129 (2004) MATHEMATICA BOHEMICA No. 1, 1–9

ON A CERTAIN CONSTRUCTION OF LATTICE EXPANSIONS

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(Received December 3, 2002)

Abstract. We obtain a simple construction for particular subclasses of several varieties of lattice expansions. The construction allows a unified approach to the characterization of the subdirectly irreducible algebras

Keywords: lattice expansions, subdirectly irreducibles, triples

MSC 2000: 06D05

1. Introduction

In [5], C. Chen and G. Grätzer introduced a method of construction of Stone lattices from Boolean algebras and distributive lattices. This was a construction by means of the so-called "triples" which were successfully used in constructions of distributive p-algebras [11], pseudocomplemented semilattices [10] and $\mathbf{K}_2$-algebras (a subvariety of MS-algebras) [3], [12], [8].

In this article we first obtain a simple construction for particular subclasses of several varieties, including Ockham algebras with De Morgan skeleton, MS-algebras, demi-p-lattices, p-algebras, Stone lattices and semi-De Morgan algebras (Section 3). The construction allows a unified approach to the characterization of the subdirectly irreducible algebras (Section 4). In Section 5 we generalize this construction. Several examples illustrate the results.

Supported by Agencia Cordoba Ciencia and SECYT (UNC).

2. Preliminaries

For notation and assumed results about lattice theory we refer the reader to [1]. Let L be a lattice expansion. ∇^L is the universal congruence on L and Δ^L is the trivial congruence on L. $\mathrm{Con}(L)$ denotes the congruence lattice of L. By $\theta(x, y)$ we denote the principal congruence relation generated by the pair $(x, y) \in L^2$. We use $\theta_{\mathrm{lat}}(x, y)$ to denote the principal lattice congruence generated by (x, y).

We are interested in several subvarieties of the variety of semi-De Morgan algebras, introduced by Sankappanavar in [15]. A *semi-De Morgan algebra* is an algebra $\langle L, \wedge, \vee, ^*, 0, 1\rangle$ satisfying:

$\langle L, \wedge, \vee, 0, 1\rangle$ is a distributive lattice with 0 and 1,

(C1) $(x \vee y)^* = x^* \wedge y^*$,

(C2) $(x \wedge y)^{**} = x^{**} \wedge y^{**}$,

(C3) $0^* = 1$,

(C4) $1^* = 0$,

(C5) $x^{***} = x^*$.

A semi-De Morgan algebra L is a *distributive demi-pseudocomplemented lattice (demi-p-lattice,* for brevity) if L satisfies $x^* \wedge x^{**} = 0$. We refer the reader to [14] for the basic properties of demi-p-lattices. A demi-p-lattice L is a *distributive p-algebra* if L satisfies the condition $x \wedge a = 0$ iff $a \leqslant x^*$. A distributive p-algebra is a Stone algebra if it satisfies the identity $x^* \vee x^{**} = 1$.

On the other hand, a semi-De Morgan algebra L is an *Ockham lattice with De Morgan skeleton* [2] if it satisfies the equation $(x \wedge y)^* = x^* \vee y^*$. This variety is denoted by $\mathcal{K}_{1,1}$. If, in addition, L satisfies $x \leqslant x^{**}$, then L is called an MS-algebra. We refer the reader to [3] and [4] for the basic properties of MS-algebras. An MS-algebra is a De Morgan algebra if $x = x^{**}$ for every $x \in L$. We recall that the subdirectly irreducible De Morgan algebras are $\mathbf{2}$, $\mathbf{3}$, and $\mathbf{M}_0$, where $\mathbf{n}$ is the chain of n elements for $n = 2, 3$, and $\mathbf{M}_0$ is the algebra with universe $\mathbf{2} \times \mathbf{2}$ and $(0, 0)^* = (1, 1)$, $(0, 1)^* = (0, 1)$, $(1, 0)^* = (1, 0)$, $(1, 1)^* = (0, 0)$.

