МАТЕМАТИЧКИ ВЕСНИК
49 (1997), 163–172

UDK 517.983.2
оригинални научни рад
research paper

ON CONTINUITY OF THE MOORE-PENROSE AND DRAZIN INVERSES

Vladimir Rakočević

Abstract. Contrary to the usual inverse of a square matrix, it is well known that the Moore-Penrose and Drazin inverses of a matrix are not necessarily continuous functions of the elements of the matrix. This paper is a short summary of some results concerning the continuity of the Moore-Penrose and Drazin inverses.

1. Introduction and preliminaries

Let X and Y be infinite-dimensional Banach spaces, and $B(X,Y)$ the set of all bounded linear operators from X into Y. We shall write $B(X)$ instead of $B(X,X)$. For an element T in $B(X,Y)$ let $N(T)$ and $R(T)$ denote, respectvely, the null space and the range of T. Recall that the *reduced minimum modulus* of T, $\gamma(T)$, is defined by

$$\gamma(T) = \inf\{\, \|Tz\|/\operatorname{dist}(z, N(T)) : \operatorname{dist}(z, N(T)) > 0 \,\}. \tag{1.1}$$

$R(T)$ is closed if and only if $\gamma(T) > 0$ ([10], [19]). If there is S in $B(Y,X)$, such that $TST = T$, then $R(T)$ is closed and $\gamma(T) \geq 1/\|S\|$ [5, Lemma 4]. Let us remark that $1/\gamma(T) = k(T)$, where $k(T) = \sup\{\inf\{\|z\| : Tz = y\} : y \in R(T), \|y\| = 1\,\}$.

Let A_n, $(n = 1, 2, \dots)$ and A be operators in $B(X,Y)$. We then write $A_n \to^s A$ if the sequence A_n converges to A *strongly*.

For T in $B(X,Y)$ set $\alpha(T) = \dim N(T)$ and $\beta(T) = \dim X/R(T)$. Recall that an operator $T \in B(X,Y)$ is *semi-Fredholm* if $R(T)$ is closed and at least one of $\alpha(T)$ and $\beta(T)$ is finite. Let $\Phi_+(X,Y)$ ($\Phi_-(X,Y)$) denote the set of *upper (lower) semi-Fredholm* operators, i.e., the set of semi-Fredholm operators with $\alpha(T) < \infty$ ($\beta(T) < \infty$). An operator T is *Fredholm* if it is both upper semi-Fredholm and lower semi-Fredholm. Let $\Phi(X,Y)$ denote the set of Fredholm operators, i.e., $\Phi(X,Y) = \Phi_+(X,Y) \cap \Phi_-(X,Y)$. We shall write $\Phi(X)$ instead of $\Phi(X,X)$, and $\Phi_\pm(X)$ instead of $\Phi_\pm(X,X)$ ([3], [6], [10]).

AMS Subject Classification: 47 A 05, 47 A 53, 15 A 09

Communicated at the 4th Symposium on Mathematical Analysis and Its Applications, Aranđelovac 1997.

Supp. by the Sci. Fund of Serbia, g. n. 04M03, through Matematički institut

Recall that if $T \in B(X)$, then $a(T)$ $(d(T))$, the *ascent (descent)* of $T \in B(X)$, is the smallest non-negative integer n such that $N(T^n) = N(T^{n+1})$ $(R(T^n) = R(T^{n+1}))$. If no such n exists, than $a(T) = \infty$ $(d(T) = \infty)$.

If M and N are two closed subspaces of the Banach space X we set

$$\delta(M, N) = \sup\{\, \mathrm{dist}\,(u, N) : u \in M, \|u\| = 1\,\} \tag{1.2}$$

and

$$\hat{\delta}(M, N) = \max\,[\delta(M, N), \delta(N, M)]. \tag{1.3}$$

$\hat{\delta}$ is called the *gap* (or *opening*) between the M and N [10, p. 197].

