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LINEAR CONNECTIONS COMPATIBLE WITH THE $F(3,1)$-STRUCTURE ON THE LAGRANGIAN SPACE

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Abstract. In this paper the F-structure, satisfying $F^3+F=0$ on the Lagrangian space, is examined. The construction of this structure is given as the prolongation of f_v-structure defined on $T_V(E)$ using the almost product or almost complex structure on $T(E)$. Moreover, the metric tensor G, with respect to which F is an isometry, is constructed as wekk as the connection compatible with such structures.

1. Introduction

Let $\mathcal{M}$ be an n-dimensional and E $2n$-dimensional differentiable manifold and let $\eta=(E,\pi,\mathcal{M})$ be vector bundle with $\pi E=\mathcal{M}$. The differential structures (U,ϕ,R^{2n}) are vector charts of the vector bundles η. Hence the canonical coordinates on $\pi^{-1}(U)$ are $(x^1,\dots,x^n,y^1,\dots,y^n)=(x^i,y^a)$, $i=1,2,\dots,n$; $a=1,\dots,n$. Transformation maps on E are

$$x^{i'}=x^{i'}(x^1,x^2,\dots,x^n),\ y^{a'}=M^{a'}_a(x^1,\dots,x^n)y^a=M^{a'}_a(x^i)y^a$$
$$\operatorname{rank}\left[\frac{\partial x^{i'}}{\partial x^i}\right]=n,\ \operatorname{rank}\left[\frac{\partial y^{a'}}{\partial y^a}\right]=\ \operatorname{rank}\ M^{a'}_a=n.$$

The inverse transformations are

$$x^i=x^i(x^{1'},x^{2'},\dots,x^{n'}),\ y^a=M^a_{a'}(x^{i'},\dots,x^{n'})y^{a'},\ \text{ where }\ M^a_{a'}M^{a'}_b=\delta^a_b.$$

The local natural bases of the tangent space $T(E)$ are $\{\partial_i,\partial_a\}$,

$$\partial_a=\frac{\partial}{\partial y^a}=M^{a'}_a(x^i)\partial_{a'},\ \ \partial_i=\frac{\partial}{\partial x^i}=\frac{\partial x^{i'}}{\partial x^i}\partial_{i'}+(\partial_i M^{a'}_b(x^i))y^b\partial_{a'}.$$

The nonlinear connection on E is distribution $N:u\in E\to N_u\subset T_u(E)$ which is supplementary to the distribution V,

$$T_u(E)=N_u\oplus V_u,\quad \forall_u\in E. \tag{1.1}$$

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They are localy determined by $\delta_i = \partial_i - N_i^a \partial_a$. The local bases adapted to decompositions in (1.1) is $\{\delta_i, \partial_a\}$.

It is easy to prove [5] that on $\{\delta_i, \partial_a\}$

$$\delta_{i'} = \delta_i \frac{\partial x^i}{\partial x^{i'}}, \quad \partial_{a'} = \frac{\partial y^a}{\partial y^{a'}} \partial_a.$$

The subspace of $T(E)$ spaned by $\{\delta_i\}$ will be denoted by $T_H(E)$ and the subspace spaned by $\{\partial_a\}$ will be denoted by $T_V(E)$, $T(E) = T_H(E) \oplus T_V(E)$, $\dim T_H(E) = n = \dim T_V(E)$.

Definition 1.1 If the Riemannian metric structure on $T(E)$ is given by $G = g_{ij}(x^i,\ y^a) dx^i \otimes dx^j + g_{ab}(x^i,\ y^a) \delta y^a \otimes \delta y^b$ where $g_{ij}(x^i, y^a) = g_{ij}(x^i)$, $g_{ab} = \frac{1}{2} \partial_a \partial_b L(x^i,\ y^a)$ and $L(x^i,\ y^a)$ is a Lagrange function, then such a space will be called a Lagrangian space.

Let $X \in T(E)$, then $X = X^i \delta_i + \bar{X}^a \partial_a$ and the automorphism $P : \mathcal{X}(T(E)) \to \mathcal{X}(T(E))$ defined by $PX = \bar{X}^i \delta_i + X^a \partial_a$ is the natural almost product structure on $T(E)$ i.e, $P^2 = I$. If we denote by v and h the projection morphisms of $T(E)$ to $T_V(E)$ and $T_H(E)$ respectively, we have $P \circ h = v \circ P$.

