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Some Geometric Properties for a Class of Non-Lipschitz Domains

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Abstract. In this paper, we introduce a class $\mathcal{C}$, of domains of $\mathbb{R}^N$, $N \geq 2$, which satisfy a geometric property of the inward normal (such domains are not Lipschitz, in general). We begin by giving various results concerning this property, and we show the stability of the solution of the Dirichlet problem when the domain varies in $\mathcal{C}$.

Contents

1. Introduction

Most of the work concerning the stability of the solution of a boundary value problem with the Laplacian operator was done by V. Keldyš in the 40's and is presented in his original paper [12], see also [9], [14], [10]. In [6], the author considered the class of all open domains satisfying the restricted cone property with a given height and angle of the cone, say ε (each open domain is said to satisfy the ε-cone property) which are stable in the sense of Keldyš (see Theorem 4.1). For other kind of constraints, see for example [4], [5], [16]. In this paper, we introduce a very simple constraint involving a geometric property of the inward normal vector. The domains we consider must satisfy the following condition: for almost every point of the boundary, the inward normal (if it exists) intersects a fixed compact convex set C. We shall call this property C-GNP. Such property is satisfied by the solution of the quadrature surface free boundary problem, see for example [1], [2], [8] and [13].

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The aim of this paper is, first, to show some important results concerning the domains which satisfy C-GNP. We will prove that the boundary of a domain Ω which satisfies C-GNP has a uniform cone property outside C. Moreover, even though cusps can be formed at the points of $\partial\Omega\cap\partial C$ (one can consider, in two dimensions, the convex $C=[-1,1]\times\{0\}$ and the domain $\Omega=B(-1,1)\cup B(1,1)$), it is shown that these cusps are not sharper than (i.e., contain) a canonical cusp (which is obtained by revolving the cusp between two touching circles of large radius around its axis). In particular, this implies that every point of $\partial\Omega$ is regular for the Dirichlet problem, as one can easily verify the Wiener criterion. We also obtain a characterization of C-GNP where we don't need the normal and give an example of a domain which satisfies C-GNP while its Steiner symmetrization doesn't have C-GNP.

Next, in Section 3, we will prove that if Ω_n is a sequence of open subsets included in a fixed ball D and satisfying C-GNP, then there exists an open subset $\Omega\subset D$ and a subsequence (still denoted by Ω_n) such that:

1. Ω_n converges to Ω in the Hausdorff sense,
2. Ω_n converges to Ω in the compact sense: every compact subset of Ω is included in Ω_n for n large enough and every compact subset of $\overline{\Omega}^c$ is included in $\overline{\Omega}_n^c$ for n large enough ($\overline{\Omega}^c$ is the exterior of Ω),
3. Ω_n converges to Ω in the sense of characteristic functions, and
4. Ω satisfies C-GNP.

In the last section, we study the behaviour of the solution of the Dirichlet problem on Ω_n when Ω_n converges to Ω. We introduce u_n the solution of the Dirichlet problem

$$\begin{cases} -\Delta u_n = f & \text{in } \Omega_n \\ u_n = 0 & \text{on } \partial\Omega_n \end{cases}$$

and we prove that, if Ω_n converges to Ω in the Hausdorff sense, the sequence u_n (extended by 0 outside Ω_n) converges strongly in $H_0^1(D)$ to u solution of the Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

$(f\in H^{-1}(D))$.

The proof relies on the notion of stability introduced by Keldyš and the convergence in the compact sense of Ω_n to Ω. It requires a particular study of the cusp points of $\partial\Omega$, and a precise computation of the capacity of the exterior of Ω, in a neighbourhood of such a point in order to obtain the stability of the set Ω.

2. The geometric normal property

In this section, we introduce for an open subset Ω of $\mathbb{R}^N$, the geometrical property of the normal with respect to convex C (noted C-GNP), and we study its various properties. It will be shown first of all that the points of the boundary of Ω which are outside of C have a property of the cone. Then for the points of $\partial\Omega$ which are on C, we prove a geometrical result (see Proposition 2.2) expressing that the outside of Ω is sufficiently consistent in term of capacity, in the neighbourhood of those points. Finally, we give a characterization of C-GNP which does not utilize the normal.

Let D be an open ball of $\mathbb{R}^N$ and C be a convex compact of D. All the open subsets with which we shall work will be included in D. For any point x of the boundary of an open subset Ω, we denote the inward normal vector to $\partial\Omega$ (if it exists) by $\nu(x)$ and set

$$\mathcal{N}_\Omega = \{x \in \partial\Omega \ : \ \nu(x) \text{ exists}\}.$$

Finally, we shall denote by $D(x, \nu(x))$, the half-line with origin x and director vector $\nu(x)$.

Definition 2.1 ([3])**.** Let C be a convex set in $\mathbb{R}^N$ and c a point of its boundary. By the normal cone to C at c we mean the set:

$$CN_c = \{y \in \mathbb{R}^N \ : \ (y - c) \cdot (z - c) \leq 0 \ \forall z \in C\}.$$

CN_c can also be seen as the set of the points of $\mathbb{R}^N$, for which c is the projection on C.

- We call a half-normal to C at c, any half-line with origin c and contained in the normal cone CN_c.
- We call a normal to C at c, a line containing one half-normal.

Remark 2.1. If c is a regular point of ∂C, then CN_c is exactly the usual normal to C at c.

Rule. Throughout this paper it is supposed that for any point c of ∂C, a particular normal Δ_c was fixed. One calls it the selected normal to C at c.

Definition 2.2. We say that an open subset Ω has a geometric normal property with respect to C (or more simply Ω satisfies the C-GNP) if:

$$\begin{cases} \text{(P1)} & \Omega \text{ contains the interior of } C. \\ \text{(P2)} & \partial\Omega \text{ is Lipschitz outside of } C. \\ \text{(P3)} & \forall\, x \in \mathcal{N}_\Omega \diagdown C,\ D(x, \nu(x)) \cap C \neq \emptyset. \\ \text{(P4)} & \text{For all selected normals, } \Delta, \text{ to } C,\ \Delta \cap \Omega \text{ is connected.} \end{cases}$$

$\mathcal{C}$ is the class of all domains which satisfy C-GNP.

We begin by showing the following proposition which will be useful thereafter.

Proposition 2.1. *Let Ω be an open subset which has C-*GNP*. Let c be some point of the boundary of C and Δ_c be a half-normal to C at c. Then:*

1. $\Delta_c \cap \Omega$ *cannot have a connected component ω such that, $c \notin \overline{\omega}$.*
2. $\Delta_c \cap \overline{\Omega}^c$ *cannot have a bounded connected component.*

Proof. We show the first point. The proof of the second is the same while working with the extremity of the connected component of $\Delta_c \cap \overline{\Omega}^c$ further from C.

Suppose by contradiction that $\Delta_c \cap \Omega$ has a connected component $]x_0, y_0[$ with c, x_0 and y_0 arranged in this order on Δ_c and $x_0 \neq c$. Let e be the unit vector of the half-line Δ_c and put the origin O in x_0. By hypothesis, O is not on C. Thus:

- On one hand, one can separate C and $\{O\}$ by an hyperplane H orthogonal to Δ_c.

- On the other hand, as O is on $\partial\Omega$, there exists a neighbourhood $V = V' \times]-\alpha, \alpha[$ of O, an orthonormal cartesian coordinate system $\mathcal{R} := (O, e_1, \dots, e_N)$ and a Lipschitz function ϕ in V' such that:
$$\Omega \cap V = \{(y', y_N) \in V \mid y_N < \phi(y')\},$$
$$(\overline{\Omega})^c \cap V = \{(y', y_N) \in V \mid y_N \geq \phi(y')\}.$$
In the cartesian coordinate system $\mathcal{R}$, we have
$$H = \{x \in \mathbb{R}^N \mid x \cdot e = \delta_0\}$$
with $\delta_0 < 0$, ($x \cdot e$ being the scalar product of x and e, in $\mathbb{R}^N$). According to what precedes,
$$C \subset \{x \in \mathbb{R}^N \mid x \cdot e < \delta_0\}.$$

Now, to continue the proof, we shall need the

Lemma 2.1. *Let ψ be a Lipschitz function in a neighbourhood V' of 0 (in $\mathbb{R}^{N-1}$). If $\psi(t, 0, .., 0) > 0$ for $t > 0$ and $\psi(0, .., 0) = 0$, then, there exists $x' \in V'$ such that $\nabla\psi(x')$ exists and $\frac{\partial \psi}{\partial x_1}(x') \geq 0$.*

Proof. Since the 1-dimensional function $f: \ s \mapsto \psi(s, 0, \dots, 0)$ is (as ψ) Lipschitz in a neighbourhood of 0, it is differentiable for almost all s and
$$f(t) = f(0) + \int_0^t f'(s)\, ds.$$
Using the function ψ, we can write
$$\psi(t, 0, \dots, 0) = \int_0^t \frac{\partial \psi}{\partial x_1}(s, 0, \dots, 0)\, ds.$$
So, if $\psi(t, 0, \dots, 0) > 0$ then there exists at least one $s \in]0, t[$ such that
$$\frac{\partial \psi}{\partial x_1}(s, 0, \dots, 0) > 0.$$
Now, if $\nabla\psi(s, 0)$ exists, the demonstration is achieved. If not, set
$$G = \{x \in V' : \nabla\psi(x) \text{ exists}\}.$$
(As ψ is Lipschitz, G is not empty and not negligible). We show that there exists $v \in G$ such that $\frac{\partial \psi}{\partial x_1}(v) \geq 0$.

Suppose by contradiction that no such point exists, i.e., $\forall\ x \in G$, $\frac{\partial \psi}{\partial x_1}(x) < 0$. This implies that, for $h > 0$ small enough and y in a neighbourhood of 0 (in $\mathbb{R}^{N-2}$),
$$\psi(s + h, y) - \psi(s, y) = \int_s^{s+h} \frac{\partial \psi}{\partial x_1}(t, y)\, dt < 0.$$
Tending y to 0, $\psi(s + h, 0) - \psi(s, 0) \leq 0$. Then, dividing by h tending h to 0, we obtain $\frac{\partial \psi}{\partial x_1}(s, 0) \leq 0$, which is absurd. $\square$

End of the proof of Proposition 2.1.

Case 1: $e = -e_N$ (by construction, $e = e_N$ is not possible). In this case, the convex C is included in the half-space $\{x_N > \delta_0\}$. As the inward normal to $\partial\Omega$ at a point x, with director vector $\left(\frac{\partial \phi}{\partial x_1}(x'), \dots, \frac{\partial \phi}{\partial x_{N-1}}(x'), -1\right)$ is in the opposite direction to C, it cannot meet C and one has the desired contradiction.

