Gamma$$
$$\text{for any } \psi \in H^{\frac{1}{2}}(\Gamma_U), \psi \geqslant 0 \text{ a.e. on } \Gamma_U.$$

**Lemma 9.1** (Cf. [6], Theorem 3.2.). *Let $G \subset \mathbb{R}^k$, $\operatorname{meas}_k G < +\infty$. Let $u_n, u \in L^2(G)$ be a sequence of functions, $u_n \to u$ in $L^2(G)$. Let $g_n, g$ be continuous functions on $\mathbb{R}$. Let $h_n$, $h$ be Nemytskii operators corresponding to the functions $g_n, g$. Let $C > 0$ be a constant such that $|g_n(\xi)| \leqslant C(1 + |\xi|)$, $|g(\xi)| \leqslant C(1 + |\xi|)$ for all $\xi \in \mathbb{R}$. Let $g_n(u_n) \to g(u)$ everywhere on $G$. Then $h_n(u_n) \to h(u)$ in $L^2(G)$.*

Proof. We have $h_n(u_n)(x) = g_n(u_n(x))$, $h(u)(x) = g(u(x))$ for all $x \in G$.

We can choose a subsequence (let us denote it $\{u_n\}$ again) that is Cauchy. We can choose again a subsequence $\{u_{n_j}\}$ such that

$$(9.4)\qquad \|u_{n_j} - u_{n_{j+1}}\|_{L^2(G)} < \frac{1}{2^j}.$$

The convergence of $g_n$ ensures $h_{n_j}(u_{n_j}) \to h(u)$ a.e. on $G$. The growth of $g_n$ and $g$ together with (9.4) gives

$$|h_{n_j}(u_{n_j})(x)| \leqslant C(1+|u_{n_j}(x)|) \leqslant C\Big(1+\sum_{j=1}^{\infty}|u_{n_{j+1}}(x)-u_{n_j}(x)|+|u_{n_1}(x)|\Big)$$
$$=: f(x) \in L^2(G).$$

It follows from the Lebesgue theorem that

$$\|h_{n_j}(u_{n_j})-h(u)\|_{L^2(G)} \to 0.$$

Therefore every subsequence of the original sequence $\{u_n\}$ contains a subsequence for which $h_{n_j}(u_{n_j})$ converges to $h(u)$ in $L^2(G)$. Our assertion is proved. □

## 10. Examples

Example 10.1. We shall investigate the Model Example from [1]. The multivalued mapping $M$ is given here by a function $m\colon \mathbb{R} \to 2^{\overline{\mathbb{R}}}$ which is singlevalued, real and continuous on $\mathbb{R}\setminus\{0\}$ and multivalued at $\xi = 0$—see (2.9).

The set $\partial\Omega$ is Lipschitz. Thus there exist a system of positive constants $a_i, b_i$, a system of sets $U_i \subset \mathbb{R}^{\mathrm{n}}$ covering $\Gamma_U$, a system of balls $B_i \subset \mathbb{R}^{\mathrm{n}-1}$ centered at 0 and a system of bi-Lipschitzian homeomorphisms $Q_i\colon U_i \to B_i \times (-a_i, b_i)$ such that $Q_i(U_i \cap \Omega) = B_i \times (0, b_i)$ and $Q_i(U_i \cap (\mathbb{R}^{\mathrm{n}} \setminus \operatorname{cl}\Omega)) = B_i \times (-a_i, 0)$, $i = 1, \ldots, M$. Let

$$\alpha_1, \ldots, \alpha_M \tag{10.1}$$

be a $C^1$-smooth partition of unity on $\Gamma_U$ subordinated to the covering $U_i$ and $\alpha_{M+1} := 1 - \sum_{i=1}^{M} \alpha_i$. Then $\operatorname{supp}\alpha_{M+1} \cap \operatorname{cl}\Gamma_U = \emptyset$. For $G_0 := \Omega \cup \bigcup_{i=1}^{M} U_i$ we have $d_0 := \operatorname{dist}(\Gamma_U, \mathbb{R}^{\mathrm{n}} \setminus G_0) > 0$. For the Lipschitz cut-off function $\eta\colon x \mapsto [1 - \frac{2}{d_0}\operatorname{dist}(x,\Omega)]^+$ and for the continuous extension $\widetilde{E}\colon \mathbb{V} \to W^{1,2}(\mathbb{R}^{\mathrm{n}})$ ensured e.g. by Theorem 2.3.10 from [5] we take $G = \mathbb{R}^{\mathrm{n}}$ and define $E := \eta\widetilde{E}$ (cf. [1], Notation 4.1). Let us recall the mollification operator $\Phi^\delta\colon \mathbb{V} \to W^{1,2}(G) \cap C^0(\operatorname{cl}\Omega)$ introduced in Notation 4.1.

In the sequel we will need

**Proposition 10.1.**

(i) *For any $v \in \mathbb{V}$, $\Phi^\delta(v)$ is a continuous function on $\operatorname{cl}\Omega$.*
(ii) *If $v_n, v \in \mathbb{V}$, $v_n \to v$ in $\mathbb{V}$ and $\delta > 0$ is fixed then $\Phi^\delta(v_n) \to \Phi^\delta(v)$ in $C^0(\operatorname{cl}\Omega)$.*
(iii) *Let $v \in \mathbb{V}$, $\delta_n \to 0_+$. Then $\Phi^{\delta_n}(v) \to Ev$ in $W_0^{1,2}(G)$.*

(iv) *Let* $v_n \rightharpoonup v$ *weakly in* $\mathbb{V}$ *and* $\delta_n \to 0_+$. *Then* $\Phi^{\delta_n}(v_n) \rightharpoonup Ev$ *weakly in* $W_0^{1,2}(G)$.

(v) *If* $v_n \rightharpoonup v$ *weakly in* $\mathbb{V}$ *and* $\delta_n \to 0_+$ *then* $\Phi^{\delta_n}(v_n) \to v$ *in* $L^2(\partial\Omega)$.

(vi) *For any* $\psi \in \mathbb{V}$, $\psi \geqslant 0$ *a.e. on* $\Gamma_U$ *there are* $w_n = w_n(\psi) \in \mathbb{V}$ *and* $\delta_n > 0$ *such that* $w_n \to \psi$ *strongly in* $\mathbb{V}$ *and* $\Phi^\delta(w_n) \geqslant 0$ *on* $\Gamma_U$ *for any* $\delta \in (0, \delta_n)$.

P r o o f. (i) and (ii) are obvious.

(iii): It follows from the definition of $\Phi^\delta$ and [5], Theorem 2.1.2 that $\Phi^{\delta_n}(v) \to Ev$ and $\frac{\partial}{\partial x_j}\Phi^{\delta_n}(v) = \Phi^{\delta_n}(\frac{\partial}{\partial x_j}v) \to \frac{\partial}{\partial x_j}Ev$ in $L^2(G)$ for any $j = 1, \dots, \mathfrak{n}$. (For the proof of the identity $\frac{\partial}{\partial x_j}\Phi^\delta(f) = \Phi^\delta(\frac{\partial}{\partial x_j}f)$ see [5], Theorem 2.2.1.) Therefore, $\Phi^{\delta_n}(v) \to Ev$ in $W_0^{1,2}(G)$.

(iv): If $v_n \rightharpoonup v$ weakly in $\mathbb{V}$ then $v_n \to v$ strongly in $L^2(\Omega)$ and $\frac{\partial}{\partial x_j}v_n \rightharpoonup \frac{\partial}{\partial x_j}v$ weakly in $L^2(\Omega)$ for any $j = 1, \dots, \mathfrak{n}$ by the embedding theorems. Let $T_n, T_0\colon L^2(G) \to L^2(G)$, $T_n f := \Phi^{\delta_n}(f)$, $T_0 f := Ef$ for any $f \in L^2(\Omega)$. It follows from [5], Theorem 2.1.2 that $T_n f \to T_0 f$ in $L^2(G)$ for any $f \in L^2(\Omega)$. $T_n, T_0$ are linear continuous operators, therefore they are uniformly bounded by the Banach-Steinhaus theorem. We obtain

$$\begin{aligned}\|T_n v_n - T_0 v\|_{L^2(G)} &\leqslant \|T_n v_n - T_n v\|_{L^2(G)} + \|T_n v - T_0 v\|_{L^2(G)}\\ &\leqslant \|T_n\|_{L(L^2(G),L^2(G))} \cdot \|v_n - v\|_{L^2(\Omega)} + \|T_n v - T_0 v\|_{L^2(G)} \to 0.\end{aligned}$$

Now, let $f_n \rightharpoonup f$ weakly in $L^2(\Omega)$ and let $g \in W^{1,2}(\Omega)$ be arbitrary. We have by using the Fubini theorem that

$$\begin{aligned}(T_n f_n - T_0 f, Eg)_{L^2(G)} &= (T_n f_n - T_n f, Eg)_{L^2(G)} + (T_n f - T_0 f, Eg)_{L^2(G)}\\ &= (Ef_n - Ef, T_n g)_{L^2(G)} + (T_n f - T_0 f, Eg)_{L^2(G)} \to 0.\end{aligned}$$

By the choice $f_n := \frac{\partial}{\partial x_j}v_n$, $f := \frac{\partial}{\partial x_j}v$ in the second part and due to the fact that $\frac{\partial}{\partial x_j}T_n f = T_n \frac{\partial}{\partial x_j}f$ for any $j = 1, \dots, \mathfrak{n}$ the proof is completed.

(v) follows from the embedding theorems and (iv).

(vi): We decompose $\psi = \psi^+ - \psi^-$, where $\psi^+, \psi^-$ denotes the positive and negative parts, respectively. We have $\psi^+, \psi^- \in \mathbb{V}$ by [3]. The "bad" term is $\psi^-$. Therefore, we can assume without loss of generality that $\psi \in \mathbb{V}$ is such that $\psi \leqslant 0$ in $\Omega$ and $\psi = 0$ a.e. on $\Gamma_U$. Let us denote $g_i = \alpha_i E\psi$, $\alpha_i$ from (10.1), $i = 1, \dots, M$. Since the boundary of $\Gamma_U$ is Lipschitz with respect to $\partial\Omega$, we can assume that $Q_i(U_i \cap \Gamma_U)$ is starshaped[1] in $B_i$ with respect to $0 \in \mathbb{R}^{\mathfrak{n}-1}$. Let us denote $g_i^r(x) := g_i(Q_i^{-1}(rQ(x)))$ for any $r \in (0,1)$ and $x \in U_i$. Let $\widetilde{\Gamma}_U$ be the extension of $\Gamma_U$ such that $g_i^r(x) = 0$ for any $x \in \widetilde{\Gamma}_U$. Thus we have constructed $\widetilde{\Gamma}_U$ such that $\operatorname{dist}(\Gamma_U, \partial\Omega \setminus \widetilde{\Gamma}_U) = \delta^{(0)} > 0$.

[1] A set $X \subset \mathbb{R}^k$ is called *starshaped* with respect to a set $Y$, $Y \subset X$, if any ray with its origin in $Y$ has a unique common point with $\partial X$—see [4].