2. $f(3,1)$-structures

Definition 2.1. We call Lagrange vertical $f_v(3,1)$-structure of rank r on $T_V(E)$ a non-null tensor field f_v on $T_V(E)$ of type (1,1) and of class C^∞ such that $f_v^3 + f_v = 0$, where rank $f_v = r$, and r is constant everywhere.

Definition 2.2. We call Lagrange horizontal $f_h(3,1)$-structure of rank r on $T_H(E)$ a non-null tensor field f_h on $T_H(E)$ of type (1,1) of class C^∞ satisfying $f_h^3 + f_h = 0$, rank $f_h = r$, where r is constant everywhere.

An $F(3,1)$-structure on $T(E)$ is a non-null tensor field F of type $\binom{11}{11}$ such that $F^3 + F = 0$, rank $F = 2r = \text{const}$.

For our study it is very convenient to consider f_v or f_h as morphisms of vector bundles [2], [3]

$$f_v : \mathcal{X}(T_V(E)) \to \mathcal{X}(T_V(E)), \quad f_h : \mathcal{X}(T_H(E)) \to \mathcal{X}(T_H(E))$$

Let f_v be a Lagrange vertical $f_v(3,1)$-structure of rank r. We define the morphisms $\boldsymbol{l}_v = -f_v^2$ and $\boldsymbol{m}_v = f_v^2 + I_{T_V(E)}$, where $I_{T_V(E)}$ denotes the identity morphism on $T_V(E)$.

It is clear that $\boldsymbol{l}_v + \boldsymbol{m}_v = I$. Also we have

$$\boldsymbol{l}_v \boldsymbol{m}_v = \boldsymbol{m}_v \boldsymbol{l}_v = -f_v^4 - f_v^2 = -f_v(f_v^3 + f_v^1) = 0,\ \boldsymbol{m}_v^2 = \boldsymbol{m}_v,\ \boldsymbol{l}_v^2 = \boldsymbol{l}_v.$$

Hence the morphisms $\boldsymbol{l}_v$, $\boldsymbol{m}_v$ applied to the $\mathcal{X}(T_V(E))$ are complementary projection morphisms. Then, there exist complementary distributions L_v and M_v

corresponding to the projection morphisms $\boldsymbol{l}_v$ and $\boldsymbol{m}_v$ respectively such that dim $L_v = r$ and dim $M_v = n - r$. It is easy to see that

$$\boldsymbol{l}_v f_v = f_v \boldsymbol{l}_v = f_v,\ \boldsymbol{m}_v f_v = f_v \boldsymbol{m}_v = 0,\ f_v^2 \boldsymbol{l}_v = \boldsymbol{l}_v f_v^2 = -I, f_v^2 \boldsymbol{m}_v = 0. \tag{2.1}$$

PROPOSITION 2.1. *If a Lagrange $f_v(3,1)$-structure of rank r is defined on $T_V(E)$, then the horizontal $f_h(3,1)$-structure of rank r is defined on $T_H(E)$ by the natural almost product structure of $T(E)$.*

Proof. If we put

$$f_h X = P f_v P X,\ \ \forall X \in T_H(E), \tag{2.2}$$

it is easy to see that $f_h^3 X = P f_v^3 P X$ and $f_h^3 + f_h = 0$, and rank $f_h = r$. ∎

PROPOSITION 2.2. *If a Lagrange $f_v(3,1)$-structure of rank r is defined on $T_V(E)$, then an $F(3,1)$-structure is defined on $T(E)$ by the natural almost product structure of $T(E)$.*

Proof. We put

$$F = f_h h + f_v v, \tag{2.3}$$

where f_h, is defined by (2.2), and h, v are the projection morphisms of $T(E)$ to $T_H(E)$ and $T_V(E)$, respectively. Then it is easy to check that

$$F^2 = f_h^2 h + f_v^2 v,\ F^3 = f_h^3 h + f_v^3 v.$$

Thus $F^3 + F = 0$. It is clear that rank $F = 2r$.

