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ON P_Σ AND WEAKLY-P_Σ SPACES

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Abstract. In this paper, we point out that properties P_Σ due to Wang [18] and strongly s-regular due to Ganster [5] are equivalent to each other. We further study these spaces and weakly-P_Σ spaces defined by the second author [15].

1. Introduction

In 1981, Wang [18] defined a weak form of regularity called P_Σ. In 1984, the second author [13] defined the notion of weakly-P_Σ spaces which is weaker than that of P_Σ spaces. Recently, Ganster [5] has introduced the class of strongly s-regular spaces which lies strictly between the class of regular spaces and the class of s-regular spaces in the sense of Maheshwari and Prasad [9]. In this paper, we point out that P_Σ and strongly s-regular are equivalent to each other. And we further investigate the properties of P_Σ and weakly-P_Σ spaces. Pre-almost open, pre-almost closed and regular-open functions are also defined and studied to obtain some preservation theorems of P_Σ and weakly-P_Σ spaces.

2. Preliminaries

Throughout this paper, spaces always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let X be a topological space and A be a subset of X. The closure of A and the interior of A in X are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A of X is said to be *semi-open* [8] if there exists an open subset U of X such that $U \subset A \subset \mathrm{Cl}(U)$. The complement of a semi-open set is said to be *semi-closed*. The *semi-closure* of A is defined as the intersection of all semi-closed sets containing A and is denoted by $\mathrm{sCl}(A)$. The *semi-interior* of A is defined as the union of all semi-open sets contained in A and is denoted by $\mathrm{sInt}(A)$. A subset A is said to be *semi-regular* [3] if it is semi-open and semi-closed. The family of all semi-open (resp. semi-regular) subsets of X is denoted by $SO(X)$ (resp. $SR(X)$). A subset A is said to be *preopen*

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[10] if $A \subset \mathrm{Int}(\mathrm{Cl}(A))$. A subset A is said to be *regular open* (resp. *regular closed*) if $A = \mathrm{Int}(\mathrm{Cl}(A))$ (resp. $A = \mathrm{Cl}(\mathrm{Int}(A))$). The family of all regular open (resp. regular closed) subsets of X is denoted by $RO(X)$ (resp. $RC(X)$). A point x of X is said to be in the *θ-semiclosure* [6] (resp. *δ-closure* [17]) of A, denoted by $\theta\text{-sCl}(A)$ (resp. $\mathrm{Cl}_\delta(A)$), if $A \cap \mathrm{Cl}(U) \neq \emptyset$ (resp. $A \cap U \neq \emptyset$), for every $U \in SO(X)$ (resp. $U \in RO(X)$) containing x. A subset A is said to be *θ-semiclosed* [6] (resp. *δ-closed* [17]) if $\theta\text{-sCl}(A) = A$ (resp. $\mathrm{Cl}_\delta(A) = A$).

DEFINITION 1. A topological space X is said to be

(a) P_Σ [18] if every open subset of X is the union of regular closed sets;

(b) *weakly-P_Σ* [13] if every regular open subset of X is the union of regular closed sets;

(c) *s^*-regular* [7] if for any semi-regular set A and any point $x \in X - A$, there exist disjoint open sets U and V such that $A \subset U$ and $x \in V$;

(d) *s-regular* [9] (resp. *semi-regular* [4]) if for each closed (resp. semi-closed) set A and any point $x \in X - A$, there exist disjoint semi-open sets U and V such that $A \subset U$ and $x \in V$;

(e) *extremally disconnected* (briefly E.D.) if $\mathrm{Cl}(U)$ is open in X, for every open set U in X;

(f) *almost regular* [15] if for any regular closed set A and any point $x \in X - A$, there exist disjoint open sets U and V such that $A \subset U$ and $x \in V$;

(g) *strongly s-regular* [5] if for each closed set A and any point $x \in X - A$, there exists an $F \in RC(X)$ such that $x \in F$ and $F \cap A = \emptyset$.