Let L be a semi-De Morgan algebra. Let $S(L) = \{x^* \colon x \in L\} = \{x \colon x = x^{**}\}$, the *skeleton* of L, and also let $G_z = \{x \in L \colon x^* = z^*\}$ for $z \in L$. The skeleton $S(L)$ can be made into a De Morgan algebra $\langle S(L), \dot\vee, \wedge, ^*, 0, 1\rangle$ by defining $a \dot\vee b = (a \vee b)^{**}$. Note that the map $x \to x^{**}$ from L to $S(L)$ is an onto homomorphism. We denote by γ^L the kernel of this homomorphism. Note that if θ is a lattice-congruence and $\theta \subseteq \gamma^L$ then $\theta \in \mathrm{Con}(L)$.

If $|G_x| \leqslant 2$ for every $x \in L$, then we say that L is 2-*regular*. The following definition was given in [6, Section 5]. If $C \subseteq L$ is the finite chain $x_1 < x_2 < \ldots < x_n$, then let

$$n^{L,C} = |\{1 \leqslant i < n \colon x_i^* = x_{i+1}^*\}|.$$

If $\{n^{L,C}\colon\ C \subseteq L$ is a finite chain$\}$ has a maximum, then we denote it by n^L. Clearly, if $n^L = 1$, then L is 2-regular and $n^L = 0$ if and only if $\gamma^L = \Delta^L$.

Lemma 1. *Let L be a (finitely) subdirectly irreducible semi-De Morgan algebra. Then $n^L \leqslant 1$.*

P r o o f. If $x < y \leqslant v < z$ and (x, y), $(v, z) \in \gamma^L$, we claim that $\sigma = \theta_{\mathrm{lat}}(x, y) \cap \theta_{\mathrm{lat}}(v, z)$ is the trivial congruence Δ^L. (Note that since $\sigma \subseteq \gamma^L$ we have $\sigma \in \mathrm{Con}(L)$). Let $(a, b) \in \sigma$. Then

$$a \wedge x = b \wedge x,\ a \vee y = b \vee y \text{ and } a \wedge v = b \wedge v,\ a \vee z = b \vee z$$

(see [7, II. 3, Theorem 3]). Since $y \leqslant v$ we have $a \wedge y = b \wedge y$. By the cancelation property for distributive lattices, $a = b$. □

We conclude the section by observing that if L is a semi-De Morgan algebra satisfying $n^L \leqslant 1$ then for every $x \in L$ we have that either $x^{**} \geqslant x$ or $x^{**} \leqslant x$.

3. A simple quadruple construction

A 2-*quadruple* will be a tuple (M, I, F, η), where

(1) M is a De Morgan algebra.

(2) I (resp. F) is an ideal (filter) of M satisfying $I \cup F = M$ and $I \cap F \neq \emptyset$.

(3) η is a proper filter of M satisfying $M - I \subseteq \eta \subseteq F$.

We associate to each 2-quadruple $T = (M, I, F, \eta)$ an algebra of type $(2, 2, 1, 0, 0)$, denoted by L_T, with universe

$$L_T = ((I - F) \times \{0\}) \cup ((I \cap F) \times \mathbf{2}) \cup ((F - I) \times \{1\}),$$

defined by the following conditions:

(a) L_T is a sublattice of $M \times \mathbf{2}$,

(b) $(a, i)^* = \begin{cases} (a^*, 1) & a^* \in \eta \\ (a^*, 0) & a^* \notin \eta. \end{cases}$

For simplicity we also denote by η the map from M to $\mathbf{2}$ defined by the condition $\eta(a) = 1 \Leftrightarrow a \in \eta$. So, $(a, i)^*$ can be defined as $(a^*, \eta(a^*))$. If we conceive of η as a map, then the condition (3) takes the following form, which will be occasionally used:

(3′) η is a $\{0, 1, \wedge\}$-preserving map from M to $\mathbf{2}$ satisfying $\eta(x) = 0$ for every $x \in I - F$, and $\eta(x) = 1$, for every $x \in F - I$.