Case 2: The vector e is not parallel to e_N. Note e' the orthogonal projection of e on the hyperplane orthogonal to e_N. One can without restriction of generality, choose a cartesian coordinate system in which $e_1 = \frac{1}{\alpha_1}e'$, $\alpha_1 > 0$. Set $e = e' + \alpha_N e_N$ ($e = \alpha_1 e_1 + \alpha_N e_N$). The hypothesis $]O, y_0[\subset \Omega$, becomes

$$t\alpha_N < \phi(t\alpha_1, 0, \dots, 0) \quad \text{for } t > 0,\ t \text{ small}.$$

Applying Lemma 2.1 to the function

$$\psi(x_1, \dots, x_{N-1}) = \phi(\alpha_1 x_1, x_2, \dots, x_{N-1}) - \alpha_N x_1,$$

there exists at least one point $x = (x', x_N)$ ($x' \in B(O, -\delta_0/2)$) where the inward normal directed by the vector $\left(\frac{\partial \phi}{\partial x_1}(x'), \dots, \frac{\partial \phi}{\partial x_{N-1}}(x'), -1\right)$ exists and $\frac{\partial \psi}{\partial x_1}(x') \geq 0$. Hence

$$\alpha_1 \frac{\partial \phi}{\partial x_1}(x') - \alpha_N \geq 0. \tag{1}$$

To conclude, show that the inward normal to $\partial\Omega$ at x cannot intersect C. Let $y = x + t\nu(x)$ ($t > 0$) be a point of this inward normal. Since $|x \cdot e| \leq ||x|| < -\delta_0/2$ and $\nu(x) \cdot e$ has the sign of $\alpha_1 \frac{\partial \phi}{\partial x_1}(x') - \alpha_N$ (which is positive according to (1)), then $y \cdot e > \delta_0/2$; which proves the result since the convex C is included in the half-space $\{x \cdot e < \delta_0\}$. □

As a corollary of Proposition 2.1, we have:

Corollary 2.1. *Let Ω be an open subset which has C-*GNP*. Let c be a point of ∂C and Δ_c be a half-normal to C at c, with director vector e. Let x be some point of $\Delta_c \cap \Omega$. Then:*

- *The interval $]c, x[$ does not meet the exterior of Ω.*
- *The half-line with origin x directed by e does not meet Ω.*

Lemma 2.2. *Let Ω be an open subset which has C-*GNP *and S be a similarity transformation* (*of ratio $k > 0$*) *then $S(\Omega)$ has the $S(C)$-*GNP*.*

The proof of this lemma is trivial and therefore is omitted.

As we have said in the begining of this paper, if Ω satisfies C-GNP then $\partial\Omega \cap \partial C$ can have cusps. We shall describe the behaviour of such points, this with the intention to prove that the eventual cusps of $\partial\Omega$ are regular in the sense of Wiener [17] (see Section 4).

Proposition 2.2. *Let C be a convex set with a nonempty interior* ($\mathrm{int}(C) \neq \emptyset$) *and $x_0 \in \partial\Omega \cap \partial C$. Let CN_0 be the normal cone to C at x_0.*

1. *If Ω satisfies C-*GNP*, then $CN_0 \cap \Omega = \emptyset$.*
2. *If CN_0 is reduced to the half-line Δ_0, let H be the hyperplane orthogonal to Δ_0 in x_0 and H^+ the open half-space limited by H and not containing C. Let R be a real number which is strictly superior to the diameter of C and ε_0 be a small strictly positive number. Put $B_{x_0} = B'(x_0, \varepsilon_0) \times \mathbb{R}$ ($B'(x_0, \varepsilon_0)$ being the $(N-1)$-dimensional ball with center x_0 and radius ε_0). Then*

$$\Omega \cap B_{x_0} \cap H^+ \subset \bigcup_{z \in H,\ |z - x_0| = R} \overline{B}(z, R) = \mathcal{B}_{x_0}.$$

Remark 2.2. If $\mathrm{int}(C) = \emptyset$, the previous result remains true if we replace CN_0 by $CN_0 \cap E$, where E is one of the half-spaces limited by an hyperplane containing C.

Remark 2.3. The interest of Proposition 2.2 is to describe the boundary $\partial\Omega$ of Ω, in the neighbourhood of x_0: the union of spheres centered in z and of radius R, form in x_0 a (hyper)surface of revolution with a perfectly characterized cusp point. This proposition says that the eventual singularity of $\partial\Omega$ in the point x_0 can be a cusp point which is included in the one of the revolution surface, described above. This geometric characterization will allow us to estimate the capacity of the exterior of Ω in the neighbourhood of the point x_0 (see Section 4).

Proof of Proposition 2.2. We first prove that the normal cone CN_0 does not intersect Ω. If CN_0 is reduced to a half-line, this is exactly the selected normal to C and the result is then an immediate consequence of the condition (P4). Now, let x_0 be a vertex of the convex C such that $x_0 \in \partial\Omega$ and let Δ_0 be the selected normal to C at x_0. Suppose that the open subset Ω meets the normal cone CN_0 and let ω be a connected component of $\Omega \cap CN_0$. By hypothesis, ω does not meet Δ_0. Now among all the planes containing Δ_0 which meet ω, one can find at least a plane P such that the 1-dimensional Lebesgue measure of the complement of $\mathcal{N}_\Omega \cap P$ is null, i.e., almost all the points of $\partial\Omega \cap P$ have a normal. For $x \in \partial(\omega \cap P)$, ν_x (resp. n_x) is the inward normal vector to $\partial\omega$ (resp. to $\partial(\omega \cap P)$) at x. In the plane P, we have the following situation: $\omega \cap P$ is a relative open subset such that $\partial(\omega \cap P) = \gamma_1 \cup \gamma_2$, where $\gamma_1 \subset \partial\Omega$ and γ_2 is included in some half-normal $\Delta_1$which limits $CN_0 \cap P$.

Let H be the supporting hyperplane orthogonal to Δ_1 and passing by x_0. Let Δ be the line $H \cap P$. Let $\mathcal{R}(x_0, e_1, \ldots, e_N)$ be the cartesian coordinate system of $\mathbb{R}^N$, where x_0 is the origin, e_1 is parallel to Δ and e_N is parallel to Δ_1. By hypothesis, the convex C is in the half-space $\{x_N \leq 0\}$. Since

$$0 = \int_{\partial} n \cdot e_N = \int_{\gamma_1} n \cdot e_N + \int_{\gamma_2} n \cdot e_N = \int_{\gamma_1} n \cdot e_N \tag{2}$$

(n is the inward normal vector to $\partial(\omega \cap P)$), then there exists a subset (of strictly positive measure) of γ_1 on which $n_x \cdot e_N \geq 0$. Then, it is easy to see that using, for $\partial\Omega$, the cartesian representation $\psi(x_1, x_2, \ldots, x_N) = 0$, one can have $\nu_x \cdot e_N \geq 0$, i.e., the inward normal vector at such point x, cannot intersect C (which is "below" the hyperplane H), contradicting thus, the property (P3).

Suppose now, that the cone CN_0 is reduced to a half-line Δ_0. To simplify, we can put the origin at x_0. Let $B_0 = B'(O, \varepsilon_0) \times \mathbb{R}$ and suppose that $(\Omega \cap B_0 \cap H^+) \setminus \mathcal{B}_{x_0}$ is not empty. Let ω be a connected component of this set. By hypothesis, ω does not intersect Δ_0. Let P be a plane containing the normal Δ_0 such that the complement of $\mathcal{N}_\Omega \cap P$ has a null measure. Once again, for $x \in \partial(\omega \cap P)$, let $\nu(x)$ (resp. $n(x)$) be the inward normal vector to $\partial\omega$ (resp. to $\partial(\omega \cap P)$) at x.

In the plane P, we have the following situation: $\omega \cap P$ is a relative open subset such that $\partial(\omega \cap P) = \gamma_1 \cup \gamma_2$, where $\gamma_1 \subset \partial\Omega$ and $\gamma_2 \subset \partial\mathcal{B}_{x_0}$. Note Δ the line $H \cap P$. Introduce, as above, a cartesian coordinate system of $\mathbb{R}^N$, $\mathcal{R}(x_0, e_1, \ldots, e_N)$, where x_0 is the origin, $e_1$is parallel to Δ (and of same direction that $\omega \cap P$) and e_N is parallel to Δ_0.

By hypothesis, Ω satisfies the condition (P3) w.r.t. C. Thus there exists $\varepsilon > 0$ (small enough) such that $\Omega \cap H^+ \cap B_0$ satisfies (P3) w.r.t. the $(N-1)$-dimensional ball $B(O, R-\varepsilon)$. This can be expressed by

$$\forall X \in \mathcal{N}_\Omega \cap H^+ \cap B_0, \quad \nu_N(X) < 0 \text{ and } \sum_{i=1}^{N-1} \left(x_i - x_N \frac{\nu_i(X)}{\nu_N(X)} \right)^2 \leq (R-\varepsilon)^2.$$

Therefore

$$\left(x_1 - x_N \frac{\nu_1(X)}{\nu_N(X)} \right)^2 \leq (R-\varepsilon)^2. \tag{3}$$

Let $n(X) = (n_1(X), n_N(X))$, be the inward normal vector (if it exists) to $\partial(\omega \cap P)$ at $X \in \gamma_1$. Using the cartesian representation $\psi(x_1, \dots, x_N) = 0$, one can have $\frac{\nu_1(X)}{\nu_N(X)} = \frac{n_1(X)}{n_N(X)}$. This together with the inequality (2) implies that

(4)

$$\forall X = (x_1, x_N) \in \gamma_1 \cap \mathcal{N}_\Omega, \quad n_N(X) < 0 \text{ and } -R+\varepsilon \leq x_1 - x_N \frac{n_1(X)}{n_N(X)} \leq R-\varepsilon,$$

or again, if we introduce the tangential vector $t(X) = (n_N(X), -n_1(X))$ to $\partial(\omega \cap P)$:

$$(R-\varepsilon) n_N(X) \leq X \cdot t(X) \leq (-R+\varepsilon) n_N(X). \tag{5}$$

Since $n_N < 0$, then $X \cdot t(X) > R n_N$ and,

$$\int_{\gamma_1} X \cdot t(X)\, ds > \int_{\gamma_1} R n_N(X)\, ds.$$

Now, as the points of γ_2 belong to some circle of center z, we have

$$X \cdot t(X) = R n_N(X).$$

This implies that

$$0 = \int_{\partial\omega} X \cdot t(X)\, ds = \int_{\gamma_1} X \cdot t\, ds + \int_{\gamma_2} X \cdot t\, ds > \int_{\partial\omega} R n_N(X)\, ds = 0,$$

which is absurd. □

Remark 2.4. An immediate consequence of Proposition 2.2 above, is that an open subset which has the C-GNP is of Caratheodory type. Recall that an open subset Ω is of Caratheodory type if, for example, any point of its boundary is limit of a sequence of points of its exterior $\overline{\Omega}^c$.