3. P_Σ and weakly P_Σ spaces

First of all, we point out that P_Σ and strongly s-regular are equivalent to each other.

THEOREM 1. (Ganster [5]) *The following are equivalent for a topological space X:*

(a) X is P_Σ.

(b) For any open subset U of X and any point $x \in U$ there exists an $F \in RC(X)$ such that $x \in F \subset U$.

(c) X is strongly s-regular.

THEOREM 2. *The following are equivalent for a topological space X:*

(a) X is weakly-P_Σ.

(b) For any regular open subset U of X and any point $x \in U$, there exists an $F \in RC(X)$ such that $x \in F \subset U$.

(c) Every regular closed set in X is the intersection of regular open sets.

(d) $\theta\text{-sCl}(A) \subset \mathrm{Cl}_\delta(A)$ for every subset A of X.

(e) Every δ-closed set of X is θ-semiclosed in X.

Proof. The proof is quite similar to that of [5, Theorem 1] and is thus omitted. ■

In [5, Theorem 2], Ganster showed that strong s-regularity is open hereditary. We shall improve this result in the following theorem.

THEOREM 3. *If X is a P_Σ spaces and Y is preopen in X, then the subspace Y is P_Σ.*

Proof. Let U be an arbitrary open susbet of Y. Then there exists an open subset V of X such that $U = V \cap Y$. Since X is a P_Σ space, we have $V = \bigcup\{ V_\alpha : \alpha \in \nabla \}$, where $V_\alpha \in RC(X)$ for each $\alpha \in \nabla$. It is easily checked that a subset is regular closed in X if and only if it is closed and semi-open in X. Therefore, $V_\alpha \cap Y$ is closed in Y for each $\alpha \in \nabla$. By [14, Lemma 2.2], we obtain $V_\alpha \cap Y \in SO(Y)$ and hence $V_\alpha \cap Y \in RC(Y)$ for each $\alpha \in \nabla$ and $U = \bigcup\{ V_\alpha \cap Y : \alpha \in \nabla \}$. This shows that the subspace Y is P_Σ. ■

COROLLARY 1. (Ganster [5]) *Strong s-regularity is open hereditary.*

THEOREM 4. *If X is a weakly-P_Σ space and Y is open in X, then the subspace Y is weakly-P_Σ.*

Proof. Let U be an arbitrary regular open subset of Y. It is shown in [11, Lemma 3] that $\mathrm{Int}_Y(\mathrm{Cl}_Y(A)) = Y \cap \mathrm{Int}(\mathrm{Cl}(A))$ for any open subset Y of X and any subset A of Y. Therefore, there exists a regular open subset V of X such that $U = Y \cap V$. Since X is a weakly-P_Σ space, we have $V = \bigcup\{ V_\alpha : \alpha \in \nabla \}$, where $V_\alpha \in RC(X)$ for each $\alpha \in \nabla$. Similarly to the proof of Theorem 3, we obtain $V_\alpha \cap Y$ is regular closed in Y for each $\alpha \in \nabla$ and $U = \bigcup\{ V_\alpha \cap Y : \alpha \in \nabla \}$. This shows that the subspace Y is weakly-P_Σ. ■

THEOREM 5. *If a space X is s-regular and s^*-regular, then it is regular.*

Proof. Let U be any open subset of X and $x \in U$. Since X is s-regular, there exists $G \in SO(X)$ such that $x \in G \subset \mathrm{sCl}(G) \subset U$ [9, Theorem 2]. It follows from Proposition 2.2 of [3] that $\mathrm{sCl}(G) \in SR(X)$. Since X is s^*-regular, there exists an open subset O of X such that $x \in O \subset \mathrm{Cl}(O) \subset \mathrm{sCl}(G)$ [7, Theorem 1]; hence $x \in O \subset \mathrm{Cl}(O) \subset U$. This shows that X is regular. ■

Since $RC(X) \subset SR(X)$, every s^*-regular space is almost regular. Ganster showed that there exists a Hausdorff strongly s-regular space which is not almost regular [5, Example 4]. By these results and Example 1 (stated below), we obtain the following property.