If $\boldsymbol{l}_h$, $\boldsymbol{m}_h$ are complementary projection morphisms of the horizontal $f_h(3,1)$-structure, which is defined by the natural almost product structure of $T(E)$, we have

$$\boldsymbol{l}_h X = -f_h^2 X = -P f_v^2 P X = P \boldsymbol{l}_v P X,\ \forall X \in T_H(E)$$
$$\boldsymbol{m}_h X = (f_h^2 + I_{T_H(E)}) X = P f_v^2 P X + P I_{T_V(E)} P X = P \boldsymbol{m}_v P X,\ \forall X \in T_H(E)$$

If $\boldsymbol{l}$, $\boldsymbol{m}$ are complementary projection morphisms of the $F(3,1)$-structure on $T(E)$, then we have

$$\begin{aligned} \boldsymbol{l} &= -F^2 = -f_h^2 h - f_v^2 v = \boldsymbol{l}_h h + \boldsymbol{l}_v v \\ \boldsymbol{m} &= F^2 + I_{T(E)} = f_h^2 h + f_v^2 v + I_{T_H(E)} h + I_{T_V(E)} v = \boldsymbol{m}_h h + \boldsymbol{m}_v v. \end{aligned} \tag{2.4}$$

Thus, if there is given a Lagrange $f_v(3,1)$-structure on $T_V(E)$ of rank r, then there exist complementary distributions L_h, M_h of $T_H(E)$, corresponding to the morphisms $\boldsymbol{l}_h$, $\boldsymbol{m}_h$ such that

$$L_h = P L_v\ ,\ M_h = P M_v. \tag{2.5}$$

Thus we have the decompositions $T(E) = T_H(E) \oplus T_V(E) = PL_v \oplus PM_v \oplus L_v \oplus M_v$. If L, M denote complementary distributions corresponding to the morphisms $\boldsymbol{l}$, $\boldsymbol{m}$ respectively then from (2.4) and (2.5) we have

$$L = PL_v \oplus L_v,\ M = PM_v \oplus M_v.$$

Let $\bar{g}_v$ be a pseudo-Riemannian metric tensor, which is symmetric, bilinear and non-degenerate on $T_V(E)$ $\bar{g}_v : \mathcal{X}(T_V(E)) \times \mathcal{X}(T_V(E)) \to \mathcal{F}(T(E))$. (For example $\bar{g}_v$ can be the vertical part of the Lagrange metric structure).

The mapping $a_v : \mathcal{X}(T_V(E)) \times \mathcal{X}(T_V(E)) \to \mathcal{F}(T(E))$ which is defined by

$$a_v(X,Y) = \frac{1}{2}[\bar{g}(\boldsymbol{l}_v X, \boldsymbol{l}_v Y) + \bar{g}(\boldsymbol{m}_v X, \boldsymbol{m}_v Y)] \;\; \forall X, Y \in \mathcal{X}(T_V(E))$$

is a pseudo-Riemannian structure on $T(E)$ such that $a_v(X,Y) = 0$, $\forall X \in \mathcal{X}(L_v)$, $Y \in \mathcal{X}(M_v)$.

THEOREM 2.1. *If a Lagrange $f_v(3,1)$-structure of rank r is defined on $T_V(E)$ then there exists a pseudo-Riemannian structure of $T_V(E)$ with respect to which the complementary distributions L_v and M_v are orthogonal and f_v is an isometry on $T_V(E)$.*

Proof. If we put $g_v(X,Y) = \frac{1}{2}[a_v(X,Y) + a_v(f_v X, f_v Y)]$, it is easy to see that

$$g_v(X,Y) = 0 \quad \forall X \in \mathcal{X}(L_v), \quad Y \in \mathcal{X}(M_v).$$

Using (2.1) we get $g_v(f_v X, f_v Y) = \frac{1}{2}[a_v(f_v X, f_v Y) + a_v(X,Y)]$. Thus f_v is an isometry with respect to g_v.

Let $X \in \mathcal{X}(L_v)$ then $f_v X$, $f_v^2 X \in \mathcal{X}(L_v)$ and

$$g_v(X, f_v X) = g_v(f_v X, f_v^2 X) = -g_v(f_v X, X).$$

Consequently $g_v(X, f_v X) = g_v(f_v X, f_v^2 X) = 0$.

We can define a maping g_h:

$$g_h(X,Y) = g_v(PX, PY), \; \forall X, Y \in \mathcal{X}(T_H(E)),$$

where g_h is a metric structure on $T_H(E)$. Using (2.5) the distributions L_h, M_h are orthogonal with respect to g_h and the horizontal $f_h(3,1)$-structure which is defined by $f_h X = P f_v P X$, $\forall X \in \mathcal{X}(T_H(E))$ is an isometry on $T_H(E)$ with respect to g_h.