Lemma 2. *Let $T = (M, I, F, \eta)$ be a 2-quadruple. Then L_T is a semi-De Morgan algebra satisfying $n^{L_T} = 1$ and $S(L_T) \cong M$.*

P r o o f. That L_T is a distributive lattice follows from (a). The conditions (C1) and (C2) can be easily checked. Conditions (C3) and (C4) hold because η is a $\{0,1\}$-preserving map. (C5) follows from the fact that M is a De Morgan algebra. We prove now that $S(L_T) \cong M$. Clearly $S(L_T) = \{(a, \eta(a)) : a \in M\}$. We check now that $a \to (a, \eta(a))$ is a De-Morgan isomorphism from M to $S(L_T)$. It is trivial that this map is a one-to-one $\{0, 1, \wedge\}$-homomorphism. By the definition of L_T, the map is onto. Moreover,

$$a \vee b \to (a \vee b, \eta(a \vee b)) = ((a \vee b)^{**}, \eta((a \vee b)^{**}))\,,$$

because M is a De Morgan algebra. But

$$((a \vee b)^{**}, \eta((a \vee b)^{**})) = ((a \vee b)^{**}, i)^{**}$$

for every i such that $((a \vee b)^{**}, i) \in L_T$. Take $i = \eta(a) \vee \eta(b)$ to conclude that the above map preserves $\vee$.

Finally, $n^{L_T} = 1$ follows from the fact that $((a, i), (b, j)) \in \gamma^{L_T}$ iff $a = b$. □

Note that different 2-quadruples can generate isomorphic algebras. For instance, consider the 2-quadruples

$$T_1 = (\mathbf{M}_0,\, \mathbf{M}_0,\, \{(0,1),(1,1)\},\, \{(0,1),(1,1)\}),$$
$$T_2 = (\mathbf{M}_0,\, \mathbf{M}_0,\, \{(1,0),(1,1)\},\, \{(1,0),(1,1)\}).$$

Both the 2-quadruples generate the only (up to isomorphism) MS-algebra having lattice reduction $\mathbf{2} \times \mathbf{3}$ and skeleton $\mathbf{M}_0$. However, the representation is unique up to *isomorphisms of 2-quadruples*.

Definition 3. The 2-quadruples $T = (M, I, F, \eta)$ and $T' = (M', I', F', \eta')$ are isomorphic if there exists an isomorphism $\alpha\colon\ M \to M'$ such that $I' = \alpha(I)$, $F' = \alpha(F)$ and $\eta' = \alpha(\eta)$.

Lemma 4. *Let L be a semi-De Morgan algebra satisfying $n^L = 1$. Then there exists (up to isomorphism) only one 2-quadruple $T = (M, I, F, \eta)$ such that $L_T \cong L$.*

P r o o f. Let $T = (S(L), (S], [S), \eta)$[1], $S = \{a \in S(L)\colon\ |G_a| = 2\}$ and

$$\eta = \{a \in S(L)\colon\ a \ \text{ is the top element of } \ G_a \ \text{ and } \ a \geqslant b \ \text{ for some } \ b \in S\}\,.$$

[1] $[S)$ is the set of x such that $x \geqslant s$ for some $s \in S$. In a similar manner $(S]$ is defined.

We claim that S is a subalgebra of $S(L)$. Suppose that $|G_a| = |G_b| = 2$, $a \neq b$. If $a < b$ then $|G_{a\wedge b}| = |G_a| = 2$ and $|G_{a\vee b}| = |G_b| = 2$. Suppose that a and b are not comparable. It is left as an exercise to the reader to check that $|G_{a\wedge b}| = 1$ or $|G_{a\vee b}| = 1$ implies that the sublattice generated by $G_a \cup G_b$ is not distributive, which is a contradiction. Thus the claim follows. Consequently, $(S]$ (resp. $[S)$) is an ideal (filter) of $S(L)$. Let $\varphi = [\eta)_L$ be the filter on L generated by η. Define $f\colon L \to L_T$ by $f(x) = (x^{**}, 1)$ if $x \in \varphi$, and $f(x) = (x^{**}, 0)$ if $x \notin \varphi$. That f is $1-1$ follows from the fact that if $(a, b) \in \gamma^L$ and $a, b \in \varphi$ then $a = b$. This is a consequence of the fact that $\bigwedge G_x \notin \varphi$ if $|G_x| = 2$. Finally, f is onto because $f(\bigwedge G_a) = (a, 0)$ and $f(\bigvee G_b) = (b, 1)$ for every $a \in I$ and $b \in F$. In order to check the case $a \in I$, use the fact that if $|G_x| = 1$, $|G_y| = 2$ and $x < y$ then $x \notin \varphi$ (if not we contradict the fact that $n^L = 1$). The rest of the lemma is easy to check. □