We have $g_i^r \to g_i$ in $W_0^{1,2}(U_i)$ for $r \to 1_-$ (this fact can be proved in a similar way as (iii) or [5], Theorem 2.1.1). For $i = 1, \dots, M$ let $O_j$, $j = 1, \dots, k_i$ be a covering of $V_i := \{x \in \mathbb{R}^{\mathfrak{n}};\ \operatorname{dist}(x, \widetilde{\Gamma}_U \cap \operatorname{supp} \alpha_i) \leqslant \delta^{(i)}\}$ in $U_i$ such that $\bigcup_{j=1}^{k_i} O_j \cap (\partial\Omega \setminus \widetilde{\Gamma}_U) = \emptyset$ and $\delta^{(i)} \in (\min\{\frac{\delta^{(0)}}{2}, \operatorname{dist}(\operatorname{supp} \alpha_i, \mathbb{R}^{\mathfrak{n}} \setminus U_i)\})$, $i = 1, \dots, M$. Let $\beta_{1,i}, \dots, \beta_{k_i,i}$ be a $C^1$-smooth partition of unity on $V_i$ subordinated to the covering $\{O_j;\ j = 1, \dots, k_i\}$, $\beta_{k_i+1,i} := 1 - \sum_{j=1}^{k_i} \beta_{j,i}$. Then $\operatorname{supp} \beta_{k_i+1,i} \cap V_i = \emptyset$.

We have $\beta_{j,i} g_i^r \in W_0^{1,2}(O_j \cap \Omega)$. Therefore there are $\varphi_n^{r,i,j}$ which are $C^1$-smooth with $\operatorname{supp} \varphi_n^{r,i,j} \subset (O_j \cap \Omega)$ and such that $\varphi_n^{r,i,j} \to \beta_{j,i} g_i^r$ in $W_0^{1,2}(O_j \cap \Omega)$ for $n \to +\infty$, $j = 1, \dots, k_i$ and $i = 1, \dots, M$—see [5]. We can choose $r_n \to 1_-$ and

$$w_n := \sum_{i=1}^{M} \Big( \sum_{j=1}^{k_i} \varphi_n^{r_n,i,j} + \beta_{k_i+1,i}\, g_i^r \Big) + \alpha_{M+1}\psi \to \psi \text{ in } \mathbb{V}$$

for $n \to +\infty$ and there are $\delta_n > 0$ such that $w_n = 0$ in a $\delta_n$-neighbourhood of $\Gamma_U$. $\square$

With help of Proposition 10.1 we verify all assumptions of Theorem 4.1 (the verification is contained in Section 12) and as a consequence of [1], Theorem 4.1 and Remark 4.2 for Example 10.1 we obtain

**Theorem 10.1.** *Let* (SIGN) *and* (1.1) *hold, let $\sigma(s)$ be a differentiable curve satisfying* (4.15), *let $d^0 \in C_p$ and let* (4.16) *hold. Let $m$ be the multivalued function from Model Example and let us assume that there exists an eigenfunction $e_p$ corresponding to the eigenvalue $\kappa_p$ of the Laplacian with* (1.3) *such that* (4.13) *is fulfilled with $e = e_p$. Then stationary spatially nonconstant weak solutions (spatial patterns) of* (SRD), (1.2) *with diffusion parameters $d_1 = \sigma_1(s)$ and $d_2 = \sigma_2(s)$ (i.e. of* (9.2), (9.3)*) bifurcate at some $s_I \in (s_0, \tilde{s}]$.*

This is actually [1], Corollary 4.1 and the proof follows from [1], Theorem 4.1 and Remark 4.2 and the fact that no nontrivial constant function can satisfy (1.3).

E x a m p l e 10.2. Let us consider the same multivalued function $m$ as in Model Example and define the corresponding mapping $M\colon \mathbb{V}^2 \to 2^{\mathbb{V}^2}$ by $M(U) = [\{0\}, M_2(v)]$,

$$M_2(v) = \Big\{ z \in \mathbb{V};\ \int_{\Omega_1} \underline{m}(v)\varphi \,\mathrm{d}x \leqslant \langle z, \varphi \rangle \leqslant \int_{\Omega_1} \overline{m}(v)\varphi \,\mathrm{d}x \text{ for all } \varphi \in \mathbb{V}, \varphi \geqslant 0 \text{ a.e. in } \Omega_1 \Big\}$$

for any $v \in \mathbb{V}$, where $\Omega_1 \subset \Omega$ is a given domain such that a $\delta_0$-neighbourhood of $\Omega_1$ (with some $\delta_0 > 0$) belongs to $\Omega$. Then a solution of (9.1) is a weak solution of the problem

$$
\begin{aligned}
&d_1\Delta u + b_{11}u + b_{12}v + n_1(u,v) = 0 && \text{in } \Omega,\\
&d_2\Delta v + b_{21}u + b_{22}v + n_2(u,v) = 0 && \text{in } \Omega \setminus \Omega_1,\\
&d_2\Delta v + b_{21}u + b_{22}v + n_2(u,v) \in m(v) && \text{in } \Omega_1
\end{aligned}
\tag{10.2}
$$

with the boundary conditions (1.3). Such a model describes a similar situation as in Example 10.1 with a source in the interior of the domain $\Omega$.

Let us define the corresponding homogeneous mapping $M_0$ by

$$M_0(U) = [\{0\}, M_{02}(v)]$$

with

$$
\begin{aligned}
&M_{02}(v) = \{z \in \mathbb{V};\ \langle z, v\rangle = 0,\ \langle z, \varphi\rangle \leqslant 0 \text{ for all } \varphi \in \mathbb{V}, \varphi \geqslant 0 \text{ in } \Omega_1\} \text{ if } v \geqslant 0 \text{ in } \Omega_1,\\
&M_{02}(v) = \emptyset \text{ if } v < 0 \text{ in a subset of } \Omega_1 \text{ of a positive measure.}
\end{aligned}
$$

Then a solution of (2.11) is a weak solution of

$$
\begin{aligned}
&d_1\Delta u + b_{11}u + b_{12}v = 0 && \text{in } \Omega,\\
&d_2\Delta v + b_{21}u + b_{22}v = 0 && \text{in } \Omega \setminus \Omega_1,\\
&d_2\Delta v + b_{21}u + b_{22}v \leqslant 0,\ v \geqslant 0,\ (d_2\Delta v + b_{21}u + b_{22}v)\, v = 0 && \text{in } \Omega_1
\end{aligned}
\tag{10.3}
$$

with boundary conditions (1.3). Note that in this example, the problem (2.11) is equivalent to (2.13) with $K = \mathbb{V} \times \{\varphi \in \mathbb{V};\ \varphi \geqslant 0 \text{ in } \Omega_1\}$.

R e m a r k 10.1. Let $v$ be a solution to (2.11) with (10.3). Let us denote $\Omega_v^+ := \{x \in \Omega_1;\ v(x) > 0\}$ and $\Omega_v^0 := \{x \in \Omega_1;\ v(x) = 0\}$. If $\partial\Omega_v^0$ is a Lipschitzian manifold then we have

$$\frac{\partial v}{\partial n} = v = 0 \quad \text{in } \partial\Omega_v^+ \cap \Omega_1$$

(for the proof of this fact see Section 12).

**Proposition 10.2.** *If $v_n \rightharpoonup v$ weakly in $\mathbb{V}$ and $\delta_n \to 0_+$ then $v_n^{\delta_n} \to v$ in $L^2(\Omega_1)$. Moreover, for any $\psi \in \mathbb{V}$, $\psi \geqslant 0$ a.e. in $\Omega_1$ there are $w_n = w_n(\psi) \in \mathbb{V}$ and $\delta_n > 0$ small such that $w_n \to \psi$ strongly in $\mathbb{V}$ and $\Phi^\delta(w_n) \geqslant 0$ in $\Omega_1$ for any $\delta \in (0, \delta_n)$.*

P r o o f. The first assertion follows from the embedding theorems and Proposition 10.1, (iv).

Similarly as in the proof of Proposition 10.1 we can assume without loss of generality that $\psi \in \mathbb{V}$ is such that $\psi \leqslant 0$ in $\Omega$ and $\psi = 0$ a.e. in $\Omega_1$. Therefore, $E\psi \in W_0^{1,2}(\widetilde{G} \setminus \operatorname{cl}\Omega_1)$ with $\widetilde{G} := \operatorname{supp} E\psi \cup \Omega$ and there exist $C^1$-smooth functions $\varphi_n$ with $\operatorname{supp}\varphi_n \subset \widetilde{G} \setminus \operatorname{cl}\Omega_1$ (this implies $\varphi_n \in \mathbb{V}$) such that $\varphi_n \to E\psi$ in $W_0^{1,2}(\widetilde{G})$—see [5]. We take $w_n := \varphi_n$ and there are $\delta_n > 0$ such that $w_n = 0$ in a $\delta_n$-neighbourhood of $\Omega_1$. $\square$

Now we define $M^\delta$ and $M_0^\delta$ in the same way as in Notation 4.1 with $\Gamma_U$ replaced by $\Omega_1$. Propositions 10.1 and 10.2 ensure the assumptions of Theorem 4.1 hold also for such $M$, $M_0$, $M^\delta$, $M_0^\delta$ corresponding to the problem (10.2).

**Theorem 10.2.** *Let* (SIGN) *hold, let $\sigma(s)$ be a differentiable curve satisfying* (4.15), *let $d^0 \in C_p$ and* (4.16) *hold. Let $m$ be the multivalued function from Model Example and let us assume that there exists an eigenfunction $e_p$ corresponding to an eigenvalue $\kappa_p$ of the Laplacian with* (1.3) *such that $e_p \leqslant -\varepsilon$ in a $\delta_0$-neighbourhood of $\Omega_1$ with some $\varepsilon > 0$. Then stationary spatially nonconstant weak solutions of the problem* (10.2), (1.3) *with diffusion parameters $d_1 = \sigma_1(s)$ and $d_2 = \sigma_2(s)$ bifurcate at some $s_I \in (s_0, \tilde{s}]$.*

Again, this follows from Theorem 4.1 and Remark 4.2 from [1] and the fact that no nontrivial constant function can satisfy (1.3).