In fact as $\partial\Omega \setminus C$ is Lipschitz then, all its points can be approached by a sequence of points of the exterior of Ω.

Now, if $x_0 \in \partial\Omega \cap C$, according to Proposition 2.2, x_0 is also in $\overline{\mathcal{B}_{x_0}}$ which is of Caratheodory type and its exterior is contained in the exterior of Ω. Consequently, there exists a sequence of points of the exterior of Ω which converges to x_0.

Remark 2.5. In two dimensions, an open subset which has C-GNP satisfies a property of the exterior segment.

Now, let us give a characterization of C-GNP where we don't need the normal.

Definition 2.3. Let C be a convex set. We say that an open subset Ω has the C-SP, if the conditions (P1), (P2), (P4) of Definition 2.2 are satisfied and

$$\text{(S)} \ \ \forall\, x \in \partial\Omega \setminus C \ \ K_x \cap \Omega = \emptyset,$$

where K_x is the closed cone defined by $\left\{ y \in \mathbb{R}^N \ : \ (y-x).(z-x) \leq 0, \ \forall\, z \in C \right\}$.

Remark 2.6. K_x is the normal cone to the convex hull of C and $\{x\}$.

Proposition 2.3. *An open subset Ω has C-GNP if and only if it satisfies the C-SP.*

To prove this proposition, we need to show the following lemma:

Lemma 2.3. *Let c be a point of ∂C and CN_c be the normal cone to C at c. Then for all $y \in CN_c$ and $y \neq c$, the half-line $\Delta(y, \overrightarrow{cy})$, with origin y and director vector $\overrightarrow{cy}$ is contained in K_y (the normal cone of the convex hull of $\{y\}$ and C).*

Proof. Let $z \in \Delta(y, \overrightarrow{cy})$, then there exists $\lambda \in \mathbb{R}^+$ such that $z - y = \lambda(y - c)$. Show that $z \in K_y$, that is to say:

$$\forall\ \psi \in C, \quad (z - y) \cdot (\psi - y) \leq 0,$$

or again,

$$\forall\ \psi \in C, \quad (y - c) \cdot (\psi - y) \leq 0.$$

Since c is the projection of y on C, one has

$$(y - c) \cdot (\psi - c) \leq 0.$$

Thus,

$$(y - c) \cdot (\psi - y) = (y - c) \cdot (\psi - c) - \|y - c\|^2 \leq 0.$$

□

Proof of Proposition 2.3. We first show that C-SP implies C-GNP: Let $x \in \mathcal{N}_\Omega \setminus C$. Denote by $\mathcal{H}_x$ the tangential hyperplane to $\partial\Omega$ at x and by $\mathcal{C}_x$ the orthogonal projection of C on $\mathcal{H}_x$. The C-GNP will be proved if we show that x belongs to $\mathcal{C}_x$. In fact, if not, there would exist an hyperplane of $\mathcal{H}_x$ (therefore an affine space in $\mathbb{R}^{N-2}$) separating strictly x of $\mathcal{C}_x$, therefore, there would exist a tangential vector $\vec{\tau}$ such that one has,

$$\forall\ y \in C, \quad \vec{\tau} \cdot (y - x) > 0.$$

Now by C-SP, one can have, for all $z \in \Omega$ and all $y \in C$, the inequality $(z - x) \cdot (y - x) > 0$. Choosing a sequence of points $z_n \in \Omega$ such that $\frac{1}{\|z_n - x\|}(z_n - x)$ converges to $-\vec{\tau}$, one obtains $-\vec{\tau} \cdot (y - x) \geq 0$, and thus the contradiction.

We now show that C-GNP implies C-SP: If $x \in \partial\Omega \setminus C$, let K_x be the cone defined above. Let c be the projection of x on C. By Lemma 2.3, the half-line $\Delta(x, \overrightarrow{cx})$ is in K_x. Therefore, according to the Corollary 2.1, $\Delta(x, \overrightarrow{cx})$ does not intersect Ω. Then, we find ouerselves in the same conditions as in the demonstration of Proposition 2.2 by replacing CN_0 by K_x and Δ_0 by $\Delta(x, \overrightarrow{cx})$. One can conclude that $\Omega \cap K_x = \emptyset$ (that returns to apply Proposition 2.2 to Ω and the convex hull of C and $\{x\}$). □

Remark 2.7. The definition of K_x implies that: if $z \in \operatorname{int}(K_x)$ and if H denotes the hyperplane passing by x and orthogonal to $\overrightarrow{xz}$, then the convex C is included in the open half-space limited by H and not containing z. Now, if z is a point (different to x) such that if H is the hyperplane defined above and C is included in the open half-space limited by H and not containing z, then $z \in \operatorname{int}(K_x)$.

This last remark allows us to state the following lemma which will be useful for later.

Lemma 2.4. *Let x be a point which does not belong to C. Let H be an hyperplane which separate strictly $\{x\}$ and C. If D_x is the half-line with origin in x, orthogonal to H and not intersecting this one, then D_x is in the interior of K_x.*

Proof. In fact, if y is a point of the half-line D_x, applying Remark 2.7 to the hyperplane H_x passing by x and parallel to H, one deduces that y belongs to $\mathrm{int}(K_x)$. $\square$

Lemma 2.4 allows us to show the following result which precises Proposition 2.1.

Corollary 2.2. *Let Ω be an open subset which satisfies the condition* (S). *Let c be a point of ∂C and CN_c the normal cone to C at c. Then any half-line with origin in c which is contained in CN_c, intersects the boundary $\partial\Omega$ at most in one point.*

Proof. Let Δ be a half-line with origin in c included in CN_c. Suppose there exists two points x and y in $\Delta \cap \partial\Omega$ (for example $y \in [c, x]$). According to Lemma 2.4, one can deduce that $x \in \mathrm{int}(K_y)$ and therefore $K_y \cap \Omega$ is not empty, contradicting the property (S). $\square$

Corollary 2.3. *Let Ω be an open subset which contains the convex C. If Ω satisfies* (P2) *and* (S)*, then it satisfies* (P4).

Proof. Let Δ be a normal to C at some point c. Suppose that $\Delta \cap \Omega$ be not connected. As this one is a relative open subset of Δ, it has at least two connected components $]a_1, a_2[$ and $]b_1, b_2[$. Now, the points a_1, a_2, b_1 and b_2 are necessarily on $\partial\Omega$. By hypothesis, at least one of the two intervals does not meet C (for example $]a_1, a_2[$). Suppose that a_1 be nearer to C than a_2, then the Corollary 2.2 applied to c and Δ gives the contradiction. $\square$

Now, we shall show the existence of an intrinsic cartesian coordinate system such that the boundary of any open subset which satisfies C-GNP can be locally represented by some Lipschitz function.

Proposition 2.4. *Let Ω be an open subset which satisfies C-GNP. Let $x \in \partial\Omega \setminus C$ and x_C its projection on C. Then, $\partial\Omega$ admits in a neighbourhood of x, a Lipschitz representation in an orthonormal cartesian coordinate system.*

Proof. Put $\mathcal{R}_x = (x, e', e_N)$ with $e_N = \frac{\overrightarrow{x_C x}}{\|\overrightarrow{x_C x}\|}$. We shall show that, in a neighbourhood of x:

1) $\partial\Omega$ is a graph in $\mathcal{R}_x$.
2) $\partial\Omega \setminus C$ is Lipschitz in $\mathcal{R}_x$.

Let $\varepsilon > 0$ be sufficiently small and $y \in \partial\Omega \cap B(x, \varepsilon)$. Put $y = (y', y_N)$ and $c = (y', 0)$.

We first show 1). Note D_c the half-line with origin in c and director vector e_N. One can show that

$$D_c \cap \partial\Omega = \{y\}.$$

By construction of K_x, the open half-line $D(x, e_N)$ with origin in x and director vector e_N is contained in the interior of K_x. As K_y continuously varies with y, the half-line $D(y, e_N)$ with origin in y and director vector e_N is in the interior of K_y, for y sufficiently close to x. Now, since Ω has C-GNP, Proposition 2.3 allows us to derive

$$\Omega \cap K_y = \emptyset. \tag{6}$$

Therefore, if $D_c = [c, y] \cup D(y, e_N)$ contains some point η of $\partial\Omega$, $\eta \neq y$, then according to (6), η would be on the interval $[c, y[$. Applying the C-SP to η, one would have

$$\Omega \cap K_\eta = \emptyset,$$

but, by Lemma 2.4 $y \in K_\eta$, which gives the contradiction.

Now, we show 2). As Ω has C-GNP, Proposition 2.3 implies that $\Omega \cap K_y = \emptyset$ or again that $K_y \subset \Omega^c$. So the graph of $\partial\Omega$ does not meet the interior of the cone K_y which contains the open half-line $D(y, e_N)$ (see above). To conclude, it remains to find an analogous cone situated under the graph of $\partial\Omega$. Put $E_y = \{\eta \ : \ y \in K_\eta\}$. Note that, according to Proposition 2.3, E_y does not contain any point of $\partial\Omega$.

Introduce H_N, the hyperplane $\{z_N = \frac{y_N}{2}\}$ and H_N^+, the closed half-space $\{z_N \geq \frac{y_N}{2}\}$. As the closed intersection $H_N^+ \cap C$ is empty, then there exists $\alpha > 0$ such that for all cartesian coordinate system $\mathcal{R}(y, e'_1, \ldots, e'_N)$ such that the angle $(\widehat{e'_N, e_N}) < \alpha$, the convex C is in the half-space $\{y'_N < \frac{y_N}{2}\}$. Let $\eta \in B(y, \frac{y_N}{2}) \cap C(y, \alpha)$, by construction, C is in the open half-space limited by the hyperplane which is orthogonal to $\overrightarrow{\eta y}$ passing by η and not containing y. Then Remark 2.7 gives $y \in K_\eta$, and

$$B\left(y, \frac{y_N}{2}\right) \cap C(y, \alpha) \subset E_y.$$

This implies the desired property. $\square$

An immediate consequence to this proposition is the following:

Corollary 2.4. *The intersection of two open subsets which have C-*GNP *is an open subset which satisfies C-*GNP.

