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On subnormal operators whose spectrum are multiply connected domains

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Abstract. Let Ω be a connected bounded domain with a finite amount of "holes" and "nice boundary". We study subnormal operators with spectrum equal to $\overline{\Omega}$, while the spectrum of their normal extensions are supported on the boundary, $\partial\Omega$.

Contents

1. Introduction

Given a pure isometry T on some Hilbert space $\mathcal{H}$ with $\operatorname{ind} T = -1$, put $\mathcal{K} = \mathcal{H} \ominus T(\mathcal{H})$. The Wold decomposition then says that T is unitarily equivalent to the shift operator S on $\oplus_{k=0}^{\infty} T^k(\mathcal{K})$. In particular, one sees that T is subnormal since S is the restriction of the bilateral shift to an invariant subspace. In other words, T is unitarily equivalent to M_z — multiplication by the independent variable z — on the Hardy space on the unit disc $H^2(\mathbb{D})$, which in turn can be identified with a subspace of $L^2(m)$, where m is the arclength measure on the unit circle $\mathbb{T}$. Analogously, had we started out with $\operatorname{ind} T = -n$, $(n \in \mathbb{N})$, we would get that T is unitarily equivalent to M_z on $\oplus_{j=1}^{n} H^2(\mathbb{D})$, or, if you wish, M_z restricted to an invariant subspace of $L^2(m, \mathbb{C}^n)$ — the L^2-space of $\mathbb{C}^n$ valued analytic functions on $\mathbb{T}$. The setting

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of the present paper is the following. Let $\Omega \subset \mathbb{C}$ be a bounded connected open set whose boundary consists of $N+1$ disjoint simple closed rectifiable nontrivial Jordan curves. Let μ be a finite Borel-measure on $\partial\Omega$, let $\mathcal{H}$ be an M_z-invariant subspace of $L^2(\mu, l^2)$ and set

$$T = M_z|_{\mathcal{H}}, \tag{1.1}$$

where $M_z|_{\mathcal{H}}$ denotes the restriction of M_z to $\mathcal{H}$. We wish to study the operator T.

It is well known that one can split $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ where $\mathcal{H}_1$ is the largest reducing subspace with respect to M_z on $L^2(\mu, l^2)$, (see Section 3). Since $M_z|_{\mathcal{H}_1}$ is well understood by multiplicity theory (see Section 2.2), we shall focus on $T = M_z|_{\mathcal{H}_2}$, that is, we will assume that $\mathcal{H} = \mathcal{H}_2$ has no reducing subspaces. Such operators are called "pure".[1] We now collect a few other observations that simplifies the analysis.

Proposition 1.1. *Suppose that μ, $\mathcal{H}$, T are as above and that $\sigma(T) \subset \operatorname{supp}\mu$. Then $\mathcal{H}$ is a reducing subspace of $L^2(\mu, l^2)$.*

Proposition 1.2. *Let N be an operator on a Hilbert space and let $\mathcal{H}$ be an N-invariant subspace. Then*

$$\partial\sigma(N|_{\mathcal{H}}) \subset \partial\sigma(N).$$

For proofs see [8], Theorem 2.11, Ch. II and Corollary 3.7, Ch. V. It is also shown that the minimal normal extension of a subnormal operator T has spectrum included in $\sigma(T)$, and hence it is no restriction to assume that $\operatorname{supp}\mu \subset \sigma(T)$. By Proposition 1.2 we know that $\partial\sigma(T) \subset \operatorname{supp}\mu$. We distinguish between three possible cases:

(i) $\sigma(T) = \operatorname{supp}\mu$,
(ii) $\sigma(T) = \overline{\Omega}$,
(iii) other.

By Proposition 1.1 we know that if (i) holds then $\mathcal{H}$ is a reducing subspace and hence T is a normal operator. Thus both T and $\mathcal{H}$ are well understood by multiplicity theory, which we briefly recall in Section 2.2. Case (iii) falls outside the scope of this article so the main part of the paper is devoted to the study of case (ii). This has previously been considered by M. B. Abrahamse and R. G. Douglas in [1], although the setting is quite different and they refer to the corresponding operators as bundle shifts. Their work requires a lot of prerequisites. Indeed, Abrahamse and Douglas's framework involves analytic vector bundles, the universal cover, Forelli's and Grauert's

[1] Again, when speaking of reducing subspace we mean with respect to M_z on $L^2(\mu, l^2)$. The usual definition of a pure operator T states that T should not have any T-reducing subspace $\mathcal{R}$ such that $T|_{\mathcal{R}}$ is normal. However, it follows by standard theory of subnormal operators (see, e.g., [8]) that such a subspace $\mathcal{R}$ is reducing for any normal extension of T. Thus T is pure if and only if $\mathcal{H}$ has no M_z-reducing subspaces. In the remainder, a subspace $\mathcal{R} \subset \mathcal{H}$ will be called reducing if it is reducing with respect to a (and hence all) normal extension of T.

theorems to mention a few ingredients. The aim of this paper and its sequel, [5], is to provide relatively simple proofs of some of the key results in [1], that only rely on classical function theory.