Example 10.3. Let $a$ be a positive constant and for any $\varphi \in \mathbb{V}$ let us denote

$$\overline{\varphi} := a \int_{\Gamma_U} \varphi(x)\,\mathrm{d}\Gamma.$$

Here e.g. $a := (\operatorname{meas}_{\mathfrak{n}-1} \Gamma_U)^{-1}$ can be taken. Let $m\colon \mathbb{R} \to 2^{\overline{\mathbb{R}}}$ be the function from Model Example. Let us define the corresponding mapping $M\colon \mathbb{V}^2 \to 2^{\mathbb{V}^2}$ by $M(U) = [\{0\}, M_2(v)]$,

$$M_2(v) = \{z \in \mathbb{V};\ \underline{m}(\overline{v})\,\overline{\varphi} \leqslant \langle z, \varphi\rangle \leqslant \overline{m}(\overline{v})\,\overline{\varphi} \ \text{ for all } \varphi \in \mathbb{V}, \overline{\varphi} \geqslant 0\} \tag{10.4}$$

for any $v \in \mathbb{V}$. Then a solution of (9.1) is a weak solution of (9.2) with the boundary conditions

$$\begin{aligned} &u = v = 0 \ \text{ on } \Gamma_D,\\ &\frac{\partial u}{\partial n} = 0,\ \frac{\partial v}{\partial n} = \text{const} \in -\frac{m(\overline{v})}{\sigma_2(s)} \text{ on } \Gamma_U,\\ &\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \ \text{ on } \Gamma_N. \end{aligned} \tag{10.5}$$

The corresponding homogeneous mapping is $M_0(U) = [\{0\}, M_{02}(v)]$ with

$$M_{02}(v) = \{z \in \mathbb{V};\ \langle z, v\rangle = 0,\ \langle z, \varphi\rangle \leqslant 0 \text{ for } \varphi \in \mathbb{V}, \overline{\varphi} \geqslant 0\} \text{ if } \overline{v} \geqslant 0,$$
$$M_{02}(v) = \emptyset \text{ if } \overline{v} < 0.$$

The associated convex cone is $K = \mathbb{V} \times K_2$, $K_2 = \{\varphi \in \mathbb{V};\ \overline{\varphi} \geqslant 0\}$ with $\emptyset \neq \operatorname{int} K_2 = \{\varphi \in \mathbb{V};\ \overline{\varphi} > 0\}$. Again, (2.11) is equivalent to (2.13).

**Theorem 10.3.** *Let* (SIGN) *hold, let $\sigma(s)$ be a differentiable curve satisfying* (4.15)*, let $d^0 \in C_p$ and* (4.16) *hold. Let $m$ be the multivalued function from Model Example and let us assume that there exists an eigenfunction $e_p$ corresponding to an eigenvalue $\kappa_p$ of the Laplacian with* (1.3) *satisfying $\int_{\Gamma_U} e_p \,\mathrm{d}\Gamma < 0$. Then stationary spatially nonconstant weak solutions of* (9.2), (10.5) *bifurcate at some $s_I \in (s_0, \tilde{s}]$.*

This follows from Theorem 4.1 and Remark 4.2 from [1] and the fact that no nontrivial constant function can satisfy (1.3).

Remark 10.2. We have $\operatorname{int} K \neq \emptyset$, therefore no regularization is necessary; we can define $M^\delta := M$, $M_0^\delta := M_0$ and we have $K^\delta = K$, $P_\tau^\delta = P_\tau$.

## 11. Another example, where sensor and source are at different points

In the situation of Examples 10.1–10.3, the homogeneous problem (2.11) is equivalent to the variational inequality (2.13). In the next example we will consider a boundary condition such that the corresponding weak homogeneous problem (2.11) is not equivalent to (2.13).

Example 11.1. (Cf. [2], Section 5.) Let $\Omega = (0,1)$, $\mathbb{V} = \{\varphi \in W_2^1(0,1);\ \varphi(0) = 0\}$. Let $x_0 \in (0,1)$ be fixed. Let us consider the multivalued function $m\colon\ \mathbb{R} \to 2^{\overline{\mathbb{R}}}$ and the corresponding singlevalued functions $\underline{m}$ and $\overline{m}$ as in Model Example. Define the related mapping $M\colon\ \mathbb{V}^2 \to 2^{\mathbb{V}^2}$, $M(U) = [\{0\}, M_2(v)]$ for $U = [u, v]$ by

$$M_2(v) = \{z \in \mathbb{V};\ \underline{m}(v(x_0))\varphi(1) \leqslant \langle z, \varphi\rangle \leqslant \overline{m}(v(x_0))\varphi(1), \varphi \in \mathbb{V}, \varphi(1) \geqslant 0\} \tag{11.1}$$

for any $v \in \mathbb{V}$. Then a solution of (9.1) is a weak solution of the problem (9.2) with the boundary conditions

$$u(0) = v(0) = u_x(1) = 0,\ v_x(1) \in -\frac{m(v(x_0))}{\sigma_2(s)}. \tag{11.2}$$

The multivalued condition in (11.2) describes e.g. a semipermeable membrane on the boundary like in Model Example but with a sensor in the interior of the domain, i.e. the sensor is located at a different point than the source. In the situation of Example 10.1, we had $x_0 = 1$ (in the case $\mathfrak{n} = 1$ and $\Omega = (0,1)$), i.e. the sensor was at the same point as the source (membrane). From this point of view, the multivalued condition in Example 11.1 is more general.

Let us define convex cones $K_{x_0} = \{\varphi \in \mathbb{V};\ \varphi(x_0) \geqslant 0\}$ and $K_1 = \{\varphi \in \mathbb{V};\ \varphi(1) \geqslant 0\}$. The corresponding homogeneous mapping $M_0$ is $M_0(U) = [\{0\}, M_{02}(v)]$, $U = [u, v]$ with

$$\begin{aligned}
&M_{02}(v) = \{0\} \quad \text{ if } v(x_0) > 0,\\
&M_{02}(v) = \{z \in \mathbb{V};\ \langle z, \varphi\rangle \leqslant 0 \text{ for all } \varphi \in K_1\} \ \text{ if } v(x_0) = 0,\\
&M_{02}(v) = \emptyset \quad \text{ if } v(x_0) < 0.
\end{aligned}$$

Then the set $K$ from (2.14) is $\mathbb{V} \times K_{x_0}$. A solution of (2.11) is a weak solution of (2.12), i.e. of

$$d_1 u_{xx} + b_{11} u + b_{12} v = 0, \quad d_2 v_{xx} + b_{21} u + b_{22} v = 0$$

with the boundary conditions

$$\begin{aligned}
&u(0) = v(0) = u_x(1) = 0,\\
&v_x(1) \geqslant 0,\ v(x_0) \geqslant 0,\ v_x(1) \cdot v(x_0) = 0.
\end{aligned} \tag{11.3}$$

A suitable penalty operator for $M$ is $P_\tau(U) = [0, P_{\tau,2}(v)]$ with

$$\langle P_{\tau,2}(v), \varphi\rangle := p_\tau(v(x_0))\varphi(1)$$

for all $v$, $\varphi \in \mathbb{V}$, where $p_\tau$ are the same functions as in Model Example. Set $\mathcal{K} = \mathbb{V} \times (K_1 \cap K_{x_0})$ and consider the condition

$$-U_0 \in E_B(d^0) \cap \operatorname{int} \mathcal{K} \tag{11.4}$$

instead of (4.14). Here, $\operatorname{int} \mathcal{K} = \mathbb{V} \times \{\varphi \in \mathcal{K};\ \varphi(x_0) > 0, \varphi(1) > 0\} \neq \emptyset$. Therefore, we can define $M^\delta := M$, $M_0^\delta := M_0$, $P_\tau^\delta := P_\tau$ for any $\delta > 0$ small. It is easy to see by using Observation 3.3 from [1] that the condition (11.4) is fulfilled for $d^0 \in C_p$ and $U_0 = [\alpha(d^0)e_p, e_p]$, $\alpha(d^0) > 0$, provided the eigenfunction $e_p$ corresponding to the eigenvalue $\kappa_p$ of Laplacian with the boundary conditions

$$u(0) = u_x(1) = 0 \tag{11.5}$$

satisfies $e_p(x_0) < 0$ and $e_p(1) < 0$. Note that the eigenvalues $\kappa_j$ are simple in the one-dimensional case. Therefore the operator $L_\delta$ from (5.9) plays a role only in the case when $d^0 = \sigma(s_0)$ is an intersection point of two different hyperbolas. Replacing $K$ by $\mathcal{K}$ at the appropriate places and (4.14) by (11.4), we can go through the whole procedure used in [1], Sections 6 and 7 and prove the assertion of Theorem 11.1 below also in this situation.

The proofs of all assertions from Sections 6 and 7 in [1] can be done analogously with the exception of the proof of the boundedness of the branch of triplets of solutions to penalty equation in $s$ (see [1], (7.6) and Lemmas 6.5–6.7) where the conditions (4.5) and (4.10) are used. However, now (4.5) and (4.10) are not satisfied for all $U \in \mathbb{V}^2$. We have to strengthen the condition (4.15) by

$$\lim_{s\to+\infty} \sigma_1(s) = +\infty, \qquad \lim_{s\to+\infty} \sigma_2(s) = +\infty \tag{11.6}$$

and prove the fact $s < s_0 + \zeta_1 < +\infty$ with some $\zeta_1 > 0$ in the following way.

Let us assume by contradiction that there exist $s_n$ and $U_n = [u_n, v_n]$ such that $s_n \to +\infty$, $\|U_n\| \to 0$, $W_n = [w_n, z_n] = \frac{U_n}{\|U_n\|} \rightharpoonup W = [w, z]$ and

$$D(\sigma(s_n))U_n - BAU_n - \frac{\tau_n}{1+\tau_n} N(U_n) + \frac{D(\sigma(s_n))}{1+\tau_n} L_\delta(s_n)U_n + P_{\tau_n}(U_n) = 0 \tag{11.7}$$

holds. The embedding theorem gives $w_n \to w$, $z_n \to z$ in $C^0([0,1])$. Writing (11.7) in the components and multiplying the first equation of (11.7) by $w_n\|U_n\|^{-1}$ we obtain

$$\begin{aligned}\sigma_1(s_n)\|w_n\|^2 - b_{11}\langle Aw_n, w_n\rangle - b_{12}\langle Az_n, w_n\rangle - \frac{\tau_n}{1+\tau_n}\Big\langle \frac{N_1(U_n)}{\|U_n\|}, w_n\Big\rangle \\ + \Big\langle \frac{\sigma_1(s_n)}{1+\tau_n} L_\delta(s_n)w_n, w_n\Big\rangle = 0.\end{aligned} \tag{11.8}$$

We have $L_\delta(s_n) \equiv 0$ for $s_n > s_0 + \eta$ directly from the definition of $L_\delta$. If we had $w \neq 0$ then the left hand side of (11.8) would tend to infinity by the assumption (11.6).