We can also define a metric tensor G on $T(E)$.

$$G(X,Y) = g_h(X,Y)h + g_v(X,Y)v. \tag{2.6}$$

The distributions L, M are orthogonal with respect to G and the $F(3,1)$-structure which is defined by $FX = f_h h + f_v v$, $X \in T(E)$ is an isometry on $T(E)$ with respect to G.

3. Linear connections compatible with $F(3,1)$-structure

It is well known that an arbitrary distribution $\mathcal{D}$ is parallel with respect to a linear connection ∇, if for any tangent field Y, ∇_Y is a transformation of $\mathcal{D}$ [4].

DEFINITION 3.1. An $f_v(3,1)$-connection on $T_V(E)$ (or a linear connection compatible with a $f_v(3,1)$-structure) is a linear connection ∇ on $T_V(E)$ with the property that distributions L_v and M_v are parallel with respect to ∇.

In a similar way ∇' is an $f_h(3,1)$-connection if distributions L_h and M_h are parallel with respect to ∇', and $\tilde{\nabla}$ is an $F(3,1)$-connection if distributions L and M are parallel with respect to $\tilde{\nabla}$.

THEOREM 3.1. *Let $\boldsymbol{l}_v$, $\boldsymbol{m}_v$ be the complementary projection morphisms of $f_v(3,1)$-structure.*

A linear connection on $T_V(E)$ is an $f_v(3,1)$-connection if and only if

$$\nabla_X \boldsymbol{l}_v = 0 \quad \forall X \in \mathcal{X}T(E).$$

Proof. Since $\boldsymbol{l}_v$ is a morphism on $T_V(E)$,

$$(\nabla_X \boldsymbol{l}_v)(Y) = \nabla_X \boldsymbol{l}_v Y - \boldsymbol{l}_v \nabla_X Y, \quad \forall X \in \mathcal{X}T(E), Y \in \mathcal{X}T_V(E). \tag{3.1}$$

If $\nabla_X \boldsymbol{l}_v = 0$, then from $\boldsymbol{l}_v + \boldsymbol{m}_v = I$, we have

$$\nabla_X \boldsymbol{m}_v Y = 0,\ \nabla_Y \boldsymbol{l}_v Y = \boldsymbol{l}_v \nabla_X Y,\ \nabla_X \boldsymbol{m}_v Y = \boldsymbol{m}_v \nabla_X Y.$$

Since $\boldsymbol{m}_v \boldsymbol{l}_v = \boldsymbol{l}_v \boldsymbol{m}_v = 0$, we have

$$\boldsymbol{m}_v \nabla_X Y = 0, \quad \forall Y \in \mathcal{X}T(L_v), X \in \mathcal{X}T(E),$$

and

$$\boldsymbol{l}_v \nabla_X Y = 0, \quad \forall Y \in \mathcal{X}T(M_v), X \in \mathcal{X}T(E).$$

Thus $\nabla_X Y \in \mathcal{X}T(L_v)$ for every $Y \in \mathcal{X}T(L_v)$ and $\nabla_X Y \in \mathcal{X}T(M_v)$ for every $Y \in \mathcal{X}T(M_v)$.

Conversely, using the decomposition $Y = \boldsymbol{l}_v Y + \boldsymbol{m}_v Y$ and (3.1) we get

$$(\nabla_X \boldsymbol{l}_v)(Y) = \nabla_X \boldsymbol{l}_v Y - \boldsymbol{l}_v \nabla_X \boldsymbol{l}_v Y - \boldsymbol{l}_v \nabla_X \boldsymbol{m}_v Y.$$

Since ∇_X is an $f_v(3,1)$-connection $\nabla_X \boldsymbol{m}_v Y \in \mathcal{X}T(M_v)$. Consequently $\boldsymbol{l}_v \nabla_Y \boldsymbol{m}_v Y = 0$, $(\nabla_X \boldsymbol{l}_v)(Y) = \nabla_X \boldsymbol{l}_v Y - \boldsymbol{l}_v \nabla_X \boldsymbol{l}_v Y = 0$, because $\boldsymbol{l}_v^2 = \boldsymbol{l}_v$.

Thus: $\nabla_X \boldsymbol{l}_v = 0, \forall X \in \mathcal{X}T(E)$.