2. The Moore-Penrose inverse

Let $\mathbb{C}^{m\times n}$ be the set of all $m \times n$ complex matrices. Let $A \in \mathbb{C}^{m\times n}$, and consider the *Moore-Penrose equations*:

$$AGA = A, \quad GAG = G, \quad (AG)^* = AG, \quad \text{and} \quad (GA)^* = GA. \tag{2.1}$$

Penrose [14, Theorem 1] has proved that above four equations have a unique solution for any A; he has called it the *generalized inverse* of A and he has written $G = A^\dagger$. The concept of a generalized inverses of an arbitrary matrix $A \in \mathbb{C}^{m\times n}$ is originaly due to Moore 1920 [13] (called by him the *general reciprocal*). R. Rado [15] has proved the equivalence of Moore's and Penrose's result, and today this inverse is known as *Moore-Penrose inverse*, shortly *M-P inverse* of A.

Let us recall that for any $A \in \mathbb{C}^{m\times n}$ we have $(A^\dagger)^\dagger = A$, $(A^*)^\dagger = (A^\dagger)^*$, $(A^*A)^\dagger = A^\dagger(A^\dagger)^*$, but in general $A^\dagger A \neq AA^\dagger$.

Let H and K be infinite-dimensional complex Hilbert spaces. It is well known (see e.g., [3], [6] or [19]) that $A \in B(H, K)$ has closed range if and only if there exists a unique operator $A^\dagger \in B(K, H)$ called the *Moore-Penrose inverse (pseudoinverse)* of A which satisfies the following properties:

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger \quad \text{and} \quad (A^\dagger A)^* = A^\dagger A. \tag{2.2}$$

Let $\mathcal{A}$ be a complex Banach algebra with identity 1. The element $a \in \mathcal{A}$ is *(von Neumann) regular* if $a \in a\mathcal{A}a$. The set of all regular elements in $\mathcal{A}$ will be denoted by $\widehat{\mathcal{A}}$. Recall that an element a in $\mathcal{A}$ is *hermitian* if $\|\exp(ita)\| = 1$ for all real t [24]. In connection with the Moore-Penrose generalized inverse, we have studied ([16], [17]) the set of elements a in $\mathcal{A}$ for which there exists an x in $\mathcal{A}$ satisfying the following conditions:

$$axa = a, \quad xax = x, \quad ax \quad \text{and} \quad xa \quad \text{are hermitian.} \tag{2.3}$$

By [16, Lemma 2.1] there is at most one x such that equations (2.3) hold. The unique x is denoted by $a^\dagger$ and called the Moore-Penrose inverse of a. Let $\mathcal{A}^\dagger$ denote the set of all elements in $\mathcal{A}$ which have Moore-Penrose inverses. Clearly $\mathcal{A}^\dagger \subset \widehat{\mathcal{A}}$, and if $\mathcal{A}$ is a C^*-algebra then $\mathcal{A}^\dagger = \widehat{\mathcal{A}}$ [7, Theorem 6]. Given an element $a \in \mathcal{A}$ let

L_a denote the *left regular representation* of a, i.e., $L_a(x) = ax, x \in \mathcal{A}$. If $a \in \mathcal{A}^\dagger$, then it is known that $\|a^\dagger\| = 1/\gamma(L_a)$ [16, Theorem 2.3]. Let us remark that Harte and Mbekhta ([7], [8]) have studied generalized inverse in C^*-algebra A, i.e., they have studied the set of elements a in $\mathcal{A}$ for which there exists an x in $\mathcal{A}$ satisfying the following conditions:

$$axa = a, \quad xax = x, \quad (ax)^* = ax \quad \text{and} \quad (xa)^* = xa. \tag{2.4}$$

Clearly (2.3) is a generalization of (2.4).

3. The Drazin inverse

Let us recall that Drazin [4] has introduced and investigated a generalized inverse (he called it *pseudoinverse*) in associative rings and semigroups, i.e., if S is an algebraic semigroup (or associative ring), then an element $a \in S$ is said to have a *Drazin* inverse if there exists $x \in S$ such that

$$a^m = a^{m+1}x \qquad \text{for some non-negative integer } m, \tag{3.1}$$

$$x = ax^2 \quad \text{and} \quad ax = xa. \tag{3.2}$$

If a has Drazin inverse, then the smallest non-negative integer m in (3.1) is called the *index* $i(a)$ of a. It is well known that there is at most one x such that equations (3.1) and (3.2) hold. The unique x is denoted by a^d and called the *Drazin* inverse of a. If S is an associative ring and $a \in S$ has Drazin inverse then a may always be written as

$$a = c + n, \tag{3.3}$$

where $c, n \in S$, c has Drazin inverse, $i(c) \leq 1$, $cn = nc = 0$, and $n^{i(a)} = 0$. The elements c, n are unique. c is called the *core* of a, and n the *nilpotent* part of a. Let us mention that in this case

$$c = a^2a^d \qquad \text{and} \qquad n = a - a^2a^d. \tag{3.4}$$

We shall refer to $c + n$ as the *core nilpotent* decomposition of a ([1], [2]).