Multiplying the second equation of (11.7) by $\varphi\|U_n\|^{-1}$ with $\varphi \in K_1$, $\varphi(1) = 0$ we obtain

$$\begin{aligned}\sigma_2(s_n)\langle z_n, \varphi\rangle - b_{21}\langle Aw_n, \varphi\rangle - b_{22}\langle Az_n, \varphi\rangle - \frac{\tau_n}{1+\tau_n}\Big\langle \frac{N_2(U_n)}{\|U_n\|}, \varphi\Big\rangle \\ + \Big\langle \frac{\sigma_2(s_n)}{1+\tau_n} L_\delta(s_n)z_n, \varphi\Big\rangle = 0\end{aligned} \tag{11.9}$$

(recall that $K_1 = \{\varphi \in \mathbb{V};\ \varphi(1) \geqslant 0\}$). We have $L_\delta(s_n) \equiv 0$ for $s_n > s_0 + \eta$ again. Dividing (11.9) by $\sigma_2(s_n)$ and letting $n \to +\infty$ we obtain $\langle z_n, \varphi\rangle \to 0$ by using

(11.6). We have $z_n \rightharpoonup z$ and it follows that $\langle z, \varphi \rangle = 0$ for arbitrary $\varphi \in \mathbb{V}$, $\varphi(1) = 0$. Thus $z_{xx} = 0$, which implies $z(x) = k \cdot x$ with some $k \in \mathbb{R}$. Note that the embedding theorem ensures $z \in C([0,1])$ and $z \in \mathbb{V}$ gives $z(0) = 0$.

Multiplying the second equation of (11.7) by $\sigma_2^{-1}(s_n) v_n \|U_n\|^{-2}$ and passing to the limit we obtain

$$(11.10) \quad 0 \leqslant \int_0^1 z_x^2 \,\mathrm{d}x = k^2 = -\lim_{n\to+\infty} \frac{p_{\tau_n}(v_n(x_0)) z_n(1)}{\sigma_2(s_n)\|U_n\|} = -k \lim_{n\to+\infty} \frac{p_{\tau_n}(v_n(x_0))}{\sigma_2(s_n)\|U_n\|}.$$

This together with the sign of $p_{\tau_n}$ implies $k \geqslant 0$. If $k = 0$ then $z = 0$ in $[0,1]$. If $k > 0$ then $z(x_0) = kx_0 > 0$ and the $C^0$-convergence of $z_n$ implies $v_n(x_0) > 0$ for $n$ large enough. But $p_{\tau_n}(v_n(x_0)) = 0$ and we obtain from (11.10) that

$$k = -\lim_{n\to+\infty} \frac{p_{\tau_n}(v_n(x_0))}{\sigma_2(s_n)\|U_n\|} = 0.$$

This contradicts $k > 0$. Therefore $z = 0$, which is a contradiction with the fact that $\|W\| = \|[w, z]\| = 1$.

We obtain

**Theorem 11.1.** *Let* (SIGN) *hold, let $\sigma(s)$ be a differentiable curve satisfying* (4.15) *and* (11.6), *let $d^0 \in C_p$ and* (4.16) *hold. Let $m$ be the multivalued function from Model Example and let us assume that there exists an eigenfunction $e_p$ corresponding to an eigenvalue $\kappa_p$ of the Laplacian with* (11.5) *satisfying $e_p(x_0) < 0$ and $e_p(1) < 0$. Then stationary spatially nonconstant weak solutions (spatial patterns) of* (9.2), (11.2) *bifurcate at some $s_I \in (s_0, +\infty)$.*

This follows from [1], Theorem 4.1 and Remark 4.2 and from the considerations above. Let us note that in this situation we have no information like $s_I < \tilde{s}$.

## 12. Validity of Propositions 4.1–4.5 for the examples investigated above

Let us consider the mappings as in Model Example. Recall that $K_2 = \{\varphi \in \mathbb{V}; \varphi \geqslant 0 \text{ a.e. on } \Gamma_U\}$. First, we need to show that a solution of (9.1) is a weak solution of the problem (9.2), (9.3). For the sake of simplicity we will write $d_1, d_2$ instead of

$\sigma_1(s), \sigma_2(s)$. The inclusion (9.1) is equivalent to the following couple of formulae:

$$\int_\Omega d_1 \sum_{j=1}^{\mathfrak{n}} u_{x_j}\varphi_{x_j} - (b_{11}u + b_{12}v + n_1(u,v))\varphi \,\mathrm{d}x = 0 \quad \text{for any } \varphi \in \mathbb{V}, \tag{12.1}$$

$$\begin{aligned} -\int_{\Gamma_U} \overline{m}(v)\psi \,\mathrm{d}\Gamma &\leqslant \int_\Omega d_2 \sum_{j=1}^{\mathfrak{n}} v_{x_j}\psi_{x_j} - (b_{21}u + b_{22}v + n_2(u,v))\psi \,\mathrm{d}x \\ &\leqslant -\int_{\Gamma_U} \underline{m}(v)\psi \,\mathrm{d}\Gamma \quad \text{for any } \psi \in K_2. \end{aligned} \tag{12.2}$$

Moreover,

$$\int_\Omega d_1 \sum_{j=1}^{\mathfrak{n}} u_{x_j}\varphi_{x_j} - \big(b_{11}u + b_{12}v + n_1(u,v)\big)\varphi \,\mathrm{d}x = 0 \quad \text{for any } \varphi \in \mathcal{D}(\Omega),$$

$$\int_\Omega d_2 \sum_{j=1}^{\mathfrak{n}} v_{x_j}\psi_{x_j} - \big(b_{21}u + b_{22}v + n_2(u,v)\big)\psi \,\mathrm{d}x = 0 \quad \text{for any } \psi \in \mathcal{D}(\Omega),$$

i.e. (9.2) is satisfied in the sense of distributions. Therefore the equations

$$d_1\Delta u + b_{11}u + b_{12}v + n_1(u,v) = 0, \tag{12.3}$$

$$d_2\Delta v + b_{21}u + b_{22}v + n_2(u,v) = 0 \tag{12.4}$$

are satisfied a.e. in $\Omega$, where all terms are represented by functions from $L^2(\Omega)$. For any $u$, $v \in \mathbb{H}$ we can use Green's formula for (12.1), (12.2) to obtain

$$\int_\Omega \big(d_1\Delta u + b_{11}u + b_{12}v + n_1(u,v)\big)\varphi \,\mathrm{d}x - d_1 \int_{\partial\Omega} \mathfrak{T}u \;\varphi \,\mathrm{d}\Gamma = 0 \quad \forall \varphi \in \mathbb{V}, \tag{12.5}$$

$$\begin{aligned} \int_{\Gamma_U} \underline{m}(v)\psi \,\mathrm{d}\Gamma &\leqslant \int_\Omega \big(d_2\Delta v + b_{21}u + b_{22}v + n_2(u,v)\big)\psi \,\mathrm{d}x \\ &\quad - d_2 \int_{\partial\Omega} \mathfrak{T}v \;\psi \,\mathrm{d}\Gamma \leqslant \int_{\Gamma_U} \overline{m}(v)\psi \,\mathrm{d}\Gamma, \quad \forall \psi \in K_2. \end{aligned} \tag{12.6}$$

With help of (12.3) and (12.4) in (12.5) and (12.6) we obtain $\mathfrak{T}u = 0$ on $\Gamma_U \cup \Gamma_N$ and

$$\int_{\Gamma_U} \underline{m}(v)\psi \,\mathrm{d}\Gamma \leqslant -d_2 \int_{\partial\Omega} \mathfrak{T}v \;\psi \,\mathrm{d}\Gamma \leqslant \int_{\Gamma_U} \overline{m}(v)\psi \,\mathrm{d}\Gamma \qquad \text{for all } \psi \in K_2. \tag{12.7}$$

It follows from the definition of $\mathbb{V}$ that $u = v = 0$ on $\Gamma_D$ in the sense of traces. The choice $\psi = 0$ on $\Gamma_U$ in (12.7) gives $\mathfrak{T}v = 0$ on $\Gamma_N$ while $\psi \geqslant 0$ on $\Gamma_U$ gives $-d_2\mathfrak{T}v \in m(v)$ on $\Gamma_U$.

In a similar way we can show that (9.1) is a weak formulation of (10.2) with (1.3) or of (9.2) with (10.5) in the situation of Example 10.2 or 10.3, respectively.

Now we shall prove Propositions 4.1–4.5, i.e. we shall verify the conditions (4.1)–(4.6) and (4.11)–(4.12) for $M^\delta$, $M_0^\delta$ and $P_\tau^\delta$ under the situation from Examples 10.1–10.3 and/or 11.1. For the sake of simplicity, we will write $v^\delta$ instead of $\Phi^\delta(v)$.

In the situation of Example 10.1 (i.e. Model Example in the sense of [1]) we define for $\tau \geqslant 0$ the functions $p_\tau$ by

$$p_\tau(\xi) := \begin{cases} 0 & \xi \geqslant 0, \\ \tau\xi & \xi \in [\xi_\tau, 0) \qquad \xi_\tau := \frac{m^0}{\sqrt{1+\tau^2}}, \\ k(\tau)m(\xi) & \xi < \xi_\tau \qquad k(\tau) := \frac{m^0}{m(\xi_\tau)}\frac{\tau}{\sqrt{1+\tau^2}}. \end{cases}$$

The definitions of $M^\delta, M_0^\delta, P_\tau^\delta$ include the triviality of their first components. Hence only the second components are essential. Definition of $M_0^\delta$ yields that $K^\delta = \{U = [u,v] \in \mathbb{V}^2;\ v^\delta \geqslant 0 \text{ on } \Gamma_U\}$ and (4.1)–(4.6) are satisfied. It is easy to see that $\operatorname{int} K^\delta = \{U = [u,v] \in \mathbb{V}^2;\ v^\delta > 0 \text{ on } \operatorname{cl}\Gamma_U\}$.

Proof of Proposition 4.1. Let $U_n \to 0$, $W_n = [\xi_n, w_n] = \frac{U_n}{\|U_n\|} \rightharpoonup W = [\xi, w]$, $Z_n = [\eta_n, z_n] \to Z = [\eta, z]$, $d_n \to d \in \mathbb{R}_+^2$, $D(d_n)W_n + Z_n \in -\frac{M^\delta(U_n)}{\|U_n\|}$. This yields for the first coordinate that $d_1^n \xi_n = -\eta_n$ and immediately $\xi_n \to \xi$ because $\eta_n \to \eta$. For the second coordinate the inclusion gives

$$\begin{aligned} -\int_{\Gamma_U} \frac{\underline{m}(v_n^\delta)}{\|U_n\|}\big[\varphi^\delta\big]^+ \mathrm{d}\Gamma + \int_{\Gamma_U} \frac{\overline{m}(v_n^\delta)}{\|U_n\|}\big[\varphi^\delta\big]^- \mathrm{d}\Gamma &\geqslant \langle d_2^n w_n + z_n, \varphi\rangle \\ \geqslant -\int_{\Gamma_U} \frac{\overline{m}(v_n^\delta)}{\|U_n\|}\big[\varphi^\delta\big]^+ \mathrm{d}\Gamma + \int_{\Gamma_U} \frac{\underline{m}(v_n^\delta)}{\|U_n\|}\big[\varphi^\delta\big]^- \mathrm{d}\Gamma &\quad \text{for all } \varphi \in \mathbb{V}. \end{aligned} \tag{12.8}$$

We obtain by using the appropriate part of (12.8) that

$$d_2^n\langle w_n, w_n\rangle \leqslant -\int_{\Gamma_U} \frac{\underline{m}(v_n^\delta)}{\|U_n\|}\big[w_n^\delta\big]^+ \mathrm{d}\Gamma + \int_{\Gamma_U} \frac{\overline{m}(v_n^\delta)}{\|U_n\|}\big[w_n^\delta\big]^- \mathrm{d}\Gamma - \langle z_n, w_n\rangle, \tag{12.9}$$

$$d_2^n\langle w_n, w\rangle \geqslant -\int_{\Gamma_U} \frac{\overline{m}(v_n^\delta)}{\|U_n\|}\big[w^\delta\big]^+ \mathrm{d}\Gamma + \int_{\Gamma_U} \frac{\underline{m}(v_n^\delta)}{\|U_n\|}\big[w^\delta\big]^- \mathrm{d}\Gamma - \langle z_n, w\rangle. \tag{12.10}$$

We have

$$\underline{m}(v_n^\delta)\big[w_n^\delta\big]^+ = 0,\ \overline{m}(v_n^\delta)\big[w_n^\delta\big]^- \leqslant 0,\ \overline{m}(v_n^\delta)\big[w^\delta\big]^+ \leqslant 0,\ \underline{m}(v_n^\delta)\big[w^\delta\big]^- \leqslant 0 \tag{12.11}$$

on $\Gamma_U$. The embedding theorem gives $v_n \to 0$, $w_n \to w$ and $v_n^\delta \to 0$ in $L^2(\partial\Omega)$ and consequently $v_n^\delta \to 0$ and $w_n^\delta \to w^\delta$ in $C^0(\operatorname{cl}\partial\Omega)$.