In a similar way we have:

PROPOSITION 3.1. *A linear connection ∇'_X on $T_H(E)$ is an $f_h(3,1)$-connection iff $\nabla'_X \boldsymbol{l}_h = 0$, $\forall X \in \mathcal{X}T(E)$.*

PROPOSITION 3.2. *A linear connection $\tilde{\nabla}_X$ on $T(E)$ is an $F(3,1)$-connection iff $\tilde{\nabla}_X \boldsymbol{l} = 0$, $\forall X \in \mathcal{X}T(E)$.*

THEOREM 3.2. *If $\bar{\nabla}_X$ is an arbitrary linear connection on $T_V(E)$ then the operator*

$$\nabla_X = f_v \bar{\nabla}_X f_v, \quad \forall X \in \mathcal{X}T(E)$$

is an $f_v(3,1)$-connection.

Proof. Applying the theorem 3.1 we have

$$(\nabla_X \boldsymbol{l}_v)Y = f_v \nabla_X f_v \boldsymbol{l}_v Y - \boldsymbol{l}_v f_v \nabla_X f_v Y, \quad \forall Y \in \mathcal{X}T_V(E).$$

Since $f_v \boldsymbol{l}_v = \boldsymbol{l}_v f_v = f_v$, we have $\nabla_X \boldsymbol{l}_v = 0$, $\forall X \in \mathcal{X}T(E)$, i.e. ∇_X is an $f_v(3,1)$-connection.

Let ∇_X be a linear connection on $T_V(E)$. We define the linear connection ∇'_X on the $T_H(E)$ by

$$\nabla'_X Y = P\nabla_X PY, \quad \forall X \in \mathcal{X}T(E), Y \in \mathcal{X}T_H(E). \tag{3.2}$$

THEOREM 3.3. *If ∇_X is an $f_v(3,1)$-connection on $T_V(E)$ then ∇'_X defined by (3.2) is a linear connection compatible with the horizontal $f_h(3,1)$-structure.*

Proof. Using (3.1), (3.2) we have

$$\begin{aligned}(\nabla'_X \boldsymbol{l}_h)Y &= \nabla'_X \boldsymbol{l}_h Y - \boldsymbol{l}_h \nabla'_X Y = P\nabla_X PP\boldsymbol{l}_v PY - P\boldsymbol{l}_v PP\nabla_X PY\\ &= P(\nabla_X \boldsymbol{l}_v PY - \boldsymbol{l}_v \nabla_X PY), \quad \forall Y \in \mathcal{X}T_H(E).\end{aligned}$$

According to theorem 3.1, $\nabla_X \boldsymbol{l}_v = 0$, thus $\nabla'_X \boldsymbol{l}_h = 0$, i.e. horizontal connection ∇'_X is compatible with the $f_h(3,1)$-structure.

Next, we define linear connection $\tilde{\nabla}_X$ on $T(E)$ by

$$\tilde{\nabla}_X Y = \nabla'_X hY + \nabla_X vY, \quad \forall X, Y \in \mathcal{X}T(E). \tag{3.3}$$

THEOREM 3.4. *If ∇_X is an $f_v(3,1)$-connection on $T_V(E)$ then $\tilde{\nabla}_X$ defined by (3.2) is a linear connection compatible with the $F(3,1)$-structure.*

Proof. We shall prove that $(\tilde{\nabla}_X \boldsymbol{l})Y = 0$, $\forall X, Y \in \mathcal{X}T(E)$.

$$\begin{aligned}&(\tilde{\nabla}_X \boldsymbol{l})Y = \nabla'_h h\boldsymbol{l}Y + \nabla_X v\boldsymbol{l}Y - \boldsymbol{l}\nabla'_X hY - \boldsymbol{l}\nabla_X vY\\ &= \nabla'_X h(\boldsymbol{l}_h h + \boldsymbol{l}_v v)Y + \nabla_X v(\boldsymbol{l}_h h + \boldsymbol{l}_v v)Y - (\boldsymbol{l}_h h + \boldsymbol{l}_v v)\nabla'_X hY - (\boldsymbol{l}_h h + \boldsymbol{l}_v v)\nabla_X vY.\end{aligned}$$

Consequently $(\tilde{\nabla}_X \boldsymbol{l})(Y) = (\nabla'_X \boldsymbol{l}_h)(hY) + (\nabla_X \boldsymbol{l}_v)vY, \quad \forall X, Y \in \mathcal{X}T(E)$.

According to theorems 3.3 and 3.1 we have $\tilde{\nabla}_X \boldsymbol{l} = 0$, $\forall X \in \mathcal{X}T(E)$ which proves the theorem. ∎

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