Recall that a square matrix always has Drazin inverse. But, if X is an infinite-dimensional complex Banach space, then it is well known that an operator $T \in B(X)$ has a Drazin inverse T^d if and only if it has finite ascent and descent. In such a case, the index of T is equal to the ascent of T ([3], [6], [11]).

4. Continuity of the Moore-Penrose inverse

Contrary to the usual inverse of a square matrix, it is well known that the Moore-Penrose generalized inverse of a matrix is not necessarily a continuous function of the elements of the matrix.

EXAMPLE 4.1. ([23]) Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

For each $\epsilon \neq 0$ we have

$$(A+\epsilon E)^{\dagger} = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix}^{\dagger} = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}.$$

Hence $A+\epsilon E \to A$, $(\epsilon \to 0)$, but $\lim_{\epsilon\to 0}(A+\epsilon E)^{\dagger}$ does not exist.

The following theorem gives necessary and sufficient conditions for continuity of the Moore-Penrose inverse of matrix.

THEOREM 4.2. ([14], [23]) *If $A_n \in \mathbb{C}^{m\times n}$, $A \in \mathbb{C}^{m\times n}$, and $A_n \to A$, then*

$$A_n^{\dagger} \to A^{\dagger} \quad \Leftrightarrow \quad \exists n_0 : \quad \mathit{rank}\, A_n = \mathit{rank}\, A \quad \textit{for } n \geq n_0. \tag{4.1}$$

The continuity of the Moore-Penrose inverse of an operator on Hilbert spaces has been studied by Izumino [9].

THEOREM 4.3. ([9, Proposition 2.3]) *Let H and K be Hilbert spaces. Let A_n be a sequence in $B(H,K)$, $A \in B(H,K)$, and $A_n \to A$. If $A_n^{\dagger}$ and $A^{\dagger}$ exist, then the following conditions are equivalent:*

$$A_n^{\dagger} \to A^{\dagger}, \tag{4.2}$$

$$\sup_n \|A_n^{\dagger}\| < \infty, \tag{4.3}$$

$$A_n^{\dagger} A_n \to A^{\dagger} A, \tag{4.4}$$

$$A_n A_n^{\dagger} \to A A^{\dagger}. \tag{4.5}$$

Concerning the strong convergence of the Moore-Penrose inverses Izumino has proved the following theorem.

THEOREM 4.4. ([9, Proposition 2.4]) *Let H and K be Hilbert spaces. Let A_n be a sequence in $B(H,K)$, $A \in B(H,K)$, and $A_n \to^s A$. If $A_n^{\dagger}$ and $A^{\dagger}$ exist, then the following conditions are equivalent:*

$$A_n^{\dagger} \to^s A^{\dagger}, \tag{4.6}$$

$$\sup_n \|A_n^{\dagger}\| < \infty, \quad A_n^{\dagger} A_n \to^s A^{\dagger} A \quad \textit{and} \quad A_n A_n^{\dagger} \to^s A A^{\dagger}. \tag{4.7}$$

REMARK 4.5. In contrast to the case of uniform convergence, Izumino [9, Remark (1)] has proved that we can not deduce the inequality $\sup_n \|A_n^{\dagger}\| < \infty$ from $A_n^{\dagger} A_n \to^s A^{\dagger} A$ (or $A_n A_n^{\dagger} \to^s A A^{\dagger}$). For example, let $A = I$ on $X = l_2$, and let

$$A_n = \operatorname{diag}\{\overbrace{1,\ldots,1}^{n}, 1/n, 1/n, \ldots\} \quad \text{on} \quad X = l_2, \quad n = 1, 2, \ldots.$$

Now, $A_n \to^s A$, and

$$A_n^\dagger = \operatorname{diag}\{\overbrace{1,\dots,1}^{n}, n, n, \dots\}, \quad n = 1, 2, \dots.$$

Hence $A_n^\dagger A_n \to^s A^\dagger A$, but $\|A_n^\dagger\| = n$, $\quad n = 1, 2, \dots$, i.e., $\sup_n \|A_n^\dagger\| = \infty$.