Now we will show that $w^\delta \geqslant 0$ on $\Gamma_U$. Let us assume by contradiction that there is an $\varepsilon_0 > 0$ and a set $\mathcal{E} \subset \Gamma_U$ with $\mathrm{meas}_{\mathrm{n}-1}\,\mathcal{E} > 0$ such that $w^\delta < -\varepsilon_0$ on $\mathcal{E}$. We have $v_n^\delta < 0$ on $\mathcal{E}$ for $n$ large enough, consequently $\overline{m}(v_n^\delta) \to m^0 < 0$ and $\frac{\overline{m}(v_n^\delta)}{v_n^\delta} \to +\infty$ on $\mathcal{E}$. Furthermore, there are $n_0$ and $c < 0$ such that for all $n \geqslant n_0$ we have $w_n^\delta < c$ on $\mathcal{E}$. Then the Fatou lemma yields

$$\begin{aligned}\limsup_{n\to+\infty} \int_{\Gamma_U} \frac{\overline{m}(v_n^\delta)}{\|U_n\|}\big[w_n^\delta\big]^- \,\mathrm{d}\Gamma &\leqslant \limsup_{n\to+\infty} \int_{\mathcal{E}} \frac{\overline{m}(v_n^\delta)}{\|U_n\|}\big[w_n^\delta\big]^- \,\mathrm{d}\Gamma \\ &\leqslant \limsup_{n\to+\infty} \int_{\mathcal{E}} \frac{\overline{m}(v_n^\delta)}{v_n^\delta}\frac{v_n^\delta}{\|U_n\|}\big[w_n^\delta\big]^- \,\mathrm{d}\Gamma \to -\infty,\end{aligned}$$

which contradicts (12.9) because $\langle z_n, w_n\rangle \to \langle z, w\rangle \in \mathbb{R}$. Therefore $w^\delta \geqslant 0$ on $\Gamma_U$ and the second integral in (12.10) vanishes. This together with (12.9) and (12.11) gives

$$\begin{aligned} d_2^n\langle w_n + z_n, w\rangle &\geqslant -\int_{\Gamma_U} \frac{\overline{m}(v_n^\delta)}{\|U_n\|}\big[w^\delta\big]^+ \,\mathrm{d}\Gamma \geqslant 0 \\ &\geqslant \int_{\Gamma_U} \frac{\overline{m}(v_n^\delta)}{\|U_n\|}\big[w_n^\delta\big]^- \,\mathrm{d}\Gamma \geqslant d_2^n\langle w_n + z_n, w_n\rangle. \end{aligned} \tag{12.12}$$

The assumptions $d_2^n \to d_2 > 0$, $z_n \to z$ together with (12.12) imply $\|w\|^2 \geqslant \limsup\limits_{n\to+\infty} \|w_n\|^2$, hence $w_n \to w$ in $\mathbb{V}$.

Now, (12.8) implies

$$\langle d_2^n w_n + z_n, \varphi\rangle \geqslant -\int_{\Gamma_U} \frac{\overline{m}(v_n^\delta)}{\|U_n\|}\varphi^\delta \,\mathrm{d}\Gamma \geqslant 0 \text{ for all } \varphi \in \mathbb{V}, \varphi^\delta \geqslant 0 \text{ on } \Gamma_U$$

and it follows that $\langle d_2 w + z, \varphi\rangle \geqslant 0$ for all $\varphi \in \mathbb{V}, \varphi^\delta \geqslant 0$ on $\Gamma_U$. By choosing $\varphi := w$ we have

$$\langle d_2 w + z, w\rangle = \lim_{n\to+\infty} \langle d_2^n w_n + z_n, w\rangle \geqslant -\int_{\Gamma_U} \frac{\overline{m}(v_n^\delta)}{\|U_n\|} w^\delta \,\mathrm{d}\Gamma \geqslant 0$$

and on the other hand, the last two inequalities in (12.12) give

$$\langle d_2 w + z, w\rangle = \lim_{n\to+\infty} \langle d_2^n w_n + z_n, w_n\rangle \leqslant \int_{\Gamma_U} \frac{\overline{m}(v_n^\delta)}{\|U_n\|}\big[w_n^\delta\big]^- \,\mathrm{d}\Gamma \leqslant 0.$$

It follows that $\langle d_2 w + z, w\rangle = 0$ and, by the definition of $M_{02}^\delta$, $d_2 w + z \in -M_{02}^\delta(w)$. $\square$

P r o o f o f P r o p o s i t i o n 4.2. We shall prove $\lim_{n\to+\infty} |\langle D^{-1}(d_n)P^\delta_{\tau_n,2}(U_n), U_n - U\rangle| = 0$. If $v_n \rightharpoonup v$ in $W^{1,2}(\Omega)$ then $v_n \to v$ in $L^2(\partial\Omega)$ and also $v_n^\delta \to v^\delta$ in $L^2(\partial\Omega)$.

Let $\tau_n \to \tau$. Let us suppose first that $\tau < +\infty$. Lemma 9.1 gives $p_{\tau_n}(v_n^\delta) \to p_\tau(v^\delta)$ in $L^2(\Gamma_U)$, which implies that $p_{\tau_n}(v_n^\delta)$ are bounded in $L^2(\Gamma_U)$. We have

$$\begin{aligned}\lim_{n\to+\infty} \big|\langle P^\delta_{\tau_n,2}(v_n), v_n - v\rangle\big| &\leqslant \lim_{n\to+\infty} \int_{\Gamma_U} |p_{\tau_n}(v_n^\delta)(v_n^\delta - v^\delta)|\,\mathrm{d}\Gamma \\ &\leqslant \lim_{n\to+\infty} \|p^{\tau_n}(v_n^\delta)\|_{L^2(\Gamma_U)} \cdot \|v_n^\delta - v^\delta\|_{L^2(\Gamma_U)} = 0.\end{aligned}$$

Now, let $\tau = +\infty$. We take a certain sufficiently small $\varepsilon > 0$ and define a continuous function $\underline{m}_\varepsilon\colon \mathbb{R} \to \mathbb{R}$ such that $\underline{m}_\varepsilon(\xi) = \underline{m}(\xi)$ for any $\xi \in (-\infty, 0]\cup[\varepsilon, +\infty)$ and $\underline{m}_\varepsilon$ is continuous and negative on $(0, \varepsilon)$. The Nemytskii theorem implies $\underline{m}_\varepsilon(v_n^\delta) \to \underline{m}_\varepsilon(v^\delta)$ in $L^2(\Gamma_U)$ for $n \to +\infty$ with $\varepsilon$ fixed. We have

$$\begin{aligned}\lim_{n\to+\infty} |\langle P^\delta_{\tau_n,2}(v_n), v_n - v\rangle| &= \lim_{n\to+\infty} \Big|\int_{\Gamma_U} p_{\tau_n}(v_n^\delta)(v_n^\delta - v^\delta)\,\mathrm{d}\Gamma\Big| \\ &\leqslant \lim_{n\to+\infty} \int_{\Gamma_U} |\underline{m}_\varepsilon(v_n^\delta)(v_n^\delta - v^\delta)|\,\mathrm{d}\Gamma \\ &\leqslant \lim_{n\to+\infty} \|\underline{m}_\varepsilon(v_n^\delta)\|_{L^2(\Gamma_U)} \cdot \|v_n^\delta - v^\delta\|_{L^2(\Gamma_U)} = 0.\end{aligned} \tag{12.13}$$

The second part of (4.12) can be proved by the same considerations by using $w_n$, $w$ and $\frac{P^\delta_{\tau_n,2}(v_n)}{\|U_n\|}$ instead of $v_n$, $v$ and $P^\delta_{\tau_n,2}(v_n)$ and with help of the assumption that $\frac{P^\delta_{\tau_n,2}(v_n)}{\|U_n\|}$ are bounded. □

P r o o f o f P r o p o s i t i o n 4.3. **A:** If $v_n \rightharpoonup v$ in $\mathbb{V}$ and $\tau_n \to \tau \in [0, +\infty)$, then $v_n \to v$ in $L^2(\partial\Omega)$. Lemma 9.1 gives $p_{\tau_n}(v_n^\delta) \to p_\tau(v^\delta)$ in $L^2(\Gamma_U)$. We have

$$\sup_{\|\varphi\|\leqslant 1} \langle P^\delta_{\tau_n,2}(v_n) - P^\delta_{\tau,2}(v), \varphi\rangle = \sup_{\|\varphi\|\leqslant 1} \int_{\Gamma_U} \big[p_{\tau_n}(v_n^\delta) - p_\tau(v^\delta)\big]\varphi^\delta\,\mathrm{d}\Gamma \to 0.$$