For a closed set S we let $R(S)$ denote the rational functions with poles outside of S, and given $\mathcal{M} \subset \mathcal{H}$ we denote by $[\mathcal{M}]_{R(S)}$ the closure of the span of $\{rf : r \in R(S) \text{ and } f \in \mathcal{M}\}$. Recall that any subnormal operator T has a minimal normal extension, and that any two such extensions are unitarily equivalent (see [8]). In particular, the multiplicity function of a minimal normal extension N of T is unique for T, and we will denote it MF_T or simply MF when T is clear from the context. (If $N = M_z$ on $\oplus_{k=1}^{\infty} L^2(\mu_k, \mathbb{C})$ then $MF_N(x) = \#\{k : x \in \mathsf{supp}\, \mu_k\}$. For more information, see [7].) The main result of this paper is the following.

Theorem 1.3. *Let T be a pure subnormal operator as above such that (ii) holds, and for some fixed $\lambda \in \Omega$ set $n = -\mathsf{ind}\,(T - \lambda)$. Then $MF_T(\cdot) = n$ μ-a.e. Moreover, there exists $f_1, \ldots, f_n$ such that $\mathcal{H} = [f_1, \ldots, f_n]_{R(\sigma(T))}$.*

It is tempting to hope that one could go further and choose the f_j's such that the $[f_j]_{R(\sigma(T))}$'s become mutually orthogonal, because then the further study of T would reduce to the case $n = 1$. When $\sigma(T) = \mathbb{D}$, i.e., when T is an isometry, this is possible by the Wold decomposition mentioned initially. This is also true for any simply connected domain and, surprisingly, also when the domain has precisely one "hole", but in general not for more holes. This is the topic of [5], which is a sequel to the present paper. It also contains some examples relevant to this article.

2. Preliminaries

This section is a collection of known results from various areas that will be used.

2.1. Complex analysis. Given a closed set $S \subset \mathbb{C}$ we will denote by $C(S)$ the Banach algebra of continuous functions on S with the supremum norm. By $A(S) \subset C(S)$ we denote the algebra of all functions that are holomorphic on the interior of S, and by $R(S) \subset C(S)$ we denote the closure of all rational functions with poles outside S. Throughout the paper we will let $\Omega \subset \mathbb{C}$ be a domain as follows:

Assumption Ω. Let $\Omega \subset \mathbb{C}$ be a bounded connected open set whose boundary consists of $N+1$ disjoint simple closed rectifiable nontrivial Jordan curves.

Denote the curves that make up $\partial\Omega$ by $\gamma_0, \ldots, \gamma_N$. Each γ_n can be chosen such that

$$\gamma_n(s) = \gamma_n(0) + \int_0^s g_n(t)dt, \quad 0 \le s \le 1,$$

for some (Borel-measurable) integrable function $g_n : [0,1] \to \mathbb{C}$. We define "arc-length measure" m for $\partial\Omega$ as the Borel measure on $\mathbb{C}$ given by

$$m(S) = \sum_{n=0}^{N} \int_{\gamma_n^{-1}(S)} |g_n(t)| dt. \tag{2.1}$$

By the a.e. Lebesgue differentiability of each g_n, it follows that $\partial\Omega$ has a tangent at m-a.e. point. In particular it makes sense to talk about nontangential limits at a.e. boundary point. Let $\alpha : \partial\Omega \to \mathbb{T}$ be the unimodular function such that $\alpha(\xi)$ points in the direction of the tangent to $\partial\Omega$ at a.e. $\xi \in \partial\Omega$ and $i\alpha(\xi)$ points inside of Ω. Given any function $f \in L^1(m, \mathbb{C})$ we define

$$\int_{\partial\Omega} f \; dz = \int f\alpha \; dm.$$

Theorem 2.1. *Let Ω satisfy Assumption Ω, let $f \in L^1(m, \mathbb{C})$ be given and define*

$$\tilde{f}(w) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta)}{\zeta - w} dz$$

for all $w \in \mathbb{C} \setminus \partial\Omega$. If $\tilde{f} \equiv 0$ on $\mathbb{C} \setminus \overline{\Omega}$, then the nontangential limits of $\tilde{f}|_\Omega$ coincide with f for m-a.e. $\xi \in \partial\Omega$.

