МАТЕМАТИЧКИ ВЕСНИК
50 (1998), 105–110

UDK 517.58
оригинални научни рад
research paper

AN APPLICATION FOR THE CHEBYSHEV POLYNOMIALS

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Abstract. Two sequences of polynomials for which all zeros, regardless of degree n, can be given by the following "simple formulae"

$$\Gamma_{n,m}(\xi) = \cot\left(\frac{(\xi+m)\pi}{n}\right) \quad \text{and} \quad \Delta_{n,m}(\xi) = \tan\left(\frac{(\xi+m)\pi}{n}\right) \quad (0 < \xi < 1)$$

($n = 1, 2, \dots$; $m = 0, 1, \dots, n-1$ and $m \neq (n-1)/2$ when $\xi = 1/2$ and n is odd in the case of $\Delta_{n,m}$) are obtained from the linear combination of the Chebyshev polynomials of the first and second kind.

1. Introduction

In our recent work [1] on the Apostol formula for the Riemann zeta function [2] the problem has arisen of the closed-form evaluation of the following finite cotangent sum

$$S_n(q;\xi) = \sum_{p=0}^{q-1} \cot^n\left(\frac{(\xi+p)\pi}{n}\right) \qquad (0 < \xi < 1) \tag{1}$$

for any positive integer n. It appears that this sum is known only when $n = 1, 2, 4$ ([3], Section 4.4.7 and [4], Sections 29.1 and 30.1). In this note we shall establish a simple way of its summation. Namely, $S_n(q;\xi)$ can be evaluated for any n by using the Newton identities (the Newton power sum formulae), since there exist the sequence of polynomials of which the numbers

$$\cot\left(\frac{(\xi+p)\pi}{n}\right) \qquad p = 0, 1, \dots, q-1$$

are zeros. The polynomials in question are constructed by making use of the Chebyshev polynomials of the first and the second kind. A similar procedure leads to the analogous polynomials of which $\tan[(\xi+p)\pi/q]$ are zeros.

AMS Subject Classification: Primary 33 C 45; Secondary 33 C 90
Keywords and phrases: Chebyshev polynomials, polynomials, zeros of polynomials

2. Statement of results

Throughout the text $T_n(x)$ and $U_n(x)$ are, respectively, the Chebyshev polynomials of the first and second kind defined on $[-1,1]$ by [5, p. 776]

$$T_n(x) = \cos(n \arccos x) = \frac{n}{2} \sum_{k=0}^{[n/2]} (-1)^k \frac{(n-k-1)!}{k!\,(n-2k)!} (2x)^{n-2k} \tag{2a}$$

and

$$U_n(x) = \frac{\sin((n+1)\arccos x)}{\sqrt{1-x^2}} = \sum_{k=0}^{[n/2]} (-1)^k \frac{(n-k)!}{k!\,(n-2k)!} (2x)^{n-2k}. \tag{2b}$$

Further, $[x]$ denotes the largest integer not exceeding x while the principal values of $\operatorname{arccot}\theta$ and of $\arctan\theta$ are defined for any real θ by $0 < \operatorname{arccot}\theta < \pi$ and by $-\pi/2 < \arctan\theta < \pi/2$. Our results are as follows.

THEOREM. *Assume that n is a positive integer and that ξ is a real, $0 < \xi < 1$. Let $T_n(x)$ and $U_n(x)$ be the Chebyshev polynomials defined in (2a,b). Consider*

$$C_n(x;\xi) = (1+x^2)^{n/2} \left[T_n\Big(\frac{x}{\sqrt{1+x^2}}\Big) - \frac{\cot(\pi\xi)}{\sqrt{1+x^2}} U_{n-1}\Big(\frac{x}{\sqrt{1+x^2}}\Big) \right] \tag{3}$$

and

$$K_n(x;\xi) = (1+x^2)^{n/2} \left[T_n\Big(\frac{1}{\sqrt{1+x^2}}\Big) - x\frac{\cot(\pi\xi)}{\sqrt{1+x^2}} U_{n-1}\Big(\frac{1}{\sqrt{1+x^2}}\Big) \right]. \tag{4}$$

Then we have:

(1) $C_n(x;\xi)$ and $K_n(x;\xi)$ are polynomials (with a parameter ξ) in a variable x, defined for any real x, such that:

(a) the polynomials $C_n(x;\xi)$ are monic of degree exactly n, given by

$$C_n(x;\xi) = \sum_{k=0}^{n} \binom{n}{k} c_k(\xi) x^{n-k}; \tag{5a}$$

(b) the polynomials $K_n(x;\xi)$ are of degree $n-1$ when $\xi = 1/2$ and n is odd and of degree n otherwise, and they are given by

$$K_n(x;\xi) = \sum_{k=0}^{n} \binom{n}{k} c_k(\xi) x^{k}. \tag{5b}$$

Here, in (a) and (b), $c_k(\xi)$ has the same meaning:

$$c_{2r}(\xi) = (-1)^r, \quad c_{2r+1}(\xi) = (-1)^{r+1}\cot(\pi\xi) \qquad (0<\xi<1, r \in \mathbf{N}_0). \tag{5c}$$

(2) For any fixed n, all zeros of $C_n(x;\xi)$ and $K_n(x,\xi)$ are respectively given by:

(a) $\Gamma_{n,m}(\xi) = \cot\left(\dfrac{(\xi+m)\pi}{n}\right)$, $m = 0,1,\dots,n-1$;

(b) $\Delta_{n,m}(\xi) = \tan\left(\dfrac{(\xi+m)\pi}{n}\right)$, $m = 0,1,\dots,n-1$, $m \neq (n-1)/2$ when $\xi = 1/2$ and n is odd.

3. Proof of the Theorem

(1a) and (2a). In order to prove these two parts of the theorem, we derive the following formulae for any real x

$$C_n(x;\xi) = (1+x^2)^{n/2}[\cos(n \operatorname{arccot} x) - \cot(\pi\xi)\sin(n \operatorname{arccot} x)] \tag{6a}$$

$$= \csc^n(\operatorname{arccot} x)[\cos(n \operatorname{arccot} x) - \cot(\pi\xi)\sin(n \operatorname{arccot} x)] \tag{6b}$$

$$= \sum_{k=0}^{[n/2]}(-1)^k\binom{n}{2k}x^{n-2k} - \cot(\pi\xi)\sum_{k=0}^{[(n-1)/2]}(-1)^k\binom{n}{2k+1}x^{n-2k-1} \tag{6c}$$

on the assumption that $\operatorname{arccot} x$ takes its principal value.

First, in view of (3), the formula in (6a) readily follows since it is easy to show, starting from the definitions of the Chebyshev polynomials in (2a,b) and using the relationship $\cos\theta = \cot\theta(1+\cot^2\theta)^{-1/2}$ that the following holds

$$\cos(n \operatorname{arccot} x) = T_n\Big(\frac{x}{\sqrt{1+x^2}}\Big), \quad \sin(n \operatorname{arccot} x) = \frac{1}{\sqrt{1+x^2}}U_{n-1}\Big(\frac{x}{\sqrt{1+x^2}}\Big)$$

for any real x. Moreover, the formula in (6b) follows at once from (6a), considering that for any real θ and integer r we have $\csc\theta = (1+\cot^2\theta)^{1/2}$ when $\theta \neq r\pi$ $(r \in \mathbf{Z})$.

Next, from (2a,b) we also have

$$(1+x^2)^{\frac{n}{2}}T_n\Big(\frac{x}{\sqrt{1+x^2}}\Big) = \frac{n}{2}\sum_{k=0}^{[n/2]}(-1)^k\frac{(n-k-1)!}{k!\,(n-2k)!}(2x)^{n-2k}(1+x^2)^k, \tag{7a}$$

$$(1+x^2)^{\frac{n-1}{2}}U_{n-1}\Big(\frac{x}{\sqrt{1+x^2}}\Big) = \sum_{k=0}^{[(n-1)/2]}(-1)^k\frac{(n-k-1)!}{k!\,(n-1-2k)!}(2x)^{n-2k-1}(1+x^2)^k, \tag{7b}$$

and it is thus clear that $C_n(x;\xi)$ (see (3)) are polynomials in x of degree n which can be expressed in terms of the finite sums in (7a,b).