Let $\tau \to +\infty$. For any sufficiently small $\varepsilon > 0$ we define a function $\underline{m}_\varepsilon\colon \mathbb{R} \to \mathbb{R}$ such that $\underline{m}_\varepsilon(\xi) = \underline{m}(\xi)$ for any $\xi \in (-\infty, 0]\cup[\varepsilon, +\infty)$, $\underline{m}_\varepsilon$ is continuous and negative on $(0, \varepsilon)$, $\underline{m}_{\varepsilon_1} \leqslant \underline{m}_{\varepsilon_2}$ for $\varepsilon_1 \geqslant \varepsilon_2$ and $\underline{m}_\varepsilon \to \underline{m}$ with $\varepsilon \to 0$. Similarly, let us define continuous functions $\overline{m}_\varepsilon\colon \mathbb{R} \to \mathbb{R}$ by $\overline{m}_\varepsilon := p_{1/\varepsilon}$. Then $\overline{m}_\varepsilon(\xi) = \overline{m}(\xi)$ for $\xi \geqslant 0$, $\overline{m}_\varepsilon(\xi) \geqslant \overline{m}(\xi)$ for $\xi < 0$, $\overline{m}_{\varepsilon_1} \geqslant \overline{m}_{\varepsilon_2}$ for $\varepsilon_1 \geqslant \varepsilon_2$ and $\overline{m}_\varepsilon \to \overline{m}$ with $\varepsilon \to 0$. Therefore, for all $\varepsilon > 0$ we have $p_{\tau_n} \leqslant \overline{m}_\varepsilon$ on $\mathbb{R}$ for $n$ large enough. For such $n$ we have

$$\begin{aligned}&\int_{\Gamma_U} \underline{m}_\varepsilon(v_n^\delta)\big[\varphi^\delta\big]^+\,\mathrm{d}\Gamma - \int_{\Gamma_U} \overline{m}_\varepsilon(v_n^\delta)\big[\varphi^\delta\big]^-\,\mathrm{d}\Gamma \\ &\qquad\leqslant \int_{\Gamma_U} p_{\tau_n}(v_n^\delta)\big[\varphi^\delta\big]^+\,\mathrm{d}\Gamma - \int_{\Gamma_U} p_{\tau_n}(v_n^\delta)\big[\varphi^\delta\big]^-\,\mathrm{d}\Gamma = \langle P^\delta_{\tau_n,2}(v_n), \varphi\rangle\end{aligned}$$

for any $\varphi \in \mathbb{V}$. The Nemytskii theorem implies $\underline{m}_\varepsilon(v_n^\delta) \to \underline{m}_\varepsilon(v^\delta)$, $\overline{m}_\varepsilon(v_n^\delta) \to \overline{m}_\varepsilon(v^\delta)$ in $L^2(\Gamma_U)$ for $n \to +\infty$ with $\varepsilon$ fixed. If $P^\delta_{\tau_n,2}(v_n) \to \psi$ in $\mathbb{V}$ then the limiting process $n \to +\infty$ gives

$$\int_{\Gamma_U} \underline{m}_\varepsilon(v^\delta)\big[\varphi^\delta\big]^+ \,\mathrm{d}\Gamma - \int_{\Gamma_U} \overline{m}_\varepsilon(v^\delta)\big[\varphi^\delta\big]^- \,\mathrm{d}\Gamma \leqslant \langle z, \varphi\rangle$$

for any $\varphi \in \mathbb{V}$. The Levi theorem gives

(12.14)
$$\begin{aligned}\int_{\Gamma_U} \underline{m}(v^\delta)\big[\varphi^\delta\big]^+ \,\mathrm{d}\Gamma &- \int_{\Gamma_U} \overline{m}(v^\delta)\big[\varphi^\delta\big]^- \,\mathrm{d}\Gamma \\ &= \lim_{\varepsilon\to 0_+} \int_{\Gamma_U} \underline{m}_\varepsilon(v^\delta)\big[\varphi^\delta\big]^+ \,\mathrm{d}\Gamma - \int_{\Gamma_U} \overline{m}_\varepsilon(v^\delta)\big[\varphi^\delta\big]^- \,\mathrm{d}\Gamma \leqslant \langle z, \varphi\rangle\end{aligned}$$

for any $\varphi \in \mathbb{V}$.

In a similar way we can prove

$$\int_{\Gamma_U} \overline{m}(v^\delta)[\varphi^\delta]^+ \,\mathrm{d}\Gamma - \int_{\Gamma_U} \underline{m}(v^\delta)[\varphi^\delta]^- \,\mathrm{d}\Gamma \geqslant \langle z, \varphi\rangle \tag{12.15}$$

for any $\varphi \in \mathbb{V}$. The inequalities in (12.14) and (12.15) imply $z \in M_2^\delta(v)$.

**B:** The assumptions $v_n \to 0$ in $\mathbb{V}$, $w_n := \frac{v_n}{\|U_n\|} \rightharpoonup w$ in $\mathbb{V}$ and $\tau_n \to 0$ together with the embedding theorem give that $v_n \to 0$ and $w_n \to w$ in $L^2(\partial\Omega)$ and $w_n^\delta \to w^\delta$ in $C^0(\operatorname{cl}\Gamma_U)$. For a fixed $\tau_0$ and $n$ large enough we have $v_n^\delta > \xi_{\tau_0}$ on $\Gamma_U$ and

$$\Big|\frac{p_{\tau_n}(v_n^\delta)}{v_n^\delta}\Big| \leqslant \Big|\frac{\tau_n v_n^\delta}{v_n^\delta}\Big| = \tau_n \to 0 \quad \text{ on } \Gamma_U.$$

We obtain

$$\begin{aligned}\Big|\sup_{\|\varphi\|\leqslant 1}\Big\langle \frac{P^\delta_{\tau_n,2}(v_n)}{\|U_n\|}, \varphi\Big\rangle\Big| &= \Big|\sup_{\|\varphi\|\leqslant 1}\int_{\Gamma_U} \frac{p_{\tau_n}(v_n^\delta)}{\|U_n\|}\varphi^\delta \,\mathrm{d}\Gamma\Big| = \Big|\sup_{\|\varphi\|\leqslant 1}\int_{\Gamma_U} \frac{p_{\tau_n}(v_n^\delta)}{v_n^\delta} w_n^\delta \varphi^\delta \,\mathrm{d}\Gamma\Big| \\ &\leqslant \sup_{\|\varphi\|\leqslant 1} \tau_n \cdot \|w_n^\delta\|_{L^2(\Gamma_U)} \cdot \|\varphi^\delta\|_{L^2(\Gamma_U)} \to 0.\end{aligned}$$

**C:** Let $U_n \to 0$, $W_n = \frac{U_n}{\|U_n\|} \rightharpoonup W$, $\tau_n \to \tau \in [0, +\infty)$. The embedding theorem gives that $v_n \to 0$, $w_n := \frac{v_n}{\|U_n\|} \to w$ in $L^2(\partial\Omega)$ and $v_n^\delta \to 0$, $w_n^\delta \to w^\delta$ in $C^0(\operatorname{cl}\partial\Omega)$. Set $\mathcal{E} := \{x \in \Gamma_U;\ w^\delta < 0\}$ and for $\tau \geqslant 0$ introduce $p_{\tau,0}\colon \xi \mapsto \begin{cases} 0, & \xi \geqslant 0, \\ \tau\xi, & \xi \leqslant 0. \end{cases}$, i.e. $p_{\tau,0} = p_\tau$ for $\xi \geqslant \xi_\tau$. Hence from the $C^0$-convergence of $v_n^\delta$ we have

$$\begin{aligned}\Big|\sup_{\|\varphi\|\leqslant 1}\Big\langle \frac{P^\delta_{\tau_n,2}(v_n)}{\|U_n\|} - P^\delta_{\tau_n,0,2}(w), \varphi\Big\rangle\Big| &\leqslant \sup_{\|\varphi\|\leqslant 1}\Big|\int_{\mathcal{E}} \Big[\frac{p_{\tau_n}(v_n^\delta)}{v_n^\delta} w_n^\delta - \tau w^\delta\Big]\varphi^\delta \,\mathrm{d}\Gamma\Big| \\ &\leqslant \sup_{\|\varphi\|\leqslant 1}\Big\|\frac{p_{\tau_n}(v_n^\delta)}{v_n^\delta} w_n^\delta - \tau\cdot w^\delta\Big\|_{L^2(\Gamma_U)} \cdot \|\varphi^\delta\|_{L^2(\Gamma_U)} \to 0.\end{aligned}$$

Let now $\tau_n \to +\infty$ and $\frac{P^\delta_{\tau_n}(U_n)}{\|U_n\|} \to Z$. By the same considerations as in the proof of Proposition 4.1 (using $p_{\tau_n}(v^\delta_n)$ instead of $\overline{m}(v^\delta_n)$) we can show that $w^\delta \geqslant 0$ on $\Gamma_U$: Let us suppose that there is an $\varepsilon_0 > 0$ and a set $\mathcal{E} \subset \Gamma_U$ with $\mathrm{meas}_{\mathrm{n}-1}\,\mathcal{E} > 0$ such that $w^\delta < -\varepsilon_0$ on $\mathcal{E}$. The $C^0$-convergence of $v^\delta_n$ ensures the existence of $\tau_0 \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that $v^\delta_n \in (\xi_{\tau_0}, 0)$ on $\mathcal{E}$ for all $n \geqslant n_0$. Then $\frac{p_{\tau_n}(v^\delta_n)}{v^\delta_n} = \tau_n \to +\infty$ for $n \to +\infty$ on $\mathcal{E}$. The Fatou lemma gives

$$\limsup_{n\to+\infty} \int_{\Gamma_U} \frac{p_{\tau_n}(v^\delta_n)}{\|U_n\|}\varphi^\delta \,\mathrm{d}\Gamma \leqslant \limsup_{n\to+\infty} \int_{\mathcal{E}} \frac{p_{\tau_n}(v^\delta_n)}{v^\delta_n}\frac{v^\delta_n}{\|U_n\|}\varphi^\delta \,\mathrm{d}\Gamma = -\infty \tag{12.16}$$

for any $\varphi \in \mathbb{V}$ such that $\varphi^\delta \geqslant 0$ on $\Gamma_U$ and $\varphi^\delta > 0$ on $\mathcal{E}$, which is the contradiction with

$$\int_{\Gamma_U} \frac{p_{\tau_n}(v^\delta_n)}{\|U_n\|}\varphi^\delta \,\mathrm{d}\Gamma \to \langle z, \varphi\rangle. \tag{12.17}$$

This implies $w^\delta \geqslant 0$ on $\Gamma_U$, i.e. $w \in K^\delta_2$. Moreover, (12.17) and the sign of $p_\tau$ give $\langle z, \varphi\rangle \leqslant 0$ for all $\varphi \in K^\delta_2$ and for $\varphi := w$ we obtain $\langle z, w\rangle \leqslant 0$. On the other hand, the choice $\varphi := w_n$ implies

$$\langle z, w\rangle = \lim_{n\to+\infty} \int_{\Gamma_U} \frac{p_{\tau_n}(v^\delta_n)}{\|U_n\|} w^\delta_n \,\mathrm{d}\Gamma \geqslant 0,$$

because the signs of $p_{\tau_n}(v^\delta_n)$ and $w^\delta_n$ are the same on $\Gamma_U$. We obtain $\langle z, w\rangle = 0$, therefore $z \in M^\delta_{02}(w)$ by definition. □