Proof. We will only outline the details. Let ξ be a point on $\partial\Omega$, let n and t_ξ be such that $\xi = \gamma_n(t_\xi)$ and suppose that g_n is Lebesgue differentiable at t_ξ. This assumption holds for m-a.e. ξ, (see, e.g., [6]). Moreover, by reparametrizing the curves with the arc-length, we may assume that $\alpha(\xi) = g_n(t_\xi)$ holds for a.e. pair ξ and t_ξ as above. Let L be the line tangent to $\partial\Omega$ through ξ, let Γ_ξ be a fixed nontangential cone with vertex at ξ, and let $w \in \Gamma_\xi$. Let $\hat{w}$ be the mirror of w in L. Let I be a small open interval around t_ξ. Note that

$$\tilde{f}(w) = \frac{1}{2\pi i} \int_{\partial\Omega} f(\zeta) \left(\frac{1}{\zeta - w} - \frac{1}{\zeta - \hat{w}} \right) dz$$

for w sufficiently near ξ. We also have

$$\begin{aligned} \limsup_{\substack{w\to\xi \\ w\in\Gamma_\xi\cap\Omega}} \tilde{f}(w) - f(\xi) \\ = \limsup_{\substack{w\to\xi \\ w\in\Gamma_\xi\cap\Omega}} \left[\frac{1}{2\pi i} \int_{\gamma_n(I)} f(\zeta) \left(\frac{1}{\zeta - w} - \frac{1}{\zeta - \hat{w}} \right) dz - f(\xi) \right] \end{aligned} \tag{2.2}$$

because the kernel goes to zero uniformly on $\partial\Omega \setminus \gamma_n(I)$. Note that $\xi + \alpha(\xi)I$ is a piece of L. Tedious estimates then show that the right hand side of (2.2) is bounded by

$$\limsup_{\substack{w\to\xi\\ w\in\Gamma_\xi\cap\Omega}}\left[\frac{1}{2\pi i}\int_I f(\gamma_n(t))\left(\frac{1}{\xi+\alpha(\xi)t-w}-\frac{1}{\xi+\alpha(\xi)t-\hat{w}}\right)dt-f(\xi)\right]+\epsilon(I),$$

where $\epsilon(I)$ goes to zero when the length of I does. However, the new kernel is precisely the Poisson kernel for L, so the $\limsup$ equals 0 by the classical Fatou's theorem, given that $f(\xi)$ is the Lebesgue derivative of $f\circ\gamma_n$ at t_ξ. Since this is true for m-a.e. ξ, the proof is complete. □

Suppose, for the next statement only, that $N=0$, i.e., that Ω is simply connected and let $\phi:\mathbb{D}\to\Omega$ be an analytic bijection, as given by the Riemann mapping theorem. Let θ denote the arc-length measure for $\mathbb{T}$. The following result is a combination of theorems by F. and M. Riesz and Carathéodory, (see Section 2.C and 2.D of [14]).

Theorem 2.2. *Let ϕ be as above, where Ω is simply connected and satisfies Assumption Ω. Then ϕ extends to a continuous bijective function on $\overline{\mathbb{D}}$ such that $\phi|_{\mathbb{T}}$ is a parametrization of $\partial\Omega$. Moreover, $\phi'\in H^1(\mathbb{D})$ and given any $f\in L^1(m)$ we have*

$$\int f dm=\int_{\mathbb{T}} f\circ\phi\ |\phi'|\ d\theta.$$

With ϕ and Ω as in Theorem 2.2, Lindelf's theorem says that a function f on Ω has nontangential limits m-*a.e.* if and only if $f\circ\phi$ has nontangential limits θ-a.e. In particular, Privalov's theorem holds on Ω, i.e., a meromorphic function on Ω can not have nontangential limits equal to zero on a set of positive measure. This fact easily extends to domains Ω whose boundary has several components, and will be used frequently in the coming proofs.

Recall that a set $E\subset\partial\Omega$ is called a peak-set for $A(\overline{\Omega})$ if there exists an $f\in A(\overline{\Omega})$ such that $f(\zeta)=1$ for all $\zeta\in E$ and $|f(\zeta)|<1$ for all $\zeta\in\overline{\Omega}\setminus E$. The next theorem is a combination of results from [12] and [13].

Theorem 2.3. *Let Ω be as in Assumption Ω. Then*

$$A(\overline{\Omega})=R(\overline{\Omega}).$$

Moreover, if $E\subset\partial\Omega$ is closed, then E is a peak set for $A(\overline{\Omega})$ if and only if $m(E)=0$.