Proof. Let Ω_1 and Ω_2 be two open subsets which contain the interior of C. If Δ is a selected normal to C. By hypothesis, $\Delta \cap \Omega_1$ and $\Delta \cap \Omega_2$ are connected (there are intervals of Δ) then their intersection is an interval of Δ and is, therefore, connected and $\Omega_1 \cap \Omega_2$ satisfies (P4). It remains now to prove that if Ω_1 and Ω_2 satisfy (P2) and (P3), then $\Omega_1 \cap \Omega_2$ also. Let Γ_I be the boundary of $\Omega_1 \cap \Omega_2$. If $x \in \Gamma_I$ where the inward normal exists, this one is necessarily the inward normal at x to $\partial\Omega_1$ or to $\partial\Omega_2$, so (P3) will be verified. To conclude, it suffices to show that if Ω_1 and Ω_2 have C-GNP, then $\Gamma_I \setminus C$ is Lipschitz (in general, if Ω_1 and Ω_2 are two Lipschitz open subsets, their intersection is an open subset which is not necessarily Lipschitz). Let $x \in \Gamma_I$, then in the cartesian coordinate system $\mathcal{R}_x$ defined by Proposition 2.4, there exists a neighbourhood V_x of x and two Lipschitz functions ϕ_1 and ϕ_2 representing respectively $\partial\Omega_1 \cap V_x$ and $\partial\Omega_2 \cap V_x$ in $\mathcal{R}_x$. Consequently if we put $\Phi_I = \inf(\phi_1, \phi_2)$, then Φ_I is a Lipschitz representation of $\Gamma_I \cap V_x$ in $\mathcal{R}_x$. $\square$

Remark 2.8. In general, the union of two open subsets which satisfy the property (P4) does not satisfy it. One can consider in two dimensions the convex $C = [-1, 1] \times \{0\}$. Let Ω be the union of the two open discs D_{-1} and D_1 centred respectively in -1 and 1 and of radius $1/2$. If the selected normal to C at the two extremities is the line $y = 0$ then D_{-1} and D_1 satisfy (P4) while Ω does not.

Now we shall precise the result obtained above. We want to prove that for an open subset Ω which has C-GNP, $\partial\Omega\setminus C$ satisfies the cone property. More precisely we shall show the following proposition.

Proposition 2.5. *Let Ω be an open subset which has C-GNP and x_0 a point of $\partial\Omega\setminus C$. Then, there exists an unitary vector η and a real number ε (strictly positive) which depend only on x_0 and C and such that*

$$\forall\, y \in B(x_0,\varepsilon)\cap\overline{\Omega} \qquad C(y,\eta,\varepsilon)\subset\Omega,$$

where $C(y,\eta,\varepsilon)$ is the cone with vertex y, of direction η and angle to the vertex and of height ε:

$$C(y,\eta,\varepsilon) = \left\{x\in\mathbb{R}^N \ ;\ |x-y|\le\varepsilon \ \textit{and}\ |(x-y)\cdot\eta| \ge |x-y|\cos\varepsilon\right\}.$$

Proof. Denote by δ, the distance of x_0 to C. The C-GNP allows us to work in the cartesian coordinate system $\mathcal{R}_0$ with origin in O (the projection of x_0 on C) and which has the last vector of coordinates, $\overrightarrow{e_N} = \overrightarrow{Ox_0}/\delta$ such that $\{x_N = 0\}$ is the supporting hyperplane to C at O. One can complete the base by choosing an orthonormal base of $\{x_N = 0\}$.

Let ϕ be the Lipschitz representation of $\partial\Omega\cap B(x_0,\alpha)$ in $\mathcal{R}_0$. One can suppose that $\alpha < \delta/2$. Then

$$\partial\Omega\cap B(x_0,\alpha) = \{(x',x_N)\in B(x_0,\alpha)\ ;\ x_N = \phi(x')\},$$

and

$$\Omega\cap B(x_0,\alpha) = \{(x',x_N)\in B(x_0,\alpha)\ ;\ x_N < \phi(x')\}.$$

Let $R > 0$ be large enough in order that the intersection between the cone with vertex y supported by C and the hyperplane $x_N = 0$ will be contained in the $(N-1)$-dimensional ball $B'(O,R)$, for all $y \in B(x_0,\delta/2)$. Start with characterizing analytically the geometric property of the normal. At every point $\xi = (\xi',\xi_N)\in \partial\Omega\cap B(x_0,\alpha)$ where the normal ν_ξ exists, the half-line with origin ξ and directed vector ν_ξ intercects the hyperplane $\{x_N = 0\}$ inside the ball $B'(O,R)$:

$$\forall\xi\in\mathcal{N}_\Omega\cap B(x_0,\alpha), \qquad |\xi' + \phi(\xi')\nabla\phi(\xi')| \le R. \tag{7}$$

This implies that the function

$$\xi \longmapsto \phi^2(\xi') + |\xi'|^2$$

is $2R$-Lipschitz in $B'(O,\alpha)$. Hence ϕ^2 is $(2R+2\alpha)$-Lipschitz and if $\phi \ge \delta/2$ we obtain that ϕ is $(2R+2\alpha)/\delta$-Lipschitz and therefore satisfies an uniform cone property (see [7]). But as it is not absolutely evident that the geometric characteristics of the cone can be chosen independently of Ω, we continue the demonstration.

Let us fix $\varepsilon > 0$ sufficiently small that

$$2\varepsilon + (1+\tan^2\varepsilon)\varepsilon + 2\tan\varepsilon < \delta,$$

(it is clear that ε depends only on x_0 and C, by the intermediary of R and δ).

Choose, as direction of the cone, $\eta = -e_N$. Let $y \in B(x_0,\varepsilon)\cap\Omega$, $y = (y',\delta+y_N)$ with

$$|y'|^2 + y_N^2 < \varepsilon^2 \ \text{ and }\ \delta + y_N \le \phi(y'). \tag{8}$$

Now, let $z \in C(y, \eta, \varepsilon)$:

(9) $\quad z = y + (h', -h_N)$ with, $|h'|^2 + h_N^2 < \varepsilon^2$, $|h'| \leq (\tan\varepsilon)h_N$ and $h_N > 0$.

We shall prove the uniform cone property, that is to say that, $z \in \Omega$, or again that,

$$(\delta + y_N - h_N)^2 < (\phi(y' + h'))^2.$$

Using (7), we can have

$$(\phi(y' + h'))^2 \geq (\phi(y'))^2 - 2R|h'| + |y'|^2 - |y' + h'|^2.$$

And from (8),

$$\begin{aligned}(\phi(y' + h'))^2 &\geq (\delta + y_N)^2 - 2R|h'| + |y'|^2 - |y' + h'|^2\\ &\geq (\delta + y_N)^2 - 2R|h'| - |h'|^2 - 2\varepsilon|h'|.\end{aligned}$$

Now, according to (9), one gets

$$(\phi(y' + h'))^2 \geq (\delta + y_N)^2 - 2(R + \varepsilon)\tan(\varepsilon)h_N - \tan^2(\varepsilon)h_N^2.$$

Using the definition of ε, we obtain

$$(\phi(y' + h'))^2 \geq (\delta + y_N)^2 + h_N^2 - 2h_N(\delta - \varepsilon) \geq (\delta + y_N - h_N)^2.$$

which is the result. $\square$

Let us now see if the class $\mathcal{C}$ is stable by Steiner symmetrization.

Definition 2.4. Let Ω be an open subset of $\mathbb{R}^2$. Assume that Ω is convex in the direction Oy. For $\alpha \in \mathbb{R}$, let Ω_α be the segment$\{(\alpha, y) \in \mathbb{R}^2\ ;\ (\alpha, y) \in \Omega\}$. The Steiner symmetrization of Ω is

$$\Omega^* = \left\{(\alpha, t) \in \mathbb{R}^2\ ;\ \alpha \in \mathbb{R} \text{ and } |t| < \frac{|\Omega_\alpha|}{2}\right\}.$$

Lemma 2.5. *Let Ω be an open subset of $\mathbb{R}^2$ which contains the convex $C = [-1, 1] \times \{0\}$. If Ω satisfies C-GNP, then Ω is convex in the direction Oy.*

Proof. The proof is an immediate consequence of Definition 2.2. Let H^+ and H^- be the half-planes separated by the axis Ox. Let $x \in \partial\Omega$. As Ω satisfies C-GNP from Proposition 2.3, we deduce that the vertical segment over x is in the closed cone K_x. In the same way, if there exists z in the vertical segment under x we have $x \in K_z$ which contradicts Lemma 2.4. Therefore Ω is convex in the direction Oy and $\partial\Omega \cap H^+$ and $\partial\Omega \cap H^-$ are two graphs. $\square$

Proposition 2.6. *There exists an open subset $\Omega \subset \mathbb{R}^2$ which satisfies C-GNP and such that its Steiner symmetrization doesn't satisfy C-GNP.*

Proof. Using the notations of the previous proof, let φ_1 (resp. φ_2) be the representation of $\partial\Omega \cap H^+$ (resp. of $\partial\Omega \cap H^-$). It is easy to see that C-GNP is equivalent to

$$-1 \leq x + \varphi_i(x)\varphi_i'(x) \leq 1,\ i = 1,\ 2,$$

for all x such that the derivative $\varphi_i'(x)$ exists.

Let $r \in]0,1[$ and put

$$\varphi_1(x) = \sqrt{(2+r)^2 - (x-1)^2}$$

$$\varphi_2(x) = \begin{cases} \sqrt{r^2 - (x+1)^2} & \text{if } x \in [-1-r, -1] \\ \sqrt{4 - (x-1)^2} & \text{if } x \in [-1, +\infty[. \end{cases}$$

The symmetrized Ω^* (of Ω) is limited by the graph of $\frac{\varphi_1+\varphi_2}{2}$ in H^+ and the graph of $-\frac{\varphi_1+\varphi_2}{2}$ in H^-. Now, if $x = -1+r$ with r sufficiently small, a simple calculation shows that

$$x + \left(\frac{\varphi_1+\varphi_2}{2}\right)(x)\left(\frac{\varphi_1+\varphi_2}{2}\right)'(x) \simeq -1 + \frac{1}{2r}\left(r + \sqrt{4r+r^2}\right),$$

which tends to $+\infty$ when r tends to 0. Therefore, Ω^* doesn't satisfy C-GNP. $\square$

When we symmetrize an open set Ω (which satisfies C-GNP) by the Steiner continuous symmetrization, the same property is satisfied by its symmetrized Ω_t, for t small, at the points of the boundary $\partial\Omega$ whose normal meets the relative interior of C.