Proof of Proposition 4.4. **A:** Let $U_n \to 0$, $W_n = \frac{U_n}{\|U_n\|} \rightharpoonup W \notin K^\delta$, $\tau_n \to \tau_0 > 0$ and $V \in \operatorname{int} K^\delta$. The embedding theorems give $v_n \to 0$ and $w_n \to w$ in $L^2(\partial\Omega)$ and $v^\delta_n \to v^\delta$, $w^\delta_n \to w^\delta$ in $C^0(\operatorname{cl}\Gamma_U)$. The assumption $W \notin K^\delta$ ensures the existence of an $\varepsilon_0 > 0$ and a set $\mathcal{E} \subset \Gamma_U$ with $\mathrm{meas}_{\mathrm{n}-1}\,\mathcal{E} > 0$ such that $w^\delta < -\varepsilon_0$ on $\mathcal{E}$. Then there exist $\tau_0 \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that $v^\delta_n \in (\xi_{\tau_0}, 0)$ on $\mathcal{E}$ for all $n \geqslant n_0$. The assumption $V = [y, z] \in \operatorname{int} K^\delta$ means $z^\delta > 0$ on $\Gamma_U$. For $\varepsilon_n \to 0_+$ we obtain

$$\begin{aligned}\limsup_{n\to+\infty} \Big\langle \frac{P^\delta_{\tau_n}(U_n)}{\|U_n\|}, V\Big\rangle &= \limsup_{n\to+\infty} \Big\langle \frac{P^\delta_{\tau_n,2}(v_n)}{\|U_n\|}, z\Big\rangle = \limsup_{n\to+\infty} \int_{\Gamma_U} \frac{p_{\tau_n}(v^\delta_n)}{\|U_n\|} z^\delta \,\mathrm{d}\Gamma \\ &\leqslant \limsup_{n\to+\infty} \int_{\mathcal{E}} \frac{p_{\tau_n}(v^\delta_n)}{v^\delta_n} w^\delta_n z^\delta \,\mathrm{d}\Gamma \leqslant \limsup_{n\to+\infty} \int_{\mathcal{E}} \frac{(\tau_0 - \varepsilon_n)\cdot v^\delta_n}{v^\delta_n} w^\delta_n z^\delta \,\mathrm{d}\Gamma < 0.\end{aligned} \tag{12.18}$$

**B:** The proof of the second part is similar—in the final line of (12.18) we use the fact that

$$\frac{p_{\tau_n}(v^\delta_n)}{\tau_n v^\delta_n} = \frac{\tau_n v^\delta_n}{\tau_n v^\delta_n} = 1$$

on $\mathcal{E}$ for $n$ large enough. □

Proof of Proposition 4.5, (4.11). Let $\delta_n \to 0_+$, $U_n = [u_n, v_n] \rightharpoonup U$, $Z_n = [\eta_n, z_n] \to Z$, $d_n \to d \in \mathbb{R}^2_+$, $D(d_n)U_n + Z_n \in -M^{\delta_n}(U_n)$. The first part of the inclusion is the equation

$$d_1^n u_n + \eta_n = 0$$

and we have immediately $u_n \to u$ because $\eta_n \to \eta$. The second part of the inclusion gives

(12.19)
$$-\int_{\Gamma_U} \underline{m}(v_n^{\delta_n})\big[\varphi^{\delta_n}\big]^+ \mathrm{d}\Gamma + \int_{\Gamma_U} \overline{m}(v_n^{\delta_n})\big[\varphi^{\delta_n}\big]^- \mathrm{d}\Gamma \geqslant \langle d_2^n v_n + z_n, \varphi\rangle$$
$$\geqslant -\int_{\Gamma_U} \overline{m}(v_n^{\delta_n})\big[\varphi^{\delta_n}\big]^+ \mathrm{d}\Gamma + \int_{\Gamma_U} \underline{m}(v_n^{\delta_n})\big[\varphi^{\delta_n}\big]^- \mathrm{d}\Gamma \quad \text{for all } \varphi \in \mathbb{V}.$$

The embedding theorem together with Proposition 10.1, (iv) give $v_n \to v$ and $v_n^{\delta_n} \to v$ in $L^2(\partial\Omega)$. By using the appropriate part of (12.19) we obtain that

$$\langle d_2^n v_n + z_n, v_n\rangle \leqslant -\int_{\Gamma_U} \underline{m}(v_n^{\delta_n})\big[v_n^{\delta_n}\big]^+ \mathrm{d}\Gamma + \int_{\Gamma_U} \overline{m}(v_n^{\delta_n})\big[v_n^{\delta_n}\big]^- \mathrm{d}\Gamma, \tag{12.20}$$

$$d_2^n \langle v_n + z_n, v\rangle \geqslant -\int_{\Gamma_U} \overline{m}(v_n^{\delta_n})\big[v^{\delta_n}\big]^+ \mathrm{d}\Gamma + \int_{\Gamma_U} \underline{m}(v_n^{\delta_n})\big[v^{\delta_n}\big]^- \mathrm{d}\Gamma. \tag{12.21}$$

The terms have the following signs on $\Gamma_U$:

$$\underline{m}(v_n^{\delta_n})\big[v_n^{\delta_n}\big]^+ = 0,\ \overline{m}(v_n^{\delta_n})\big[v_n^{\delta_n}\big]^- \leqslant 0,\ \overline{m}(v_n^{\delta_n})\big[v^{\delta_n}\big]^+ \leqslant 0,\ \underline{m}(v_n^{\delta_n})\big[v^{\delta_n}\big]^- \leqslant 0. \tag{12.22}$$

For any fixed $\varepsilon > 0$ small let us define continuous functions $\underline{m}_\varepsilon, \overline{m}_\varepsilon\colon \mathbb{R} \to \mathbb{R}$ such that $\underline{m}_\varepsilon(\xi) = \underline{m}(\xi)$ for any $\xi \in (-\infty, 0] \cup [\varepsilon, +\infty)$ and $\underline{m}_\varepsilon$ is negative on $(0, \varepsilon)$, and $\overline{m}_\varepsilon(\xi) = \overline{m}(\xi)$ for any $\xi \in (-\infty, -\varepsilon] \cup [0, +\infty)$ and $\overline{m}_\varepsilon$ is negative on $(-\varepsilon, 0)$ and such that they converge monotonously to $\underline{m}$ or $\overline{m}$, respectively, for $\varepsilon \to 0_+$. It follows from (12.20), (12.21), (12.22) and from the above definitions of $\underline{m}_\varepsilon$ and $\overline{m}_\varepsilon$ that

$$\langle d_2^n v_n + z_n, v_n\rangle \leqslant \int_{\Gamma_U} \overline{m}_\varepsilon(v_n^{\delta_n})\big[v_n^{\delta_n}\big]^- \mathrm{d}\Gamma, \tag{12.23}$$

$$d_2^n \langle v_n + z_n, v\rangle \geqslant -\int_{\Gamma_U} \overline{m}_\varepsilon(v_n^{\delta_n})\big[v^{\delta_n}\big]^+ \mathrm{d}\Gamma + \int_{\Gamma_U} \underline{m}_\varepsilon(v_n^{\delta_n})\big[v^{\delta_n}\big]^- \mathrm{d}\Gamma. \tag{12.24}$$

The limiting process for $n \to +\infty$ in (12.23), (12.24) by using the Nemytskii theorem gives

$$\limsup_{n\to+\infty} \langle d_2^n v_n + z_n, v_n\rangle \leqslant \int_{\Gamma_U} \overline{m}_\varepsilon(v) v^- \mathrm{d}\Gamma,$$
$$d_2\|v\|^2 + \langle z, v\rangle \geqslant -\int_{\Gamma_U} \overline{m}_\varepsilon(v) v^+ \mathrm{d}\Gamma + \int_{\Gamma_U} \underline{m}_\varepsilon(v) v^- \mathrm{d}\Gamma$$

and the limiting process for $\varepsilon \to 0_+$ by using the Levi theorem implies

$$
\begin{aligned}
d_2 \limsup_{n\to+\infty} \|v_n\|^2 + \langle z, v\rangle &\leqslant \int_{\Gamma_U} \overline{m}(v) v^- \,\mathrm{d}\Gamma, \\
d_2\|v\|^2 + \langle z, v\rangle &\geqslant -\int_{\Gamma_U} \overline{m}(v) v^+ \,\mathrm{d}\Gamma + \int_{\Gamma_U} \underline{m}(v) v^- \,\mathrm{d}\Gamma.
\end{aligned} \tag{12.25}
$$

We have $\overline{m}(v)v^+ = 0$, $\overline{m}(v)v^- = \underline{m}(v)v^-$ and therefore (12.25) gives $\|v\|^2 \geqslant \limsup_{n\to+\infty} \|v_n\|^2$, which implies $v_n \to v$ strongly in $\mathbb{V}$.

Similarly as above, we can estimate (12.19) both from below and above by using $\underline{m}_\varepsilon$ and $\overline{m}_\varepsilon$ to obtain

$$
\begin{aligned}
&-\int_{\Gamma_U} \underline{m}_\varepsilon(v_n^{\delta_n}) \big[\varphi^{\delta_n}\big]^+ \mathrm{d}\Gamma + \int_{\Gamma_U} \overline{m}_\varepsilon(v_n^{\delta_n}) \big[\varphi^{\delta_n}\big]^- \mathrm{d}\Gamma \geqslant \langle d_2^n v_n + z_n, \varphi\rangle \\
&\qquad \geqslant -\int_{\Gamma_U} \overline{m}_\varepsilon(v_n^{\delta_n}) \big[\varphi^{\delta_n}\big]^+ \,\mathrm{d}\Gamma + \int_{\Gamma_U} \underline{m}_\varepsilon(v_n^{\delta_n}) \big[\varphi^{\delta_n}\big]^- \,\mathrm{d}\Gamma \quad \text{for all } \varphi \in \mathbb{V}.
\end{aligned} \tag{12.26}
$$

The "double" limiting process in (12.26) (first for $n \to +\infty$, then for $\varepsilon \to 0_+$) gives

$$
\begin{aligned}
&-\int_{\Gamma_U} \underline{m}(v)\varphi^+ \mathrm{d}\Gamma + \int_{\Gamma_U} \overline{m}(v)\varphi^- \mathrm{d}\Gamma \geqslant \langle d_2 v + z, \varphi\rangle \\
&\qquad\qquad \geqslant -\int_{\Gamma_U} \overline{m}(v)\varphi^+ \mathrm{d}\Gamma + \int_{\Gamma_U} \underline{m}(v)\varphi^- \mathrm{d}\Gamma \quad \text{for all } \varphi \in \mathbb{V},
\end{aligned} \tag{12.27}
$$

which is equivalent to

$$
-\int_{\Gamma_U} \underline{m}(v)\varphi \,\mathrm{d}\Gamma \geqslant \langle d_2 v + z, \varphi\rangle \geqslant -\int_{\Gamma_U} \overline{m}(v)\varphi \,\mathrm{d}\Gamma \qquad \text{for all } \varphi \in K_2
$$

and we have $D(d)U + Z \in -M(U)$ by definition. □

Proof of Proposition 4.5, (4.12). Let $\delta_n \to 0_+$, $U_n \rightharpoonup U$, $Z_n \to Z$, $d_n \to d \in \mathbb{R}^2_+$, $D(d_n)U_n + Z_n \in -M_0^{\delta_n}(U_n)$. Again, as in the proof of (4.11), the first part of the inclusion gives $u_n \to u$ strongly. The second part of the inclusion gives

$$
\langle d_2^n v_n + z_n, v_n\rangle = 0, \tag{12.28}
$$

$$
\langle d_2^n v_n + z_n, \varphi\rangle \geqslant 0 \quad \text{for all } \varphi \in \mathbb{V},\ \varphi^{\delta_n} \geqslant 0 \text{ on } \Gamma_U. \tag{12.29}
$$

The embedding theorem gives $v_n \to v$ and $v_n^{\delta_n} \to v$ in $L^2(\partial\Omega)$. We have $v_n^{\delta_n} \geqslant 0$ on $\Gamma_U$ and therefore also $v \geqslant 0$ on $\Gamma_U$. We can choose a subsequence (let us denote it $v_n$ again) and Proposition 10.1, (vi) ensures the existence of $w_n = w_n(v) \in \mathbb{V}$ such

that $w_n^{\delta_n} \geqslant 0$ on $\Gamma_U$ and $w_n \to v$ strongly in $\mathbb{V}$. We can put $\varphi := w_n$ in (12.29) to obtain with help of (12.28) that

$$\langle d_2^n v_n + z_n, w_n\rangle \geqslant 0 = \langle d_2^n v_n + z_n, v_n\rangle.$$

The assumptions $d_2^n \to d_2 > 0$, $z_n \to z$ and the fact $w_n \to v$ imply $\limsup\limits_{n\to+\infty} \|v_n\|^2 \leqslant \|v\|^2$, therefore $v_n \to v$ strongly in $\mathbb{V}$.