Now we show that some of the previous results could be presented in general Banach algebras.

Theorem 4.6. ([17, Theorem 2.5]) *Let $\mathcal{A}$ be a complex Banach algebra, $\{a_n\}$ a sequence in $\mathcal{A}^\dagger$, and let $a_n \to a \in \mathcal{A}^\dagger$. Then the following conditions are equivalent:*

$$a_n^\dagger \to a^\dagger, \tag{4.8}$$

$$\sup_n \|a_n^\dagger\| < \infty, \quad \hat{\delta}(N(L_{a_n^\dagger}), N(L_{a^\dagger})) \to 0, \quad \hat{\delta}(R(L_{a_n^\dagger}, R(L_{a^\dagger})) \to 0, \tag{4.9}$$

$$a_n^\dagger a_n \to a^\dagger a \quad \textit{and} \quad \hat{\delta}(N(L_{a_n^\dagger}), N(L_{a^\dagger})) \to 0, \tag{4.10}$$

$$a_n a_n^\dagger \to a a^\dagger \quad \textit{and} \quad \hat{\delta}(R(L_{a_n^\dagger}), R(L_{a^\dagger})) \to 0. \tag{4.11}$$

If $\mathcal{A}$ is a C^*-algebra, then the previous results could be presented in a simpler form.

Theorem 4.7. ([8, Theorem 6], [18, Theorem 2.2]) *Let $\mathcal{A}$ be a C^*-algebra, $\{a_n\}$ a sequence in $\mathcal{A}^\dagger$, and let $a_n \to a \in \mathcal{A}^\dagger$. Then the following conditions are equivalent:*

$$a_n^\dagger \to a^\dagger, \tag{4.12}$$

$$\sup_n \|a_n^\dagger\| < \infty, \tag{4.13}$$

$$a_n^\dagger a_n \to a^\dagger a, \tag{4.14}$$

$$a_n a_n^\dagger \to a a^\dagger. \tag{4.15}$$

5. Continuity of the Drazin inverse

It is well known that the Drazin inverse of a matrix is not necessarily a continuous function of the elements of the matrix.

Example 5.1. ([1, Example 1]) Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1/n \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $A_n \to A$, $A_n^d = A_n^{-1}$, $A^d = A^{-1}$, and hence $A_n^d \to A^d$. Let us remark that $\operatorname{rank} A_n > \operatorname{rank} A$ for all n, and by Theorem 4.2 we have that $A_n^\dagger \not\to A^\dagger$.

EXAMPLE 5.2. ([1, Example 2]) Let

$$A_n = \begin{pmatrix} 1/n & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then

$$A_n^d = \begin{pmatrix} n & n^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A^d = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence $A_n \to A$, rank $A_n =$ rank A, and $i\,(A_n) = i\,(A)$, but $A_n^d \not\to A^d$. Let us remark that by Theorem 4.2 we have that $A_n^\dagger \to A^\dagger$.

The following theorem gives necessary and sufficient conditions for continuity of the Drazin inverse of matrix.

THEOREM 5.3. ([1, Theorem 2]) *Suppose that $A_n \in \mathbb{C}^{m\times m}$, $A \in \mathbb{C}^{m\times m}$, and $A_n \to A$. Let $A = C + N$ and $A_n = C_n + N_n$, $n = 1, 2, \ldots$, be the core nilpotent decompositions of A and A_n, $n = 1, 2, \ldots$, respectively. Then*

$$A_n^d \to A^d \quad \Leftrightarrow \quad \exists n_0 : \quad rank\, C_n = rank\, C \quad for\ n \geq n_0. \tag{5.1}$$

In [22] we have studied the continuity of the Drazin inverse of bounded linear operator on Banach space X, i.e., the continuity of the map $A \mapsto A^d$, $A \in B(X)$. Then, among other things, as a corollary we have got the well known result of Campbell and Meyer for the continuity of the Drazin inverse of square matrix (Theorem 5.3). Let us mention that Campbell and Meyer in their proof used the continuity of the Moore Penrose inverse of matrix. It seems that our proof of [22, Theorem 2.2, Corollary 3.5] is more natural, because it does not invoke the Moore-Penrose inverse and the definition of Drazin inverse does not clearly involve the notion of the Moore-Penrose inverse.