Definition 2.5. Let Ω be an open set in $\mathbb{R}^2$ which is convex in the direction Oy. The Steiner continuous symmetrization consists in centering each segment $[y_1, y_2]$ parallel with the axis Oy (y_1 and y_2 belong to $\partial\Omega$) with a speed equal to the distance from the center of $[y_1, y_2]$ to the axis $\{x = 0\}$, i.e., that if $\partial\Omega$ is given by two functions ϕ_1 and ϕ_2 then for all t ($t \in [0,1]$) the boundary $\partial\Omega_t$ of its symmetrized Ω_t will be given by the functions ϕ_1^t and ϕ_2^t such that

$$\begin{cases} \phi_1^t = \phi_1 - \frac{t}{2}(\phi_1 - \phi_2) \\ \phi_2^t = \phi_2 + \frac{t}{2}(\phi_1 - \phi_2). \end{cases}$$

Definition 2.6. An arc γ centered in $(-1,0)$ or $(1,0)$ is said to be of Type I if it is not included in $\{(x,y) \in \mathbb{R}^2 : x \le -1\} \cup \{(x,y) \in \mathbb{R}^2 : x \ge 1\}$.

Proposition 2.7. *Let Ω be an open set which strictly contains the segment C and satisfying C-GNP. If $\partial\Omega$ does not contain arcs of Type I, then for t sufficiently small, Ω_t satisfies also C-GNP.*

Proof. Suppose that $\partial\Omega$ is given by two functions ϕ_1 and ϕ_2, then, for t small enough, its symmetrized $\partial\Omega_t$ will be given by the functions ϕ_1^t and ϕ_2^t such that

$$\begin{cases} \phi_1^t = \phi_1 - \frac{t}{2}(\phi_1 - \phi_2) \\ \phi_2^t = \phi_2 + \frac{t}{2}(\phi_1 - \phi_2). \end{cases}$$

For $x \in \partial\Omega_t$, the inward normal meets C if and only if

$$-1 \le x + \phi_i^t(x)(\phi_i^t)'(x) \le 1\ ,\ i = 1,2.$$

It is thus a question of checking if

$$-1 \le x + \phi_1(x)\phi_1'(x) - \frac{t}{2}[2\phi_1(x)\phi_1'(x) - (\phi_1'(x)\phi_2(x) + \phi_1(x)\phi_2'(x))] \\ + \frac{t^2}{4}(\phi_1(x) - \phi_2(x))(\phi_1'(x) - \phi_2'(x)) \le 1.$$

Now we place ourselves in a point of the boundary $\partial\Omega$, of x-coordinate x_0 and such that the interior normal cuts the segment C, for example, at $(-1, 0)$. One has

$$x_0 + \phi_1(x_0)\phi_1'(x_0) = -1,$$

and

$$\phi_2'(x_0) \geq -\frac{1 + x_0}{\phi_2(x_0)}.$$

One thus deduces that at the point of x-coordinate x_0,

$$\phi_1'(x_0)\phi_2(x_0) + \phi_1(x_0)\phi_2'(x_0) \geq -(1 + x_0)\left(\frac{\phi_2(x_0)}{\phi_1(x_0)} + \frac{\phi_1(x_0)}{\phi_2(x_0)}\right).$$

But

$$\left(\frac{\phi_2(x_0)}{\phi_1(x_0)} + \frac{\phi_1(x_0)}{\phi_2(x_0)}\right) \geq 2$$

(and even > 2 if $\phi_1(x_0) \neq \phi_2(x_0)$, the equality corresponds to the case where we don't move). Therefore if $-(1 + x_0) > 0$ one has

$$\phi_1'(x_0)\phi_2(x_0) + \phi_1(x_0)\phi_2'(x_0) \geq -2(1 + x_0) = 2\phi_1(x_0)\phi_2'(x_0)$$

and the term between brackets in the previous inequality is negative and consequently this one is checked at the point of x-coordinate x_0. In addition, since $x_0 + \phi(x_0)\phi_1'(x_0) = -1$ and t is rather small then

$$\begin{aligned} &x_0 + \phi_1(x_0)\phi_1'(x_0) - \frac{t}{2}[2\phi_1(x_0)\phi_1'(x_0) - (\phi_1'(x_0)\phi_2(x_0) + \phi_1(x_0)\phi_2'(x_0))] \\ &+ \frac{t^2}{4}(\phi_1(x_0) - \phi_2(x_0))(\phi_1'(x_0) - \phi_2'(x_0)) \leq 1. \end{aligned}$$

Which gives the result. $\square$

3. The geometric normal property and the convergence of domains

In this section, we start by recalling three notions of convergence we can define on the open subsets of $\mathbb{R}^N$. Next, we prove that the class $\mathcal{C}$ of all open subsets satisfying C-GNP is compact for the Hausdorff convergence. We finish by showing that the three considered convergences are equivalent on $\mathcal{C}$.

In all the following, we consider a fixed ball D centered to the origin and of sufficiently large radius to be able to contain the convex compact C, and all the open subsets we shall use.

Definition 3.1. Let K_1 and K_2 be two compact subsets of D. One calls a Hausdorff distance of K_1 and K_2 and we denote by it $d_H(K_1, K_2)$, the following positive number:

$$d_H(K_1, K_2) = \max\left[\rho(K_1, K_2), \rho(K_2, K_1)\right],$$

where $\rho(K_i, K_j) = \max_{x \in K_i} d(x, K_j) \quad i, j = 1, 2$ and $d(x, K_j) = \min_{y \in K_j} |x - y|$.

Definition 3.2. Let Ω_n be a sequence of open subsets of D and Ω be an open subset of D. Let K_n and K be their complements in $\overline{D}$. One says that the sequence Ω_n converges in the Hausdorff sense to Ω and we denote by $\Omega_n \xrightarrow{H} \Omega$ if $\lim\limits_{n\to+\infty} d_H(K_n, K) = 0$.

The notion of convergence that follows is less classical than the Hausdorff convergence, but it is very useful for the stability of the solution of elliptic problems when the domain varies, e.g., [10] and [12].

Definition 3.3. Let Ω_n be a sequence of open subsets of D and Ω be an open subset of D. One says that the sequence Ω_n converges in the compact sense to Ω and we denote by $\Omega_n \xrightarrow{K} \Omega$ if:

- Every compact subset of Ω is included in Ω_n, for *n large enough.*
- Every compact subset of $\overline{\Omega}^c$ is included in $\overline{\Omega}_n^c$, for *n large enough.*

Definition 3.4. Let Ω_n be a sequence of open subsets of D and Ω be an open subset of D. One says that the sequence Ω_n converges in the sense of characteristic functions to Ω and we denote by $\Omega_n \xrightarrow{L} \Omega$ if χ_{Ω_n} converges to χ_Ω in $L^p_{\text{loc}}(\mathbb{R}^N)$, $p \neq \infty$, (χ_Ω is the characteristic function of Ω).

Now, we recall some elementary results concerning the Hausdorff convergence. First of all, the three following propositions which are very classical, see for example [4] or [15].

Proposition 3.1. *The set of open subsets confined in the ball D is relatively compact for the Hausdorff convergence.*

Remark 3.1. The previous proposition is very important because it allows us to extract from each sequence of open subsets of D, a subsequence which converges in the Hausdorff sense.

Proposition 3.2. *If Ω_n is a sequence of open subsets of D and Ω is an open subset of D such that $\Omega_n \xrightarrow{H} \Omega$, then:*

(i) *Every compact subset of Ω is included in Ω_n for n large enough.*
(ii) *For all x in $\partial\Omega$, $\lim\limits_{n\to+\infty} d(x, \partial\Omega_n) = 0$.*

Proposition 3.3. *If Ω_n is a sequence of open subsets of D and Ω is an open subset of D such that $\Omega_n \xrightarrow{H} \Omega$, then:*

(iii) $\lim \operatorname{meas}(\Omega \setminus \Omega_n) = 0$.
(iv) $\chi_\Omega \leq \liminf\limits_{n\to+\infty} \chi_{\Omega_n}$.

Show now the following theorem.

Theorem 3.1. *Let Ω_n be a sequence of open subsets of D satisfying the C-GNP. Then there exists an open subset Ω of D and a subsequence (still denoted by Ω_n) such that:*

- *Ω_n converges to Ω in the Hausdorff sense.*
- *Ω_n converges to Ω in the compact sense.*
- *Ω_n converges to Ω in the sense of characteristic functions.*
- *Ω satisfies C-GNP.*

We start by showing that Hausdorff convergence implies compact convergence.

Proposition 3.4. *Let Ω_n be a sequence of open subsets of D, which have C-GNP. If the sequence Ω_n converges in the Hausdorff sense to an open subset Ω, then it converges in the compact sense to Ω.*

Proof. One knows by Proposition 3.2 that for all compact $K \subset \Omega$, one has $K \subset \Omega_n$, for n large enough. It remains to show an analogous property for the compacts situated in the exterior of Ω. Let L be a compact set of D (one can always suppose that the interior of L is not empty) $L \subset \overline{\Omega}^c$. Let Ω_n be an eventual open subset of the sequence such that $\overline{\Omega}_n \cap L \neq \emptyset$. In the case where $L \subset \Omega_n$, one has immediately $d_H(\Omega_n, \Omega) \geq \alpha$, where α is the radius of a ball included in L. As $\lim\limits_{n\to+\infty} d_H(K_n, K) = 0$, the set of n such that $L \subset \Omega_n$, is finite.

In the other case one has $\partial\Omega_n \cap L \neq \emptyset$. Let x be some point of this intersection. According to the Proposition 3.2, there exists $\varepsilon > 0$ and an unitary vector η (both are independent of n) such that the cone $C(x, \eta, \varepsilon)$ is included in Ω_n. As one can always suppose ε small enough so that $C(x, \eta, \varepsilon) \cap \Omega = \emptyset$, one can derive that $d_H(\Omega_n, \Omega) \geq \beta$ (where β is the radius of some ball included in the cone $C(x, \eta, \varepsilon)$). As $\lim\limits_{n\to+\infty} d_H(\Omega_n, \Omega) = 0$, the set of n such that $\partial\Omega_n \cap L \neq \emptyset$ is finite. This achieves the demonstration. □

The following proposition will be useful later.

Proposition 3.5. *Let Ω_n be a sequence of open subsets which converges, in the compact sense, to an open subset Ω. If $\partial\Omega$ has a null measure, then Ω_n converges to Ω in the sense of chracteristic functions.*

Proof. If χ_n is the characteristic function of Ω_n, χ_n is bounded in $L^\infty(D)$, so it converges to a function $l \in L^\infty(D)$ for the w*-$L^\infty(D)$ convergence. Let K be a compact set in Ω. By the compact convergence, there exists $N_K \in \mathbb{N}$ such that $K \subset \Omega_n$, for all $n \geq N_K$. Therefore, $\chi_n = 1$ on K for all $n \geq N_K$ and thus, $l = 1$ on K. As this is valid for all compact K of Ω, one has $l = 1$ on Ω. In the same way considering some compact $L \subset (\overline{\Omega})^c$, one has $l = 0$ on $(\overline{\Omega})^c$. As $D = \Omega \cup (\overline{\Omega})^c \cup \partial\Omega$ and $\text{meas}(\partial\Omega) = 0$, then one has $l = \chi_\Omega$ almost everywhere (χ_Ω being the characteristic of Ω). Therefore, $\chi_n \overset{*}{\rightharpoonup} \chi_\Omega$ and thus, as all the open subsets are confined in D, χ_n converges strongly, to χ_Ω in $L^1(D)$. □

According to Proposition 3.1, one knows that $\mathcal{C}$ is relatively compact for the Hausdorff convergence. It remains to show the closure of $\mathcal{C}$, that is to say that if Ω_n is a sequence of open sets in $\mathcal{C}$ which converges in the Hausdorff sense to an open set Ω, this one belongs to $\mathcal{C}$. For this, we need to prove the following:

1. Ω is Lipschitz outside of C.
2. At points of $\partial\Omega$, where the inward normal exists, this one intersects C.
3. For all selected normals Δ to C, $\Delta \cap \Omega$ is connected.