Lemma 5. *Let $T = (M, I, F, \eta)$ be a 2-quadruple. Then*

(i) *L_T is an Ockham lattice with De Morgan skeleton iff η is a prime filter.*
(ii) *L_T is an MS-algebra iff η is a prime filter and $F = \eta$.*
(iii) *L_T is a demi-p-lattice iff M is a Boolean algebra.*
(iv) *L_T is a p-algebra iff M is a Boolean algebra, $I = M$ and $\eta = F$.*
(v) *L_T is a Stone algebra iff $M = I$ is a Boolean algebra and $\eta = F$ is a prime filter.*

P r o o f. Use the following facts: (a) η is a prime filter iff L_T satisfies the De-Morgan laws. (b) $F = \eta$ iff L_T satisfies $x \leqslant x^{**}$ for every $x \in L_T$. (c) M is a Boolean algebra iff L_T satisfies the equation $x^* \wedge x^{**} = 0$. In order to prove (iv), note that a p-algebra is a demi-p-lattice satisfying $x \leqslant x^{**}$ for every x. □

N o t a t i o n. The 2-quadruples of the right-hand side of (i) of Lemma 5 (resp. (ii), (iii), (iv) and (v)) will be called $\mathcal{K}_{1,1}$-tuples (resp. MS-tuples, demi-p-tuples, p-tuples and Stone tuples).

For instance, in the following table we show a set of non isomorphic 2-quadruples generating all MS-algebras with $S(L) \in \{\mathbf{2}, \mathbf{3}, \mathbf{M}_0\}$ and $n^L = 1$. Here a is the middle element of the chain **3.** In the next section we will show that these 2-quadruples are exactly those corresponding to the subdirectly irreducible MS-algebras that are not De Morgan algebras.

2-quadruple T	*lattice reduction of L_T*
$(\mathbf{2}, \mathbf{2}, \{1\}, \{1\})$	$\mathbf{3}$
$(\mathbf{3}, \mathbf{3}, \{1\}, \{1\})$	$\mathbf{4}$
$(\mathbf{3}, \{0, a\}, \{a, 1\}, \{a, 1\})$	$\mathbf{4}$
$(\mathbf{3}, \mathbf{3}, \{a, 1\}, \{a, 1\})$	$\mathbf{1} \oplus (\mathbf{2} \times \mathbf{2})$
$(\mathbf{M}_0, \mathbf{M}_0, \{(0,1),(1,1)\}, \{(0,1),(1,1)\})$	$\mathbf{3} \times \mathbf{2}$

On the other hand, if $\mathbf{n}$ represents the chain of n elements and $n > 4$, then an algebra L with lattice reduction $\mathbf{n}$ and skeleton $\mathbf{3}$ can not be represented by a 2-quadruple, because $n^L > 1$.

4. Subdirectly irreducible algebras

Let $\mathcal{V}$ be any of the varieties from Lemma 5. By Lemmas 1 and 3.3, in order to give a complete characterization of the (finitely) subdirectly irreducible members of $\mathcal{V}$, it is sufficient to determine the class of $\mathcal{V}$-tuples T such that L_T is (finitely) subdirectly irreducible. In this section we do this for each of those varieties, and also for the class of all semi-De Morgan algebras.

The classes of subdirectly irreducible of the above mentioned varieties were determined by the following authors:

p-algebras: Lakser, [13],

demi-p-lattices: Sankappanavar, [14],

MS-algebras: Blyth and Varlet, [3],

semi-De Morgan algebras (topological characterization): Hobby, [9].

The following theorem follows from [6, Theorem 5.6].