Now we give the main result of [22].

THEOREM 5.4. ([22, Theorem 2.2]) *Let X be an infinite-dimensional complex Banach space. Let $\{A_n\}$ be a sequence in $B(X)$, and let $A_n \to A \in B(X)$. Suppose that A and A_n, $n = 1, 2, \ldots$, have Drazin inverses A^d and A_n^d, $n = 1, 2, \ldots$, respectively, and let $A = C + N$ and $A_n = C_n + N_n$, $n = 1, 2, \ldots$, be the core nilpotent decompositions of A and A_n, $n = 1, 2, \ldots$, respectively. Then the following conditions are equivalent:*

$$A_n^d \to A^d, \tag{5.2}$$

$$A_n^d A_n \to A^d A, \tag{5.3}$$

$$\sup_n \|A_n^d\| < \infty, \tag{5.4}$$

$$\hat{\delta}(N(C_n), N(C)) \to 0, \quad \textit{and} \quad \hat{\delta}(R(C_n), R(C)) \to 0, \tag{5.5}$$

$$C_n \to C \quad \textit{and} \quad \hat{\delta}(N(C_n), N(C)) \to 0, \tag{5.6}$$

$$C_n \to C \quad \textit{and} \quad \hat{\delta}(R(C_n), R(C)) \to 0, \tag{5.7}$$

$$C_n \to C \quad \textit{and} \quad \sup_n k(C_n) < \infty, \tag{5.8}$$

$$\sup_n k(C_n) < \infty. \tag{5.9}$$

As a corollary we get.

Corollary 5.5. ([22, Corollary 3.3]) *Let A_n, A, C_n, C, N_n and N be as above in Theorem 5.4. If the index of A_n, $n = 1, 2, \ldots$, is bounded, i.e., if $\sup_n i(A_n) < \infty$, then the following conditions are equivalent:*

$$A_n^d \to A^d, \tag{5.10}$$

$$\hat{\delta}(N(C_n), N(C)) \to 0, \tag{5.11}$$

$$\hat{\delta}(R(C_n), R(C)) \to 0. \tag{5.12}$$

Now we study the continuity of the Drazin inverse when operators have finite dimensional null spaces and when their ranges have finite codimension. It is clear that if T is semi-Fredholm and has Drazin inverse, then T is Fredholm. Let us remark that the convergence problem of Drazin inverses is connected with perturbation problem of them (see [20] and [21] for the recent results on perturbations of semi-Fredholm operators with finite ascent or descent).

Corollary 5.6. ([22, Corollary 3.4]) *Let A_n, A, C_n, C, N_n and N be as above in Theorem 5.4. If the index of A_n, $n = 1, 2, \ldots$, is bounded and $C, C_n \in \Phi(X)$, $n = 1, 2, \ldots$, then the following conditions are equivalent:*

$$A_n^d \to A^d, \tag{5.13}$$

$$\hat{\delta}(N(C_n), N(C)) \to 0, \tag{5.14}$$

$$\hat{\delta}(R(C_n), R(C)) \to 0. \tag{5.15}$$

$$\textit{there is an integer } n_0 \textit{ such that} \quad \alpha(C_n) = \alpha(C), \quad \textit{for } n \geq n_0, \tag{5.16}$$

$$\textit{there is an integer } n_0 \textit{ such that} \quad \beta(C_n) = \beta(C), \quad \textit{for } n \geq n_0. \tag{5.17}$$

Let us remark, that now it is clear that as a corollary of Corollary 5.6 we get Theorem 5.3, the result of Campbell and Meyer.

We give next an equivalent condition for the strong convergence of Drazin inverses, which is to be compared with Theorem 5.4.