It is clear that if $\Omega_n \xrightarrow{H} \Omega$ and $\Delta \cap \Omega_n$ is connected, then $\Delta \cap \Omega$ is also connected (because the limit, in the Hausdoff sense, for a sequence of intervals is an interval).

We now show the first point.

Theorem 3.2. *Let Ω_n be a sequence of open subsets of D which have C-*GNP *and Ω be an open subset of D. If $\Omega_n \xrightarrow{H} \Omega$, then $\partial\Omega$ is Lipschitz outside of the convex C.*

Proof. Let $x \in \partial\Omega \backslash C$. According to (ii) of Proposition 3.2, there exists a sequence of points $x_n \in \partial\Omega_n \backslash C$ which converges to x. Denote by τ_n the translation of vector $\overrightarrow{x_n x}$. It is clear that if $\Omega_n \xrightarrow{H} \Omega,\ \tau_n(\Omega_n) \xrightarrow{H} \Omega$. So we can suppose that $x \in \partial\Omega_n$, for all n.

For each n, $\partial\Omega_n$ can be represented, in a neighbourhood of x, by some Lipschitz representation ϕ_n, in the cartesian coordinate system $(O, e_1, \ldots, e_N)$ where O is the projection of x on C and $e_N = \frac{\overrightarrow{Ox}}{|\overrightarrow{Ox}|}$.

According to Proposition 2.5, all the Ω_n satisfy a uniform property of the ε-cone, in the neighbourhood of x, with ε independent of n (it depends only on x and on C). Using the result obtained by D. Chenais in [7], one can fix a $(N-1)$-dimensional closed ball $B'(O, \alpha)$ on which all the functions ϕ_n are defined.

As in the proof of Proposition 2.5, if $\phi_n \geq \delta/2$ then ϕ_n is $(2R + 2\alpha)/\delta$-Lipschitz, proving that the functions ϕ_n are uniformly Lipschitz. One can use Ascoli's theorem to extract from ϕ_n, a subsequence (still denoted ϕ_n) which converges uniformly to a Lipschitz function ϕ. It remains now, to show that the function ϕ is a representation of $\partial\Omega$ in a neighborhood of x, that is to say that in a neighborhood $\mathcal{V}$ of x, one has

$$\partial\Omega \cap \mathcal{V} = \{(y', y_N) \in B'(O, \alpha) \times \mathbb{R} \ :\ y_N = \phi(y')\} := \mathrm{Gra}(\phi) \tag{10}$$

$$\Omega \cap \mathcal{V} = \{(y', y_N) \in B'(O, \alpha) \times \mathbb{R} \ :\ y_N < \phi(y')\}. \tag{11}$$

Let $y = (y', y_N) \in \partial\Omega \cap \mathcal{V}$. By Proposition 3.2, there exists a sequence of points $y^n = ((y^n)', y^n_N) \in \partial\Omega_n$ which converges to y. Since $y^n_N = \phi_n((y^n)')$, the uniform convegence of ϕ_n to ϕ implies that $y_N = \phi(y')$, i.e., $\partial\Omega \cap \mathcal{V} \subset \mathrm{Gra}(\phi)$. For the inverse inclusion, one can remark that, for all fixed point y' in $B'(O, \alpha)$, there exists only one point of $\mathrm{Gra}(\phi)$ which is "above" y'. This one is necessarily the point (y', y_N) which belongs to $\partial\Omega \cap \mathcal{V}$.

Now, let $y = (y', y_N) \in \Omega$. According to Proposition 3.2, one knows that there exists $n_0 \in \mathbb{N}$ such that $y \in \Omega_n,\ \forall n \geq n_0$. Therfore, $y_N < \phi_n(y')$ and, tending n to infinity, $y_N \leq \phi(y')$. As $y \notin \partial\Omega,\ y_N < \phi(y')$.

At last, let $y = (y', y_N)$ with $y_N < \phi(y')$. One can prove that $y \in \Omega$. If this is not true, then either:

1. $y \in \partial\Omega$ which is impossible according to what precedes, or
2. y is in the exterior of Ω. This is also impossible because $\Omega_n \xrightarrow{K} \Omega$ which implies that $y \in \overline{\Omega}^c_n$ for n large enough, and so $y_N > \phi_n(y')$ and, at the limit $y_N \geq \phi(y')$.

□

Theorem 3.3. *Let Ω_n be a sequence of open subsets of D which have C-*GNP *and Ω be an open subset of D. If $\Omega_n \xrightarrow{H} \Omega$, then Ω satisfies C-*GNP.

Proof. To prove that Ω satisfies C-GNP is equivalent to show that Ω satisfies the C-SP.

Let $x \in \partial\Omega \backslash C$. As $\Omega_n \xrightarrow{H} \Omega$, by Proposition 3.2 there exists a sequence $x_n \in \partial\Omega_n \backslash C$ which converges to x. This implies in particular that the sequence of the closed cones K_{x_n}(defined in Definition 2.3) converges in the Hausdorff sense to the closed cone K_x. As the intersection is continuous for the Hausdorff convergence, $\Omega_n \cap K_{x_n} \xrightarrow{H} \Omega \cap K_x$. But $\Omega_n \cap K_{x_n} = \emptyset$ so $\Omega \cap K_x = \emptyset$ (according to Proposition 3.2) and Ω satisfies the C-SP. $\square$

As a consequence of what precedes, we have the following propositions which prove that the three convergences are equivalent on $\mathcal{C}$.

Proposition 3.6. *Let Ω_n be a sequence of open subsets of D which have C-*GNP *and Ω be an open subset of D. If $\Omega_n \xrightarrow{K} \Omega$, then $\Omega_n \xrightarrow{H} \Omega$.*

Proof. Since the Ω_n are confined in D, then Proposition 3.1, gives us the existence of an open subset $\widetilde{\Omega}$ and a subsequence (still denoted Ω_n) such that Ω_n converges in the Hausdorff sense to $\widetilde{\Omega}$. We now show that $\widetilde{\Omega} = \Omega$. Since Ω has C-GNP, then Ω is of Caratheodory type $(\text{int}(\overline{\Omega}) = \Omega)$. To conclude, it suffices to prove that $\Omega \subset \widetilde{\Omega} \subset \overline{\Omega}$. We start by showing the first inclusion. Suppose there exists $x \in \widetilde{\Omega} \cap (\overline{\Omega})^c$, then:

1. On one hand, the compact convergence of Ω_n to Ω implies the existence of an integer $N_x \in \mathbb{N}$ such that
$$\forall\, n \geq N_x, \quad x \in (\overline{\Omega}_n)^c.$$
2. On the other hand, as $\Omega_n \xrightarrow{H} \widetilde{\Omega}$, then there exists $N'_x \in \mathbb{N}$ s.t. $\forall\, n \geq N'_x,\ x \in \Omega_n$. Consequently,
$$\forall\, n \geq \max(N_x, N'_x)\ ,\ x \in \Omega_n \cap (\overline{\Omega}_n)^c,$$
which is absurd.

We now show that $\Omega \subset \widetilde{\Omega}$. Let $x \in \Omega$, there exists a closed ball $\overline{B}(x)$ centered in x s.t. $\overline{B}(x) \subset \Omega_n$. But $\Omega_n \xrightarrow{H} \widetilde{\Omega}$, so $B(x) \subset \widetilde{\Omega}$ and thus, $x \in \widetilde{\Omega}$. $\square$

Proposition 3.7. *Let Ω_n be a sequence of open subsets of D which have C-*GNP *and Ω be an open subset of D. If $\Omega_n \xrightarrow{H} \Omega$, then $\Omega_n \xrightarrow{L} \Omega$.*

Proof. As $\Omega_n \xrightarrow{H} \Omega$, then according to Proposition 3.4, $\Omega_n \xrightarrow{K} \Omega$. Since $\partial\Omega \subset (\partial\Omega \setminus C) \cup \partial C$ and ∂C and $\partial\Omega \setminus C$ have null measures (since they are Lipschitz), then $\text{meas}(\partial\Omega) = 0$. Then, Proposition 3.5 gives the result. $\square$

Proposition 3.8. *Let Ω_n be a sequence of open subsets of D which have C-*GNP *and Ω be an open subset of D. If $\Omega_n \xrightarrow{L} \Omega$, then $\Omega_n \xrightarrow{H} \Omega$ and Ω has C-*GNP.

Proof. Let Ω_n be a sequence of open subsets of D converging in the sense of characteristic functions to Ω. The Ω_n are confined in D, so, even if to take a subsequence of Ω_n, $\Omega_n \xrightarrow{H} \widetilde{\Omega}$ and $\widetilde{\Omega}$ has C-GNP. Now, as the Ω_n have C-GNP, $\Omega_n \xrightarrow{K} \widetilde{\Omega}$ (see Proposition 3.4). Because Ω and $\widetilde{\Omega}$ are of Caratheodory type, Proposition 3.5 implies that $\widetilde{\Omega} = \Omega$, and thus Ω satisfies C-GNP. $\square$

4. Stability of the solution of Dirichlet problem

By Remark 2.5 in two dimensions, an open subset which has C-GNP satisfies the exterior segment property. Therefore, it is stable in the sense of Keldyš (see Theorem 4.1). In all the following we suppose that $N \geq 3$.

Generally, Hausdorff convergence is not sufficient to get the stability of the solution of a boundary value problem like the Dirichlet one, when the domain varies. In our case, nevertheless, we were able to prove (Proposition 3.4) that in the class $\mathcal{C}$, Hausdorff convergence implies compact convergence which is the main tool for proving the stability of the solution of Dirichlet problem. However, this is not sufficient to assure *a priori* the desired result of stability. In fact, it is necessary that the limit domain has a minimum of regularity in order to get this stability result. Now, if an open subset Ω satisfies C-GNP, it is Lipschitz outside of C that is more than the wished regularity. Unfortunatly, we have seen, in Proposition 2.1, that the boundary of Ω could have cusps which are on ∂C. In general, at cusp point, an open subset can be regular or irregular in the sense of Wiener (Definition 4.2). This depends on the geometric shape of cusps and, in particular, it depends on the "quantity of material" (in terms of capacity) contained in the exterior of Ω, in a neighbourhood of such points. We will make a precise computation of the capacity of the exterior of Ω in a neighbourhood of cusps in order to be sure that C-GNP implies their regularity. Before that, let us recall some definitions and results which are very useful for proving the stability of the solution of the Dirichlet problem, when the domain varies.