Now, let $\psi \in \mathbb{V}$ be arbitrary such that $\psi \geqslant 0$ a.e. on $\Gamma_U$. Let $w_n = w_n(\psi)$ be the functions from Proposition 10.1, (vi) corresponding to $\psi$. Then the choice $\varphi := w_n(\psi)$ in (12.29) and the limiting process in (12.28) and (12.29) (we have $w_n(\psi) \to \psi$) gives $d_2 v + z \in -M_{02}(v)$. □

In the situation of Example 10.2, the verification of validity of Propositions 4.1–4.5 can be done analogously as in Model Example.

Proof of Remark 10.1. It follows from the last part of (10.3) that

$$d_2\Delta v + b_{21}u + b_{22}v = 0 \quad \text{in } \Omega_v^+.$$

Multiplying this equation by an arbitrary $\varphi \in \mathbb{V}$, $\varphi \geqslant 0$ in $\Omega_1$, integrating over $\Omega_v^+$ and using Green's formula we obtain

(12.30)

$$\begin{aligned} 0 &= \int_{\partial\Omega_v^+} d_2\mathfrak{T}v\varphi \,\mathrm{d}\Gamma + \int_{\Omega_v^+} -d_2\sum_{j=1}^{\mathfrak{n}} v_{x_j}\varphi_{x_j} + (b_{21}u + b_{22}v)\varphi \,\mathrm{d}x \\ &= \int_{\partial\Omega_v^+} d_2\mathfrak{T}v\varphi \,\mathrm{d}\Gamma + \int_{\Omega_1} -d_2\sum_{j=1}^{\mathfrak{n}} v_{x_j}\varphi_{x_j} + (b_{21}u + b_{22}v)\varphi \,\mathrm{d}x \\ &\qquad - \int_{\Omega_v^0} -d_2\sum_{j=1}^{\mathfrak{n}} v_{x_j}\varphi_{x_j} + (b_{21}u + b_{22}v)\varphi \,\mathrm{d}x. \end{aligned}$$

It follows from (2.11) and the definition of $M_{02}(v)$ that

$$\int_{\Omega_1} -d_2\sum_{j=1}^{\mathfrak{n}} v_{x_j}\varphi_{x_j} + (b_{21}u + b_{22}v)\varphi \,\mathrm{d}x \leqslant 0 \quad \text{for any } \varphi \in \mathbb{V}, \varphi \geqslant 0 \text{ in } \Omega_1.$$

This fact together with (12.30) implies

$$\int_{\partial\Omega_v^+} d_2\mathfrak{T}v\varphi \,\mathrm{d}\Gamma - \int_{\Omega_v^0} -d_2\sum_{j=1}^{\mathfrak{n}} v_{x_j}\varphi_{x_j} + (b_{21}u + b_{22}v)\varphi \,\mathrm{d}x \geqslant 0 \quad \text{for any } \varphi \in \mathbb{V}, \varphi \geqslant 0 \text{ in } \Omega_1. \tag{12.31}$$

Clearly, we have $\nabla v = 0$ in $\Omega_v^0$ and the second term in (12.31) vanishes. As a test function in (12.31) we can choose $\varphi_n \in \mathbb{V}$, $\varphi_n \geqslant 0$ in $\Omega_1$, $\varphi_n = 1$ in $\operatorname{cl}\Omega_v^+$, $\varphi_n = 0$ in $\Omega_v^n \subset \Omega_v^0$, $\operatorname{meas}(\Omega_v^0 \setminus \Omega_v^n) \to 0$ for $n \to +\infty$ to get

$$\int_{\partial\Omega_v^+} \mathfrak{T}v \, \mathrm{d}\Gamma \geqslant 0.$$

But $\mathfrak{T}v \leqslant 0$ on $\partial\Omega_v^+$ by the second condition in the last part of (10.3). Therefore $\int_{\partial\Omega_v^+} \mathfrak{T}v\varphi \, \mathrm{d}\Gamma \leqslant 0$ and, consequently, $\int_{\partial\Omega_v^+} \mathfrak{T}v\varphi \, \mathrm{d}\Gamma = 0$ for any $\varphi \geqslant 0$ in $\Omega_1$. Thus we obtain $\mathfrak{T}v = 0$ on $\partial\Omega_v^+$. Finally, it is easy to see that $v = 0$ on $\partial\Omega_v^+$. $\square$

In the situation of Example 10.3 we have $\operatorname{int} K \neq \emptyset$ and there is no need to regularize via $\Phi^\delta$.

P r o o f of the fact $\frac{\partial v}{\partial n} = \text{const}$ on $\Gamma_U$ from (10.5), i.e. of $\int_{\Gamma_U} \mathfrak{T}v\varphi \, \mathrm{d}\Gamma = C\,\overline{\varphi}$ for some $C \in \mathbb{R}$. In a similar way as at the beginning of Section 12 we can prove that (9.1) is a weak solution of (9.2) with (10.5) and we obtain

$$\underline{m}(\overline{v})\overline{\varphi} \leqslant -d_2 \int_{\Gamma_U} \mathfrak{T}v\varphi \, \mathrm{d}\Gamma \leqslant \overline{m}(\overline{v})\overline{\varphi} \qquad \text{for all } \varphi \in \mathbb{V}, \overline{\varphi} \geqslant 0 \tag{12.32}$$

(cf. (12.7)). The choice $\varphi \in \mathbb{V}$, $\overline{\varphi} = 0$ gives $\int_{\Gamma_U} \mathfrak{T}v\varphi \, \mathrm{d}\Gamma = 0$. If $\mathfrak{T}v$ were nonconstant on $\Gamma_U$ then we would find $\varphi_1, \varphi_2 \in \mathbb{V}$, $\overline{\varphi}_1 = \overline{\varphi}_2$ and $C_1, C_2 \in \mathbb{R}$ such that

$$\int_{\Gamma_U} \mathfrak{T}v\varphi_j \, \mathrm{d}\Gamma = C_j\,\overline{\varphi}_j, \quad j = 1, 2.$$

Then $\int_{\Gamma_U} \mathfrak{T}v(\varphi_1 - \varphi_2) \, \mathrm{d}\Gamma = (C_1 - C_2)\overline{\varphi}_1 \neq 0$, which is a contradiction. $\square$

The verification of validity of Propositions 4.1–4.5 can be done analogously as in Model Example by using the functional $\overline{\varphi}$ instead of $\varphi^\delta$.

In the situation of Example 11.1 (where $\Omega = (0,1)$) the embedding theorem guarantees nonempty interiors of the sets $K$, $K_1$ and $\mathcal{K}$, so we need not regularize (9.1).

P r o o f o f P r o p o s i t i o n 4.1. Let $U_n \to 0$, $W_n = [\xi_n, w_n] = \frac{U_n}{\|U_n\|} \rightharpoonup W = [\xi, w]$, $Z_n = [\eta_n, z_n] \to Z = [\eta, z]$, $d_n \to d \in \mathbb{R}_+^2$ and $D(d_n)W_n + Z_n \in -\frac{M(U_n)}{\|U_n\|}$. Analogously to the proof of Proposition 4.1 for Example 10.1 performed earlier we can prove $w(x_0) \geqslant 0$. Let us define a linear continuous functional $\mathcal{L}$ on $\mathbb{V}$ by $\mathcal{L}\varphi = \varphi(1)$ and $\mathbb{V} = \operatorname{Ker}\mathcal{L} \oplus \mathbb{V}_0$ with $\dim \mathbb{V}_0 = 1$. It follows from the definition of $M$ that

$$-\frac{\underline{m}(v_n(x_0))}{\|U_n\|}\varphi(1) \geqslant \langle d_2^n w_n + z_n, \varphi\rangle \geqslant -\frac{\overline{m}(v_n(x_0))}{\|U_n\|}\varphi(1) \qquad \text{for all } \varphi \in K_1. \tag{12.33}$$

Therefore $\langle d_2^n w_n + z_n, \varphi\rangle = 0$ for any $\varphi \in \operatorname{Ker}\mathcal{L}$, i.e. $d_2^n w_n + z_n \in \mathbb{V}_0$. This together with the assumed convergences gives $w_n \to w$ in $\mathbb{V}$.

In the case $w(x_0) > 0$ the embedding theorem yields $w_n(x_0) > 0$ for $n$ large enough. Then (12.33) is equivalent to $d_2^n w_n + z_n = 0$ and by the limiting process for $n \to +\infty$ we obtain $d_2 w + z = 0$, i.e. $d_2 w + z \in -M_{02}(w)$. In the case $w(x_0) = 0$ it follows from (12.33) and the sign of $\overline{m}$ that $\langle d_2 w + z, \varphi \rangle = \lim_{n \to +\infty} \langle d_2^n w_n + z_n, \varphi \rangle \geqslant 0$ for any $\varphi \in K_1$, i.e. we have $d_2 w + z \in -M_{02}(w)$ again. $\square$

The verification of validity of Propositions 4.1–4.5 can be done analogously as in Model Example.

A c k n o w l e d g e m e n t. I would like to thank Dr. J. Jarušek for his important and helpful comments and suggestions, Dr. M. Kučera for his remarks and for the proof of constantness of $\frac{\partial v}{\partial n}$ in (10.5) and Dr. D. Medková for her ideas concerning the limiting process $\delta \to 0$.

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*Author's address*: *Jan Eisner*, Mathematical Institute, Academy of Sciences of the Czech Republic, Žitná 25, Praha 1, Czech Republic, e-mail: `eisner@math.cas.cz`.