REMARK 1. s^*-regularity is independent of strong s-regularity and also s-regularity.

EXAMPLE 1. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}, X\}$. Then (X, τ) is an s^*-regular space. And it is not s-regular since a subset $\{a, b\}$ is closed and not semi-open in (X, τ).

THEOREM 6. *A topological space X is s^*-regular if and only if it is E.D.*

Proof. Necessity. Let X be s^*-regular and V a nonempty open set in X. Then we have $\mathrm{Cl}(V) \in RC(X)$ and $RC(X) \subset SR(X)$. For each $x \in \mathrm{Cl}(V)$, there exists an open set U_x, such that $x \in U_x \subset \mathrm{Cl}(U_x) \subset \mathrm{Cl}(V)$. Therefore, $\mathrm{Cl}(V) = \bigcup\{\, U_x \ : \ x \in \mathrm{Cl}(V)\,\}$ is open in X. This shows that X is E.D.

Sufficiency. Let X be E.D. and A be any semi-regular set in X. Since A is semi-open, by [3, Proposition 2.4] we have $\mathrm{sCl}(A) = \mathrm{Cl}(A)$ and hence $A = \mathrm{sCl}(A) = \mathrm{Cl}(A) = \mathrm{Cl}(\mathrm{Int}(A))$. This shows that A is open and closed in X. Therefore, X is s^*-regular. ■

COROLLARY 2. (Ganster [5]) *The following are equivalent for an E.D. space X:*

(a) X is regular.

(b) X is strongly s-regular.

(c) X is s-regular.

Proof. The proof follows immediately from Theorems 5 and 6. ■

We have the following diagram related to separation axioms defined in §2.

$$\begin{tikzcd}
\text{extremally disconnected} \arrow[d, shift left] & \text{regular} \arrow[d] & \text{semi-regular} \arrow[d] \\
s^*\text{-regular} \arrow[u, shift left] \arrow[d] & \text{strongly } s\text{-regular} \arrow[r] \arrow[d] & s\text{-regular} \\
\text{almost regular} \arrow[r] & \text{weakly-}P_\Sigma &
\end{tikzcd}$$

REMARK 2. None of implications in the above diagram is reversible as shown by the following:

(a) Dorsett [4] pointed out that semi-regularity is independent of regularity and is strictly stronger than s-regularity.

(b) In Examples 1 and 2 of [5], Ganster showed that strong s-regularity lies strictly between regularity and s-regularity. We should note that the term "semi-regular" in [5, Example 2] is different from "semi-regular" in the sense of Dorsett [4].

(c) By [5, Example 4] and Example 1, the both notions of almost regular and strongly s-regular are strictly stronger than that of weakly-P_Σ.

(d) The real numbers with the usual topology is a regular space which is not E.D. Therefore, by Example 1 "E.D." and "regular" are independent of each other. And also almost regularity does not always imply s^*-regularity.

In [5, Theorem 3], Ganster showed that strong s-regularity is productive. We obtain the similar result about weakly-P_Σ spaces.

THEOREM 7. *If (X_α, τ_α) is a weakly-P_Σ space for each $\alpha \in \nabla$, then the product space $(X, \tau) = \prod\{\,(X_\alpha, \tau_\alpha) \ : \ \alpha \in \nabla\,\}$ is weakly-P_Σ.*

Proof. Let W be an arbitrary regular open set in (X, τ) and $x \in W$. Then we have $x \in \prod\{\, U_\alpha \ : \ \alpha \in \nabla\,\} \subset W$, where U_α is open in (X_α, τ_α) for each $\alpha \in \nabla$ and there exists a finite subset ∇_0 of ∇ such that $U_\alpha = X_\alpha$ whenever $\alpha \in \nabla - \nabla_0$.