Definition 4.1. The Sobolev capacity is defined as follows:

- For a compact subset K of $\mathbb{R}^N$

$$\mathrm{Cap}(K) = \inf\left\{\int_{\mathbb{R}^N} |\nabla\varphi|^2 \ ; \ \varphi \in C_0^\infty(\mathbb{R}^N),\ \varphi \geq 1 \text{ on } K\right\}.$$

- For an open subset G of $\mathbb{R}^N$

$$\mathrm{Cap}(G) = \sup\{\mathrm{Cap}(K) \ : \ K \subseteq G,\ K \text{ is compact}\}.$$

- For any set E of $\mathbb{R}^N$

$$\mathrm{Cap}(E) = \inf\{\mathrm{Cap}(G) \ : \ G \supseteq E,\ G \text{ is open}\}.$$

One says that a property takes place quasi everywhere (or more simply q.e.) if it takes place in the complement of a set of null capacity.

We now give a regularity criterion called the Wiener criterion [17].

Definition 4.2 (Wiener criterion)**.** Let Ω be an open subset of $\mathbb{R}^N$, $N \geq 3$, and x_0 a point of $\partial\Omega$. For all $q,\ q \in]0,1[$, let

$$G_n = \overline{\Omega}^c \cap \left\{x \ : \ q^{n+1} \leq |x - x_0| \leq q^n\right\}.$$

Then, x_0 is regular if and only if the series of general term $\frac{\mathrm{Cap}(G_n)}{q^{n(N-2)}}$ is divergent.

Now we return to our question concerning the stability of the solution of the Dirichlet problem when the domain varies. This stability is related to the stability (in the sense of Keldyš) of the domain Ω intervening in such a problem. The fundamental question now, is to know which condition the domain Ω must satisfy in order that a function which vanishes *q.e.* on the exterior of Ω, vanishes in the complement of Ω (that is to say, that such a function belongs to $H_0^1(\Omega)$). This is exactly what the Wiener criterion says. To be more precise let us give the following results, e.g., [10], [12] and [14].

Definition 4.3 (Keldyš)**.** An open set Ω is said to be stable if and only if all points of its boundary (except a set of null capacity) are regular in the Wiener sense.

Theorem 4.1. *Let Ω_n be a sequence of open subsets of D and let u_n be the solution of the Dirichlet problem*

$$(P_n)\begin{cases} -\Delta u_n = f \quad in \quad \Omega_n \\ u_n = 0 \quad on \quad \partial\Omega_n. \end{cases}$$

If $\Omega_n \xrightarrow{K} \Omega$ and Ω is stable in the sense of Keldyš, then u_n converges strongly in $H_0^1(D)$ to a function u, the solution of the Dirichlet problem on Ω :

$$(P)\begin{cases} -\Delta u = f \quad in \quad \Omega \\ u = 0 \quad on \quad \partial\Omega \end{cases}$$

(the functions u_n and u are extended by zero in D, $f \in H^{-1}(D)$).

Before applying this result to our problem, we shall give a necessary and sufficient condition so that a cusp of surface of revolution will be regular in Wiener sense. Without loss of generality, one can suppose this point in the origin. Such point is given, in spherical coordinates, by

$$(PR)\begin{cases} \theta = f(\rho), \quad \rho \leq 1, \quad f(0) = 0 \text{ and } f(1) < \pi, \\ \quad \text{where } f(\rho) \text{ is an increasing function.} \end{cases}$$

Theorem 4.2. *Let Ω be an open subset of $\mathbb{R}^N$ and $O \in \partial\Omega$ be given by (PR).*

- *If $N = 3$, then O is regular if and only if the series of general term $\frac{1}{\ln f(q^n)}$ is divergent.*
- *If $N \geq 4$, then O is regular if and only if the series of general term $[f(q^n)]^{N-3}$ is divergent.*

Proof. Suppose first, that $N = 3$. Let q be a real number such that $0 < q < 1$, and let $\theta_n = f(q^n)$. Then the set

$$G_n \;=\; \overline{\Omega}^c \cap \{x : \; q^{n+1} \leq |x| \leq q^n\}$$

is on one hand included in the right circular cylinder with height q^n and of basis radius $q^n \sin\theta_n$, or again in the ellipsoid $\mathcal{E}_n$ of half-axes $2q^n \sin\theta_n$, $2q^n \sin\theta_n$, $2q^n$. On the other hand, it contains the right circular cylinder with height $q^n - q^{n+1}$ and of basis radius $q^{n+1}\sin\theta_{n+1}$, or again, it contains the ellipsoid $\mathcal{F}_n$ of half-axes $q^{n+1}\sin\theta_{n+1}$, $q^{n+1}\sin\theta_{n+1}$, $q^n(1-q)/2$.

Now, compute the capacity of these two ellipsoids. For this, let us consider a general ellipsoid $\mathcal{E}$

$$\mathcal{E} := \left\{ X = (x,y,z) \in \mathbb{R}^3 \;\middle|\; \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} < 1 \right\}. \tag{12}$$

Introduce the capacitary potential

$$U(x,y,z) \;=\; p\int_\lambda^\infty \frac{dt}{[(a^2+t)(b^2+t)(c^2+t)]^{\frac{1}{2}}}, \tag{13}$$

where λ is the biggest root of the equation

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1, \tag{14}$$

and p is such that $U=1$ on $\partial\mathcal{E}$, that is to say

$$\frac{1}{p}=\int_0^\infty \frac{dt}{[(a^2+t)(b^2+t)(c^2+t)]^{\frac{1}{2}}}. \tag{15}$$

We recall that U is the solution of

$$\begin{cases} \Delta U=0 \ \text{ in } \mathbb{R}^3\setminus\bar{\mathcal{E}} \\ U=1 \ \text{ on } \partial\mathcal{E} \\ U=O(\frac{1}{r}) \quad \text{if } r\to\infty. \end{cases}$$

Note that the capacity of the set $\mathcal{E}$ is given by

$$\mathrm{Cap}(\mathcal{E}) \;=\; 8\pi\left(\int_0^\infty \frac{dt}{[(a^2+t)(b^2+t)(c^2+t)]^{\frac{1}{2}}}\right)^{-1}.$$

In fact,

$$\mathrm{Cap}(\mathcal{E})=\int_{\partial\mathcal{E}}|\nabla U|.$$

By (14), one gets

$$\left(\frac{\partial\lambda}{\partial x}\right)_{|\lambda=0}=\frac{2x}{a^2(x^2/a^4+y^2/b^4+z^2/c^4)}.$$

(one can do the same for y and z).

Therefore on $\partial\mathcal{E}$, one has

$$\frac{\partial u}{\partial x}=-p\frac{\frac{\partial\lambda}{\partial x}}{abc}=-\frac{p}{abc}\,\frac{2x}{a^2(x^2/a^4+y^2/b^4+z^2/c^4)}$$

(one can do the same for y and z).

Now, if $X=(x,y,z)$ is on $\partial\mathcal{E}$ and $\vec{n}(X)$ is the normal vector to $\partial\mathcal{E}$ at X, then

$$\vec{n}(X)=\frac{1}{\sqrt{x^2/a^4+y^2/b^4+z^2/c^4}}\left(x/a^2,y/b^2,z/c^2\right).$$

One obtains therefore,

$$|\nabla u|=\frac{2p}{abc}X\cdot\vec{n}(X).$$

Hence, (see also [17], [12], [11]),

$$\mathrm{Cap}(\mathcal{E})=\int_{\partial\mathcal{E}}|\nabla U| \;=\; \frac{2p}{abc}\int_{\partial\mathcal{E}}X\cdot n \;=\; \frac{2p}{abc}3\mathrm{Vol}(\mathcal{E})=8\pi p.$$

Where $\mathrm{Vol}(\mathcal{E})$, denotes the volume of $\mathcal{E}$.

Now, as we work with ellipsoids which have two equal axes, we are in the case where $a=c\le b$. We have to compute an integral of the form

$$I_3:=\int_0^\infty\frac{dt}{(a^2+t)(b^2+t)^{\frac{1}{2}}}.$$

Using the variable change $s^2 = b^2 + t$, we obtain

$$I_3 = \int_b^\infty \frac{2ds}{s^2 - (b^2 - a^2)}.$$

If we put, $w^2 = b^2 - a^2$, we get

$$I_3 = \frac{1}{w} \ln \frac{b - w}{b + w}.$$

If $\mathcal{E} = \mathcal{E}_n$, $a = 2q^n \sin\theta_n$, $b = 2q^n$ and $w = 2q^n \cos\theta_n$, I_3 becomes

$$I_3 = \frac{1}{2q^n \cos\theta_n} \ln \frac{1 - \cos\theta_n}{1 + \cos\theta_n},$$

and thus,

$$\operatorname{Cap}(\mathcal{E}_n) = c_3 \frac{2q^n \cos\theta_n}{\ln\cot\frac{\theta_n}{2}}.$$

In the same way, the capacity of $\mathcal{F}_n$ is given by

$$\operatorname{Cap}(\mathcal{F}_n) = c_3 \frac{2q^{n+1} \cos\theta_{n+1}}{\ln\cot\frac{\theta_{n+1}}{2}},$$

One can, therefore, deduces

$$c_3 \frac{2q \cos\theta_{n+1}}{\ln\cot\frac{\theta_{n+1}}{2}} \;<\; \frac{\operatorname{Cap}(G_n)}{q^n} \;<\; c_3 \frac{2\cos\theta_n}{\ln\cot\frac{\theta_n}{2}}.$$

This shows that the point O is regular if and only if the series of general term $\frac{2\cos\theta_n}{\ln\cot\frac{\theta_n}{2}}$ diverges. As θ_n must to tend to zero, the divergence of this series is equivalent to the divergence of the series of general term $\frac{1}{\ln\theta_n} = \frac{1}{\ln f(q^n)}$.

Remark 4.1. According to what precedes, if $f(\rho) = A\rho^m$, the cusp point is regular and if $f(\rho) = A\exp(\frac{-m}{\rho})$, it is irregular.