Theorem 5.7. ([22, Theorem 3.6]) *Let $\{A_n\}$ be a sequence in $B(X)$, and let $A_n \to^s A \in B(X)$. Suppose that A and A_n, $n = 1, 2, \ldots$, have Drazin inverses A^d*

and A_n^d, $n = 1, 2, \ldots$, *respectively. Then the following conditions are equivalent:*

$$A_n^d \to^s A^d, \tag{5.18}$$

$$\sup_n \|A_n^d\| < \infty, \quad \text{and} \quad A_n^d A_n \to^s A^d A. \tag{5.19}$$

REMARK 5.8. In contrast to the case of uniform convergence, we can not deduce the inequality $\sup_n \|A_n^d\| < \infty$ from $A_n^d A_n \to^s A^d A$. For example, let let A, X and A_n be as in Remark 4.5. Clearly, $A_n \to^s A$, and

$$A_n^d = \operatorname{diag} \{\overbrace{1, \ldots, 1}^{n}, n, n, \ldots\}, \quad n = 1, 2, \ldots.$$

Now $A_n^d A_n \to^s A^d A$, but $\|A_n^d\| = n$, $n = 1, 2, \ldots$, i.e., $\sup_n \|A_n^d\| = \infty$.

Let us mention that some of the previous results could be presented in general Banach algebras. Now by Theorem 5.4 we have

THEOREM 5.9. ([22, Theorem 4.1]) *Let $\mathcal{A}$ denote a complex Banach algebra with identity 1. Let $\{a_m\}$ be a sequence in $\mathcal{A}$, and let $a_m \to a \in \mathcal{A}$. Suppose that a and a_m, $m = 1, 2, \ldots$, have Drazin inverses a^d and a_m^d, $m = 1, 2, \ldots$, respectively, and let $a = c + n$ and $a_m = c_m + n_m$, $m = 1, 2, \ldots$, be the core nilpotent decompositions of a and a_m, $m = 1, 2, \ldots$, respectively. Then the following conditions are equivalent:*

$$a_m^d \to a^d, \tag{5.20}$$

$$a_m^d a_m \to a^d a, \tag{5.21}$$

$$\sup_m \|a_m^d\| < \infty, \tag{5.22}$$

$$\hat{\delta}(N(L_{c_m}), N(L_c)) \to 0, \quad \text{and} \quad \hat{\delta}(R(L_{c_m}), R(L_c)) \to 0, \tag{5.23}$$

$$c_m \to c \quad \text{and} \quad \hat{\delta}(N(L_{c_m}), N(L_c)) \to 0, \tag{5.24}$$

$$c_m \to c \quad \text{and} \quad \hat{\delta}(R(L_{c_m}), R(L_c)) \to 0, \tag{5.25}$$

$$c_m \to c \quad \text{and} \quad \sup_m k(L_{c_m}) < \infty, \tag{5.26}$$

$$\sup_m k(L_{c_m}) < \infty. \tag{5.27}$$

Concerning the results from Section 4 we get the following theorem.

THEOREM 5.10. ([22, Theorem 4.2]) *Let $\mathcal{A}$ denote a complex Banach algebra with identity 1. Suppose that all the assumptions from Theorem 5.9 are valid, and that in addition c and c_m are in $\mathcal{A}^\dagger$, $m = 1, 2, \ldots$. Then the following conditions are equivalent:*

$$a_m^d \to a^d, \quad \hat{\delta}(N(L_{c_m^\dagger}), N(L_{c^\dagger})) \to 0, \quad \text{and} \quad \hat{\delta}(R(L_{c_m^\dagger}), R(L_{c^\dagger})) \to 0, \tag{5.28}$$

$$c_m^\dagger \to c^\dagger. \tag{5.29}$$

If $\mathcal{A}$ is a C^*–algebra, then the previous results could be presented in a simpler form.

THEOREM 5.11. ([22, Theorem 4.3]) *Let $\mathcal{A}$ ba a C^*–algebra, and suppose that all the assumptions from Theorem 5.9 are valid. Then c and c_m are in $\mathcal{A}^\dagger$, $m = 1, 2, \ldots$, and the following conditions are equivalent:*

$$a_m^d \to a^d, \tag{5.30}$$

$$c_m^\dagger \to c^\dagger. \tag{5.31}$$

REMARK 5.12. Recently Koliha [12] has introduced and investigated a generalized inverse (he called it a *generalized Drazin inverse*) in associative rings and Banach algebras, i.e., if A is a complex unital Banach algebra, then an element $a \in A$ is said to have a *generalized Drazin* inverse if there exists $x \in A$ such that

$$a - a^2x \qquad \text{is quasinilpotent,} \tag{5.32}$$

$$x = ax^2 \quad \text{and} \quad ax = xa. \tag{5.33}$$

If a has generalized Drazin inverse, then there is at most one x such that equations (5.32) and (5.33) hold. The unique x is denoted by a^D. In our opinion it is an open (and interesting) problem to find some necessary and sufficient conditions for continuity of the generalized Drazin inverse.