Therefore, we have $x \in \prod\{\,\mathrm{Int}(\mathrm{Cl}(U_\alpha)) \;:\; \alpha \in \nabla\,\} \subset W$. For each $\alpha \in \nabla_0$, we have $x_\alpha \in \mathrm{Int}(\mathrm{Cl}(U_\alpha))$ and hence $x_\alpha \in F_\alpha \subset \mathrm{Int}(\mathrm{Cl}(U_\alpha))$ for some $F_\alpha \in RC(X_\alpha, \tau_\alpha)$. Now, let $F = \prod\{\,F_\alpha \;:\; \alpha \in \nabla_0\,\} \times \prod\{\,X_\alpha \;:\; \alpha \in \nabla - \nabla_0\,\}$. Then, we obtain $F \in RC(X, \tau)$ and $x \in F \subset \prod\{\,\mathrm{Int}(\mathrm{Cl}(U_\alpha)) \;:\; \alpha \in \nabla\,\} \subset W$. This shows that (X, τ) is weakly-P_Σ. ∎

4. Preservation theorems

We shall recall definitions of some functions used in the sequel to obtain several preservation theorems.

Definition 2. A function $f\colon X \to Y$ is said to be:

(a) *almost-continuous* [16] if $f^1(V)$ is open in X for every $V \in RO(Y)$;

(b) *completely-continuous* [1] (resp. *R-map* [2]) if $f^1(V) \in RO(X)$ for every open subset V of Y (resp. $V \in RO(Y)$);

(c) *almost-open* [16] if $f(U)$ is open in Y for every $U \in RO(X)$.

Definition 3. A function $f\colon X \to Y$ is said to be:

(a) *pre-almost open* (resp. *regular open*) if $f(U) \in RO(Y)$ for every $U \in RO(X)$ (resp. open set U in X);

(b) *pre-almost closed* if $f(U) \in RC(Y)$ for every $U \in RC(X)$.

Lemma 1. (Noiri [12]) *Every almost-continuous almost-open function is an R-map.*

Theorem 8. *If $f\colon X \to Y$ is an almost-continuous and open (resp. almost-open) injection and Y is P_Σ, then X is P_Σ (resp. weakly-P_Σ).*

Proof. Let U be an arbitrary open (resp. regular open) subset of X. Then $f(U)$ is open in Y and $f(U) = \bigcup\{\,V_\alpha \;:\; \alpha \in \nabla\,\}$, where $V_\alpha \in RC(Y)$ for each $\alpha \in \nabla$. Since f is injective, we have $U = \bigcup\{\,f^1(V_\alpha) \;:\; \alpha \in \nabla\,\}$. By Lemma 1, f is an R-map and hence $f^1(V_\alpha) \in RC(X)$ for each $\alpha \in \nabla$. Therefore, X is P_Σ (resp. weakly-P_Σ). ∎

Theorem 9. *If $f\colon X \to Y$ is an almost-continuous and pre-almost open (resp. regular open) injection and Y is weakly-P_Σ, then X is weakly-P_Σ (resp. P_Σ).*

Proof. Let U be an arbitrary regular open (resp. open) set in X. Then $f(U)$ is regular open in Y and $f(U) = \bigcup\{\,V_\alpha \;:\; \alpha \in \nabla\,\}$, where $V_\alpha \in RC(Y)$ for each $\alpha \in \nabla$. Since f is injective, we have $U = \bigcup\{\,f^1(V_\alpha) \;:\; \alpha \in \nabla\,\}$. Every regular open function is pre-almost open and every pre-almost open function is almost open. Therefore, by Lemma 1, f is an R-map and hence $f^1(V_\alpha) \in RC(X)$ for each $\alpha \in \nabla$. This shows that X is weakly-P_Σ (resp. P_Σ). ∎

Theorem 10. *If $f\colon X \to Y$ is a continuous (resp. completely continuous) and pre-almost closed surjection and X is P_Σ (resp. weakly-P_Σ), then Y is P_Σ.*