Now, suppose that $N \geq 4$ and consider the ellipsoid

$$\mathcal{E}_N := \left\{ X = (x_1, \ldots, x_N) \in \mathbb{R}^N \;\middle|\; \sum_{i=1}^{i=N} \frac{x_i^2}{a_i^2} < 1 \right\},$$

and the capacitary potential

$$U(X) \;=\; p \int_\lambda^\infty \frac{dt}{[(a_1^2 + t)(a_2^2 + t)\ldots(a_N^2 + t)]^{\frac{1}{2}}},$$

where λ is the biggest root of the equation

$$\sum_{i=1}^{i=N} \frac{x_i^2}{a_i^2 + \lambda} = 1,$$

and

$$\frac{1}{p} \;=\; \int_0^\infty \frac{dt}{[(a_1^2 + t)(a_2^2 + t)\ldots(a_N^2 + t)]^{\frac{1}{2}}}.$$

As seen above, the set G_n is contained in the ellipsoid $\mathcal{E}_n$ of $(N-1)$ equal half-axes $2q^n \sin\theta_n, \ldots, 2q^n \sin\theta_n$ and $2q^n$. It contains the ellipsoid $\mathcal{F}_n$ of $(N-1)$ equal half-axes $2q^{n+1}\sin\theta_{n+1}, \dots 2q^{n+1}\sin\theta_{n+1}$ and $2q^{n+1}$. To get the capacity of the ellipsoid

$$\mathcal{E}_N := \left\{ X = (x_1, \dots, x_N) \in \mathbb{R}^N \;\middle|\; \sum_{i=1}^{i=N-1} \frac{x_i^2}{a^2} + \frac{x_N^2}{b^2} < 1 \right\},$$

we need to compute the following intgral

$$I_N \;=\; \int_0^\infty \frac{dt}{(a^2+t)^{\frac{N-1}{2}}(b^2+t)^{\frac{1}{2}}}.$$

Using the variable change $s^2 = b^2 - t$ and putting $w^2 = b^2 - a^2$, I_N becomes

$$I_N \;=\; 2\int_b^\infty \frac{ds}{(s^2-w^2)^{\frac{N-1}{2}}}.$$

Putting $s = wx$, $x = \cosh\theta$ and $t = \tanh\frac{\theta}{2}$, we get

$$I_N \;=\; \frac{2^2}{(2w)^{N-2}}\int_\alpha^1 \frac{(1-t^2)^{N-3}}{t^{N-2}},$$

with

$$\alpha = \tanh\frac{\operatorname{argch}\frac{b}{w}}{2} = \sqrt{\frac{b-w}{b+w}}.$$

Thus,

$$I_N \;=\; \frac{2^2}{(2w)^{N-2}} \sum_{k=0}^{k=N-3} C_{N-3}^k \frac{(-1)^k}{2k-N+3}\left[1 - \left(\frac{b-w}{b+w}\right)^{k-\frac{N-3}{2}}\right]$$

Now, for the ellipsoid $\mathcal{E}_n$, one has $a = 2q^n\sin\theta_n$, $b = 2q^n$ and $w = 2q^n\cos\theta_n$, therefore

$$I_N \;=\; \frac{1}{2^{2N-6}q^{n(N-2)}} \sum_{k=0}^{k=N-3} C_{N-3}^k \frac{(-1)^k}{2k-N+3}\left[1 - \left(\frac{1-\cos\theta_n}{1+\cos\theta_n}\right)^{k-\frac{N-3}{2}}\right],$$

and,

$$\frac{\operatorname{Cap}(\mathcal{E}_n)}{q^{n(N-2)}} = \frac{C_N(\cos\theta_n)^{N-2}}{\sum_{k=0}^{k=N-3} C_{N-3}^k \frac{(-1)^k}{2k-N+3}\left[1-\left(\frac{1-\cos\theta_n}{1+\cos\theta_n}\right)^{k-\frac{N-3}{2}}\right]}.$$

In the same way,

$$\frac{\operatorname{Cap}(\mathcal{F}_n)}{q^{n(N-2)}} = \frac{C_N q(\cos\theta_{n+1})^{N-2}}{\sum_{k=0}^{k=N-3} C_{N-3}^k \frac{(-1)^k}{2k-N+3}\left[1-\left(\frac{1-\cos\theta_{n+1}}{1+\cos\theta_{n+1}}\right)^{k-\frac{N-3}{2}}\right]}.$$

When θ_n tends to zero, $\frac{\operatorname{Cap}(\mathcal{E}_n)}{q^{n(N-2)}}$ and $\frac{\operatorname{Cap}(\mathcal{F}_n)}{q^{n(N-2)}}$ are equivalent to θ_n^{N-3}. As

$$\frac{\operatorname{Cap}(\mathcal{F}_n)}{q^{n(N-2)}} \;<\; \frac{\operatorname{Cap}(G_n)}{q^{n(N-2}} \;<\; \frac{\operatorname{Cap}(\mathcal{E}_n)}{q^{n(N-2)}},$$

we get

$$\frac{\operatorname{Cap}(G_n)}{q^{n(N-2)}} \sim \theta_n^{N-3},$$

or again,

$$\frac{\mathrm{Cap}(G_n)}{q^{n(N-2)}} \sim f(q^n)^{N-3}.$$

□

Remark 4.2. If $f(\rho) = A\rho^m$ or $f(\rho) = A\exp(\frac{-m}{\rho})$, then the considered cusp point is irregular. On the other hand, if $f(\rho) = \left[-\frac{1}{\ln(\rho)}\right]^{\frac{1}{N-3}}$, $(\rho \le \rho_0 < 1)$, it is regular.

Now, consider a point $x_0 \in \partial\Omega \cap \partial C$. By Proposition 2.2, this eventual cusp point is contained in some cusp point of surface of revolution, which is given in spherical coordinates by,

$$\theta = \arcsin \frac{\rho}{2R}.$$

In particular, when ρ tends to zero, one has $\theta \sim \rho/2R$. By Theorem 4.2, this cusp point is regular in the sense of Wiener. One deduces therefore, that x_0 is also regular since, in its neighbourhood, the capacity of the exterior of Ω is greater than the capacity of the set $(\mathcal{B}_{x_0})^c$ (of Proposition 2.2).

Now, Definition 4.3 and Theorems 3.1, and 4.1 allow us to state:

Theorem 4.3. *Let Ω_n be a sequence of open subsets of $\mathbb{R}^N$ $(N = 2,\ 3)$, included in D and having C-GNP. If u_n is the solution of the Dirichlet problem*

$$(P_n)\begin{cases} -\Delta u_n = f & in \ \ \Omega_n \\ u_n = 0 \ \ on & \partial\Omega_n, \end{cases}$$

then there exists a subsequence of Ω_n (still denoted Ω_n) and an open subset Ω of D which is stable in the sense of Keldyš, such that:

- $\Omega_n \xrightarrow{H} \Omega$.
- $\Omega_n \xrightarrow{K} \Omega$.
- u_n *converges strongly in $H_0^1(D)$ to a function u, solution of the Dirichlet problem on Ω:*

$$(P)\begin{cases} -\Delta u = f & in \ \ \Omega \\ u = 0 \ \ on & \partial\Omega \end{cases}$$

(*the functions u_n and u are extended by zero in D, $f \in H^{-1}(D)$*).

References

[1] M. Barkatou and A. Henrot, *Un résultat d'existence en optimisation de forme en utilisant une propriété géométrique de la normale*, ESAIM-COCV: Control, Optimization and Calculus of Variations, **2** (1997), 105–123, MR 98f:49040, Zbl 0920.49027.

[2] M. Barkatou, *Necessary and sufficient conditions of existence for a Bernoulli's free boundary problem*, submitted.

[3] M. Berger, *Géométrie Tome 3: Convexes et Polytopes, Polyèdres Réguliers, Aires et Volumes*, Publie avec le concours du Centre National de la Recherche Scientifique, Paris: Cedic/Fernand Nathan, 1978, Zbl 0423.51001.

[4] D. Bucur and J.P. Zolesio, *N-dimensional shape optimization under capacitary constraint*, J. of Diff. Eq., **123**(**2**) (1995), 504–522, MR 97g:49054, Zbl 0847.49029.

[5] D. Bucur and J.P. Zolesio, *Boundary optimization under pseudo curvature constraint*, Ann. Scuola Norm. Pisa (4), **23**(**4**) (1996), 681–700, MR 98j:49055, Zbl 0889.49026.

[6] D. Chenais, *On the existence of a solution in a domain identification problem*, J. Math. Anal. Appl., **52** (1975), 189–289, MR 52 #6526, Zbl 0317.49005.

[7] D. Chenais, *Sur une famille de variétés à bord Lipschitziennes. Application à un problème d'identification de domaines*, Ann. Inst. Fourier, **27**(**4**) (1977), 201–231, MR 57 #13463, Zbl 0333.46020.

[8] B. Gustafsson and H. Shahgholian, *Existence and geometric properties of solutions of a free boundary problem in potential theory*, J. Reine Angew. Math., **473** (1996), 137–179, MR 97e:35205, Zbl 0846.31005.

[9] L.I. Hedberg, *Spectral synthesis and stability in Sobolev spaces*, Euclidean Harmonic Analysis (Proc. Sem., Univ. Maryland, College Park, Md., 1979), Lecture Notes in Math., **779**, Springer, Berlin, 1980, 73–103, MR 82a:46033, Zbl 0469.31003.

[10] A. Henrot, *Continuity with respect to the domain for the Laplacian: A survey*, Control and Cybernetics, **23**(**3**) (1994), 427–443, MR 95i:35030, Zbl 0822.35029.

[11] J. Jeans, *The Mathematical Theory of Electricity and Magnetism* (5th ed.), Cambridge University Press, New York, 1960, MR 22 #6378.

[12] M.V. Keldyš, *On the solvability and stability of the Dirichlet problem*, Amer. Math. Soc. Transl. (2), **51** (1966), 1–73, translation from Usp. Mat. Nauk, **8** (1941), 171–231, Zbl 0179.43901.

[13] H. Shahgholian, *Quadrature surfaces as free boundaries*, Arkiv Math., **32**(**2**) (1994), 475–492, MR 96e:35190, Zbl 0827.31004.

[14] N.S. Landkof, *Foundations of Modern Potential Theory*, Die Grundlehren der Mathematischen Wissenschaften, **180**, Springer, Berlin-Heidelberg-New York, 1972, MR 50 #2520, Zbl 0253.31001.

[15] O. Pironneau, *Optimal Shape Design for Elliptic Systems*, Springer Series in Computational Physics, Springer, New York, 1984, MR 86e:49003, Zbl 0534.49001.

[16] V. Šverak, *On optimal shape design*, J. Math. Pures Appl., **72**(**6**) (1993), 537–551, MR 94j:49047, Zbl 0849.49021.

[17] N. Wiener, *The Dirichlet problem*, J. Maths Phys., **3** (1924), 127–146.

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