REFERENCES

[**1**] S. L. Campbell and C. D. Meyer, Jr., *Continuity Properties of the Drazin Pseudoinverse*, Linear Alg. Applic. **10** (1975), 77–83.

[**2**] S. L. Campbell and C. D. Meyer, Jr., *Generalized Inverses of Linear Transformations*, Pitman, 1979.

[**3**] S. R. Caradus, *Generalized Inverses and Operator Theory*, Queen's Papers in Pure and Applied Mathematics no. 50, Queen's University, Kingston, Ontario, 1978.

[**4**] M. P. Drazin, *Pseudoinverse in associative rings and semigroups*, Am. Math. Monthly **65** (1958), 506–514.

[**5**] K. H. Förster and M. A. Kaashoek, *The asymptotic behaviour of the redused minimum modulus of a Fredholm operator*, Proc. Amer. Math. Soc. **49** (1975), 123–131.

[**6**] R. Harte, *Invertibility and Singularity for Bounded Linear Operators*, Marcel Dekker, Inc., New York and Basel, 1988.

[**7**] R. Harte and M. Mbekhta, *On generalized inverses in C^*-algebras*, Studia Math. **103** (1992), 71–77.

[**8**] R. Harte and M. Mbekhta, *On generalized inverses in C^*-algebras II*, Studia Math. **106** (1993), 129–138.

[**9**] S. Izumino, *Convergence of generalized inverses and spline projectors*, J. Approx. Theory **38** (1983), 269–278

[**10**] T. Kato, *Perturbation Theory for Linear Operators*, Springer, 1976.

[**11**] C. F. King, *A note on Drazin inverses*, Pacific J. Math. **70** (1977), 383–390.

[**12**] J. J. Koliha, *A generalized Drazin inverse*, Glasgow Math. J **38** (1996), 367–381.

[**13**] E. H. Moore, *On the reciprocal of the general algebraic matrix (Abstract)*, Bull. Amer. Math. Soc **26** (1920), 394–395.

[**14**] R. Penrose, *A generalized inverse for matrices*, Proc. Cambrifge Phil. Soc. **51** (1955), 406–413.

[**15**] R. Rado, *Note on generalized inverses of matrices*, Proc. Cambridge Phil. Soc. **52** (1956), 600–601.

[**16**] V. Rakočević, *Moore-Penrose inverse in Banach algebras*, Proc. R. Ir. Acad. **88A** (1988), 57–60.

[**17**] V. Rakočević, *On the continuity of the Moore-Penrose inverse in Banach algebras*, Facta Univ. Ser. Math. Inform. **6** (1991), 133–138.

[**18**] V. Rakočević, *On the continuity of the Moore-Penrose inverse in C^*-algebras*, Math. Montisnigri **2** (1993), 89–92.

[**19**] V. Rakočević, *Funkcionalna analiza*, Naučna knjiga, Beograd 1994.

[**20**] V. Rakočević, *Semi-Fredholm operators with finite ascent or descent and perturbations*, Proc. Amer. Math. Soc. **123** (1995), 3823–3825.

[**21**] V. Rakočević, *Semi-Browder operators and perturbations*, Studia Math. **122** (1997), 131–137.

[**22**] V. Rakočević, *Continuity of the Drazin inverse*, preprint.

[**23**] G. W. Stewart, *On the continuity of the generalized inverse*, SIAM **17** (1969), 33–45.

[**24**] I. Vidav, *Eine metrische Kennzeichnung der selbstadjungieten Operatoren*, Math. Z. **66** (1956), 121–128.

(received 07.08.1997.)

University of Niš, Faculty of Philosophy, Department of Mathematics, Ćirila and Metodija 2, 18000 Niš, Yugoslavia

e-mail: vrakoc@archimed.filfak.ni.ac.yu