Theorem 6. *Let $T = (M, I, F, \eta)$ be a 2-quadruple. L_T is a (finitely) subdirectly irreducible semi-De Morgan algebra if and only if for every chain $a < b$ in M there exist $c, d \in M$ satisfying either $(c \vee a) \wedge d \notin \eta$ and $(c \vee b) \wedge d \in \eta$, or $(c \vee b^*) \wedge d \notin \eta$ and $(c \vee a^*) \wedge d \in \eta$.*

Corollary 7. (i) *Let $T = (M, I, F, \eta)$ be a 2-quadruple such that η is a prime filter. Then L_T is (finitely) subdirectly irreducible if and only if $M \in \{\mathbf{2}, \mathbf{3}, \mathbf{M}_0\}$.*

(ii) *Let $T = (M, I, F, \eta)$ be a 2-quadruple such that M is a Boolean algebra. Then L_T is (finitely) subdirectly irreducible if and only if $\eta = \{1\}$.*

Proof. (i) The "if" part follows from a simple computation, by checking that γ^{L_T} is a monolite of the congruence lattice of L_T. Suppose that L_T is a finitely subdirectly irreducible semi-De Morgan algebra. Using Theorem 6 and the fact that η is a prime filter, it can be checked that

(a) for every $a \leqslant b$, if $\eta(a) = \eta(b)$ and $\eta(a^*) = \eta(b^*)$ then $a = b$. Let $a \in M - \{0, 1\}$. We claim that $\eta(a \wedge a^*) = \eta(a \vee a^*)$. Suppose not. Thus $\eta(a \wedge a^*) = 0$ and $\eta(a \vee a^*) = 1$. Thus, for some $b \in \{a, a^*\}$, we have $\eta(b) = 0$ and $\eta(b^*) = 1$. By (a), $b = 0$, which is a contradiction. Thus we have the claim. Consequently, $a = a^*$. Since the map $a \to a^*$ from $S(L)$ to $S(L)$ is an anti-isomorphism, we have $S(L) \in \{\mathbf{2}, \mathbf{3}, \mathbf{M}_0\}$.

(ii) Let L_T be subdirectly irreducible and suppose that $a \in \eta$. Since $a \wedge a^* = 0$ and η is a proper filter, we have $a^* \notin \eta$. If $a \neq 1$, then, by Theorem 6, there exist

$c, d \in M$ such that either $(c \vee a) \wedge d \notin \eta$ and $(c \vee 1) \wedge d \in \eta$, or $(c \vee 1^*) \wedge d \notin \eta$ and $(c \vee a^*) \wedge d \in \eta$. That $(c \vee a) \wedge d \notin \eta$ and $(c \vee 1) \wedge d \in \eta$ is impossible, because η is a filter and $a \in \eta$. Suppose that $(c \vee 1^*) \wedge d \notin \eta$ and $(c \vee a^*) \wedge d \in \eta$. Then $c \wedge d \geqslant c \wedge d \wedge a = a \wedge (c \vee a^*) \wedge d \in \eta$, which is a contradiction. Hence $a = 1$.

Suppose now that $\eta = \{1\}$, and let $a < b$. We will check the condition of Theorem 6. Suppose that $a < b$. Since $a \vee b^* \neq 1$, we have $(b^* \vee a) \wedge 1 \notin \eta$ and $(b^* \vee b) \wedge 1 \in \eta$. □

Corollary 8. *Let $T = (M, I, F, \eta)$ be a 2-quadruple.*

(i) *L_T is a (finitely) subdirectly irreducible Ockham lattice with De Morgan skeleton iff η is a prime filter and $M \in \{\mathbf{2}, \mathbf{3}, \mathbf{M}_0\}$.*

(ii) *L_T is a (finitely) subdirectly irreducible MS-algebra iff η is a prime filter, $F = \eta$ and $M \in \{\mathbf{2}, \mathbf{3}, \mathbf{M}_0\}$.*

(iii) *L_T is a (finitely) subdirectly irreducible demi-p-lattice iff M is a Boolean algebra and $\eta = \{1\}$.*

(iv) *L_T is a (finitely) subdirectly irreducible p-algebra iff M is a Boolean algebra, $I = M$ and $\eta = F = \{1\}$.*