However, our polynomials can be represented in much simpler way as it is suggested in (6c). Indeed, note that the real and imaginary parts of $z = (x+i)^n$ are

$$\operatorname{Re} z = (1+x^2)^{n/2}\cos(n \operatorname{arccot} x) \quad \text{and} \quad \operatorname{Im} z = (1+x^2)^{n/2}\sin(n \operatorname{arccot} x).$$

On the other hand, upon setting $z^* = (x-i)^n$ and making use of the binomial theorem, we arrive at

$$\operatorname{Re} z = \frac{z+z^*}{2} = \sum_{k=0}^{[n/2]}(-1)^k\binom{n}{2k}x^{n-2k},$$

$$\operatorname{Im} z = \frac{z-z^*}{2i} = \sum_{k=0}^{[(n-1)/2]}(-1)^k\binom{n}{2k+1}x^{n-2k-1}$$

so that the formula in (6c) and the proposed formula in (5a), as its immediate consequence, follow without difficulty. It is obvious from (5a) that $C_n(x;\xi)$ are monic polynomials of degree exactly n.

Part (2a) is easily established by putting $x = \Gamma_{n,m}(\xi)$ in (6b) and showing that $C_n(\Gamma_{m,n}(\xi);\xi) = 0$ for any n and $m = 0, 1, \ldots, n-1$.

Observe first that in general if k is an integer, then $\operatorname{arccot}(\cot x) = x - k\pi$, $k\pi < x < (k+1)\pi$. Thus $\operatorname{arccot}(\Gamma_{n,m}(\xi)) = (\xi+m)\pi/n$, since $0 < (\xi+m)\pi/n < \pi$ $(m = 0, 1, \ldots, n-1)$ provided that $0 < \xi < 1$. Hence, on setting $x = \Gamma_{n,m}(\xi)$ in (6b), we have after simplification

$$C_n(\Gamma_{n,m}(\xi);\xi) = (-1)^m \csc^n\left(\frac{(\xi+m)\pi}{n}\right)[\cos(\pi\xi) - \cot(\pi\xi)\sin(\pi\xi)] = 0.$$

Since $C_n(x;\xi)$ is of degree n, it has exactly n zeros, and so the numbers $\Gamma_{n,m}(\xi)$ $(m = 0, 1, \ldots, n-1)$ are all zeros of $C_n(x;\xi)$. In view of the properties of the cotangent functions, these zeros are real and simple.

(1b), (2b). First, it can be shown working along the same lines as in the proof of Part (1a) that for any real x we have

$$K_n(x;\xi) = (1+x^2)^{n/2}[\cos(n\arctan x) - \cot(\pi\xi)\sin(n\arctan x)] \tag{8a}$$

$$= \sec^n(\arctan x)[\cos(n\arctan x) - \cot(\pi\xi)\sin(n\arctan x)] \tag{8b}$$

$$= \sum_{k=0}^{[n/2]}(-1)^k\binom{n}{2k}x^{2k} - \cot(\pi\xi)\sum_{k=0}^{[(n-1)/2]}(-1)^k\binom{n}{2k+1}x^{2k+1} \tag{8c}$$

if $\arctan x$ takes its principal value. To do this, we need the relationship $\cos\theta = (1+\cot^2\theta)^{-1/2}$ and to note that the use of $z = (xi+1)^n$ leads to the desired result in (8c).

Next, on rewriting (8c) the proposed formula for $K_n(x;\xi)$ in (5b) is obtained and it is trivial to verify that the polynomials $K_n(x;\xi)$ are not monic; they are of degree exactly $n-1$ when $\xi = 1/2$ and n is odd and of degree n otherwise. We also have

$$K_n(x;\xi) = x^n C_n(1/x;\xi). \tag{9}$$

To prove Part (2b) recall that, in general, if z_0 is a non-vanishing zero of the polynomial $P_n(z)$, then $1/z_0$ is a zero of the polynomial $z^n P_n(1/z)$ (see for instance [6], p. 180). Hence, because of (9) the zeros of $K_n(x;\xi)$ are the reciprocals of the zeros of $C_n(x;\xi)$, i.e. the reciprocals of $\Gamma_{n,m}(\xi)$. Clearly, the case when $\Gamma_{n,m}(\xi) = 0$ must be excluded.