Proof. Let V be an arbitrary open subset of Y. Then $f^1(V)$ is open (resp. regular open) in X. Since X is P_Σ (resp. weakly-P_Σ), $f^1(V) = \bigcup\{\, U_\alpha \ :\ \alpha \in \nabla \,\}$, where $U_\alpha \in RC(X)$ for each $\alpha \in \nabla$. Since f is surjective, we have $V = \bigcup\{\, f(U_\alpha) \ :\ \alpha \in \nabla \,\}$. Since f is pre-almost closed, $f(U_\alpha) \in RC(Y)$ for each $\alpha \in \nabla$. This shows that Y is P_Σ. ■

THEOREM 11. *If $f\colon X \to Y$ is an R-map (resp. almost-continuous) and pre-almost closed surjection and X is weakly-P_Σ (resp. P_Σ), then Y is weakly-P_Σ.*

Proof. Let V be an arbitrary regular open set in Y. Then $f^1(V)$ is regular open (resp. open) in X. Since X is weakly-P_Σ (resp. P_Σ), $f^1(V) = \bigcup\{\, U_\alpha \ :\ \alpha \in \nabla \,\}$, where $U_\alpha \in RC(X)$ for each $\alpha \in \nabla$. Since f is a pre-almost closed surjection, we have $V = \bigcup\{\, f(U_\alpha) \ :\ \alpha \in \nabla \,\}$ and $f(U_\alpha) \in RC(Y)$ for each $\alpha \in \nabla$. This shows that Y is weakly-P_Σ. ■

LEMMA 2. *If $f\colon X \to Y$ is a pre-almost open function, then for any point y of Y and any $A \in RC(X)$ containing $f^1(y)$, there exists a $B \in RC(Y)$ containing y such that $f^1(B) \subset A$.*

Proof. Let $B = Y - f(X - A)$. Then, since $f^1(y) \subset A$, it follows that $y \in B$ and $B \in RC(Y)$ because f is pre-almost open. By a straightforward calculation, we have $f^1(B) \subset A$. ■

A subset S of a space X is said to be *S-closed relative to X* [13] if for every cover $\{\, U_\alpha \ :\ \alpha \in \nabla \,\}$ of S by semi-open sets of X there exists a finite subset ∇_0 of ∇ such that $S \subset \bigcup\{\, \mathrm{Cl}(U_\alpha) \ :\ \alpha \in \nabla_0 \,\}$. It is obvious that a subset S of a space X is S-closed relative to X if and only if every cover of S by regular closed sets in X has a finite subcover.

THEOREM 12. *Let $f\colon X \to Y$ be a continuous and pre-almost open surjection such that $f^1(y)$ is S-closed relative to X for each point y of Y. If X is P_Σ (resp. weakly-P_Σ), then Y is P_Σ (resp. weakly-P_Σ).*

Proof. Let V be an arbitrary open (resp. regular open) set in Y and $y \in V$. Then, since pre-almost open sets are almost open, by Lemma 1 $f^1(V)$ is open (resp. regular open) in X. Since X is P_Σ (resp. weakly-P_Σ) and $f^1(y) \subset f^1(V)$, for each $x \in f^1(y)$, there exists $R(x) \in RC(X)$ such that $x \in R(x) \subset f^1(V)$. Since the family $\{\, R(X) \ :\ x \in f^1(y) \,\}$ is a regular closed cover of $f^1(y)$ and $f^1(y)$ is S-closed relative to X, there exists a finite number of points, say $x_1, x_2, \ldots, x_n$, such that $f^1(y) \subset \bigcup\{\, R(x_i) \ :\ 1 \leqslant i \leqslant n \,\}$. The finite union of regular closed sets is regular closed. Therefore, by Lemma 2 there exists $R \in RC(Y)$ containing y such that $f^1(R) \subset \bigcup\{\, R(x_i) \ :\ 1 \leqslant i \leqslant n \,\}$, where each $R(x_i)$ is contained in $f^1(V)$. Therefore, we obtain that $y \in R \subset V$ and Y is P_Σ (resp. weakly-P_Σ). ■

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