(v) *L_T is a (finitely) subdirectly irreducible Stone algebra iff $M = I = \mathbf{2}$ and $\eta = F = \{1\}$*[2]*.*

5. A general construction

In this section we extend the construction from Section 4 to the class of all semi-De Morgan algebras such that n^L is finite. In this Section we omit the proofs of the results. If N is a finite distributive lattice, then by $\mathcal{I}(N)$ we denote the set of intervals of N, that is,

$$\mathcal{I}(N) = \{I_{a,b} \colon a, b \in N,\ a \leqslant b\},$$

where $I_{a,b} = \{x \in N \colon a \leqslant x \leqslant b\}$. The set $\mathcal{I}(N)$ can be made into a lattice by defining $I_{u,v} \leqslant I_{w,r}$ iff for every x,

$$x \in I_{u,v} \text{ implies } x \vee w \leqslant r.$$

It is easy to check that

$$I_{a,b} \wedge I_{c,d} = I_{a \wedge c, b \wedge d},$$
$$I_{a,b} \vee I_{c,d} = I_{a \vee c, b \vee d}.$$

[2] This determines the chain of three elements, the only s.i. proper 2-regular Stone algebra.

A *quadruple* will be a tuple (M, N, φ, η), where

(1) M is a De Morgan algebra and N is a finite distributive lattice.

(2) $\varphi\colon M \to \mathcal{I}(N)$ is a lattice homomorphism satisfying $0 \in \varphi(0)$, $1 \in \varphi(1)$ and that for every $s, t \in N$ such that t covers s there exists $a \in M$ with $s, t \in \varphi(a)$.

(3) η is a $\{0, 1, \wedge\}$-preserving map from M to N such that $\eta(a) \in \varphi(a)$ for every $a \in M$.

We will occasionally use $l(a)$ and $u(a)$ to denote the elements of N satisfying that $\varphi(a) = I_{l(a),u(a)}$. We associate with each quadruple $T = (M, N, \varphi, \eta)$ an algebra of type $(2, 2, 1, 0, 0)$, denoted by L_T, with universe $L_T = \{(a, t)\colon\ t \in \varphi(a)\}$, defined by the following conditions:

(a) L_T is a sublattice of $M \times N$,

(b) $(a, i)^* = (a^*, \eta(a^*))$.

Definition 9. Two quadruples $T = (M, N, \varphi, \eta)$, $T' = (M', N', \varphi', \eta')$ are *isomorphic* if there exist isomorphisms $\alpha\colon M \to M'$ and $\beta\colon N \to N'$ such that $\varphi'(\alpha(a)) = \beta(\varphi(a))$ and $\eta'(\alpha(a)) = \beta(\eta(a))$.

Lemma 10. (i) *Let $T = (M, N, \varphi, \eta)$ be a quadruple. Then L_T is a semi-De Morgan algebra satisfying $S(L_T) \cong M$ and $n^{L_T} < \infty$.*

(ii) *Let L be a semi-De Morgan algebra satisfying $n^L < \infty$ and $S(L) \cong M$, and let*

$$N = L/(\gamma^L)^*,$$
$$\varphi(a) = \{[x](\gamma^L)^*\colon\ x^{**} = a\},\ \text{and}\ \eta(a) = [a](\gamma^L)^*,$$

where $(\gamma^L)^*$ is the pseudocomplement of γ^L in the congruence lattice of L. Then $T = (M, N, \varphi, \eta)$ is (up to isomorphism) the only quadruple satisfying $L_T \cong L$.

Lemma 11. *Let $T = (M, N, \varphi, \eta)$ be a quadruple. Then*

(i) *L_T is a Ockham lattice with De Morgan skeleton iff η preserves $\vee$.*

(ii) *L_T is a MS-algebra iff η preserves $\vee$ and $\eta(a) = u(a)$ for every $a \in M$.*

(iii) *L_T is a demi-p-lattice iff M is a Boolean algebra.*

(iv) *L_T is a p-algebra iff M is a Boolean algebra and $\eta(a) = u(a)$ for every $a \in M$.*

A c k n o w l e d g m e n t s. We would like to thank the referee for several corrections which improve the presentation of the paper.

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