This completes the proof of the Theorem. ■

4. Concluding remarks

An interesting application of the Chebyshev polynomials, which we have failed to find in the literature (for instance, in the standard text on these "classical orthogonal polynomilas" [7]), is described in this note: two sequences of polynomials, $C_n(x;\xi)$ and $K_n(x;\xi)$, are obtained, for which all zeros, regardless of the degree, can be given by a single "simple formula". This porcedure, however, fails to give similar polynomials when applied to the sine, cosine and their reciprocals.

Recall that similarly all zeros τ_m and υ_m of the Chebyshev polynomilas $T_n(x)$ and $U_n(x)$ are also given by a "simple formula" (a single analytical expression) [5, Entry 22.16.4] as follows:

$$\tau_m = \cos\frac{(2m-1)\pi}{2n}, \quad \upsilon_m = \cos\frac{m\pi}{n+1} \qquad (m = 1, 2, \ldots, n).$$

Several examples of $C_n(x;\xi)$ and $K_n(x;\xi)$, $0 < \xi < 1$, are:

$$\begin{aligned}
&C_1(x;\xi) = x - \cot(\pi\xi), \qquad K_1(x;\xi) = -\cot(\pi\xi)x + 1,\\
&C_2(x;\xi) = x^2 - 2\cot(\pi\xi)x - 1, \qquad K_2(x;\xi) = -x^2 - 2\cot(\pi\xi)x + 1,\\
&C_3(x;\xi) = x^3 - 3\cot(\pi\xi)x^2 - 3x + \cot(\pi\xi),\\
&K_3(x;\xi) = \cot(\pi\xi)x^3 - 3x^2 - 3\cot(\pi\xi)x + 1,\\
&C_4(x;\xi) = x^4 - 4\cot(\pi\xi)x^3 - 6x^2 + 4\cot(\pi\xi)x + 1,\\
&K_4(x;\xi) = x^4 + 4\cot(\pi\xi)x^3 - 6x^2 - 4\cot(\pi\xi)x + 1.
\end{aligned}$$

A straightforward consequence of the results is that the cotangent sum $S_n(q;\xi)$ in (1) is in fact the power sum of the zeros of $C_q(x;\xi)$ and this enables the closed-form evaluation of $S_n(q;\xi)$ for any n by using the Newton identities [8, p. 179], for instance. This gives for example

$$\begin{aligned}
S_3(q;\xi) &= q^3(\cot^3(\pi\xi) + \cot(\pi\xi)) - q\cot(\pi\xi),\\
S_5(q;\xi) &= q^5(\cot^5(\pi\xi) + \tfrac{5}{3}\cot^3(\pi\xi) + \tfrac{2}{3}\cot(\pi\xi))\\
&\quad - q^3(\tfrac{5}{3}\cot^3(\pi\xi) + \tfrac{5}{3}\cot^3(\pi\xi)) + q\cot(\pi\xi).
\end{aligned}$$

Finally, note that the sums deduced here

$$\begin{aligned}
\sum_{k=0}^{[n/2]}(-1)^k\binom{n}{2k}x^{n-2k} &= (1+x^2)^{n/2}T_n\Big(\frac{x}{\sqrt{1+x^2}}\Big)\\
&= (1+x^2)^{n/2}\cos(n\operatorname{arccot} x) = \csc^n(\operatorname{arccot} x)\cos(n\operatorname{arccot} x)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=0}^{[(n-1)/2]}(-1)^k\binom{n}{2k+1}x^{n-2k-1} &= (1+x^2)^{(n-1)/2}U_{n-1}\Big(\frac{x}{\sqrt{1+x^2}}\Big)\\
&= (1+x^2)^{n/2}\sin(n\operatorname{arccot} x) = \csc^n(\operatorname{arccot} x)\sin(n\operatorname{arccot} x)
\end{aligned}$$

(see (3) in conjunction with (6)) are not listed in [3, section 10.28], the most extensive compilation of sums, and in [4, Section 4.2.3]. It is interesting that Ramanujan [9, Entry 21, p. 32] showed the polynomial nature of $(1+x^2)^{n/2}\sin(n\arctan x)$ without deriving the summation formula

$$\sum_{k=0}^{[(n-1)/2]}(-1)^k\binom{n}{2k+1}x^{2k+1} = x(1+x^2)^{(n-1)/2}U_{n-1}\Big(\frac{1}{\sqrt{1+x^2}}\Big)$$

(see (4) in conjunction with (8a) and (8c)).

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