2.2. Normal operators. Recall that the concept of weak and strong measurability coincide for functions with values in $l^2(\mathbb{N})$, (see [15], sec 3.11). We thus say that $f:\mathbb{C}\to l^2$ is (Borel-) measurable if the function $\langle f(\cdot),x\rangle_{l^2}$ is (Borel-) measurable for all $x\in l^2$, which in turn is equivalent to saying that each "coordinate function" $f_i=\langle f(\cdot),e_i\rangle_{l^2}$ is measurable, where $\{e_i\}_{i\geq 0}$ denotes the standard orthonormal basis for l^2. Let μ denote a compactly supported finite positive Borel measure on $\mathbb{C}$. We will denote by $L^2(\mu,l^2)$

the space of all measurable l^2-valued functions f on $\mathbb{C}$ such that

$$\|f\|^2 = \int \|f\|_{l^2}^2 \, d\mu < \infty.$$

To simplify the notation, given $f, g \in L^2(\mu, l^2)$, we will often denote the function $\langle f(\cdot), g(\cdot)\rangle_{l^2}$ by $f \cdot \bar{g}$, and similarly we write $|f|$ instead of $\|f(\cdot)\|_{l^2}$. Let z be the function $z(\zeta) = \zeta$, $\zeta \in \mathbb{C}$, and let M_z denote the operator of multiplication by the independent variable on $L^2(\mu, l^2)$, i.e.,

$$M_z(f) = zf.$$

Let N be a normal operator and let $MF(\cdot)$ denote the multiplicity function of N, (cf. [7]). Set

$$\sigma_n = \{\zeta \in \mathbb{C} : MF(\zeta) \geq n\}$$

and let χ_{σ_n} be the characteristic function of σ_n. By standard multiplicity theory, N is unitarily equivalent to M_z on $\oplus_{i=1}^{\infty} L^2(\chi_{\sigma_i}\mu, \mathbb{C})$, where μ is the scalar valued spectral measure of N. The latter space can clearly be viewed as a reducing subspace of $L^2(\mu, l^2)$. (Recall that by reducing we mean reducing with respect to M_z.) Conversely, if $\mathcal{M}$ is a reducing subspace of $L^2(\mu, l^2)$, then $M_z|_{\mathcal{M}}$ is a normal operator.

The next result characterizes all reducing subspaces of $L^2(\mu, l^2)$, and the next again characterizes operators commutating with M_z restricted to such subspaces. Although they are not precisely stated in the below form, these results follows from the theory in [10]. Let $\mathcal{B}(l^2)$ denote the space of operators on l^2 and recall that a function $\phi : \mathbb{C} \to \mathcal{B}(l^2)$ is called SOT-measurable if $\phi(\cdot)(f(\cdot))$ is measurable for every measurable function $f : \mathbb{C} \to l^2$.

Proposition 2.4. *Let $\mathcal{M}$ be a reducing subspace of $L^2(\mu, l^2)$. Then there is an SOT-measurable function $P_{\mathcal{M}}$ on $\mathbb{C}$ whose values are orthogonal projections in l^2 such that*

$$\mathcal{M} = \Big\{ f \in L^2(\mu, l^2) : f(\zeta) \in \mathsf{Ran} P_{\mathcal{M}}(\zeta) \text{ for } \mu\text{-a.e. } \zeta \in \mathbb{C} \Big\}. \tag{2.3}$$

Given a reducing subspace $\mathcal{M}$, we will without comment associate with it $P_{\mathcal{M}}$ as above. Next, we will characterize all operators between subspaces $\mathcal{M}$ and $\mathcal{M}'$ as above, that commute with M_z restricted to the respective subspaces. Given a function $f : \mathbb{C} \to \mathbb{R}^+$ we will let $\mathsf{ess\text{-}sup}_\mu(f)$ denote the essential supremum of f with respect to μ. If ϕ is SOT-measurable and $\mathsf{ess\text{-}sup}(\|\phi(\cdot)\|) < \infty$, then ϕ clearly defines a bounded operator Φ on $L^2(\mu, l^2)$ via $\Phi(f) = \phi(\cdot) f(\cdot)$.

Corollary 2.5. *Let $\mathcal{M}$ and $\mathcal{M}'$ be reducing subspaces of $L^2(\mu, l^2)$ and let*

$$\Phi : \mathcal{M} \to \mathcal{M}'$$

be such that $\Phi M_z|_{\mathcal{M}} = M_z|_{\mathcal{M}'}\Phi$. Then $\forall \zeta \in \mathbb{C}$, $\exists \phi(\zeta) : \mathsf{Ran} P_{\mathcal{M}}(\zeta) \to \mathsf{Ran} P_{\mathcal{M}'}(\zeta)$ such that $\mathsf{ess\text{-}sup}_\mu(\|\phi(\cdot)\|) = \|\Phi\|$ and

$$\Phi(f)(\zeta) = \phi(\zeta) f(\zeta)$$

for every $f \in \mathcal{M}$ and μ-a.e. ζ. Moreover, Φ^* corresponds in the same way with $\phi^*(\cdot)$ and if Φ has either of the properties injective, bounded below, dense range, surjective or isometric, then the same is true for μ-a.e. $\phi(\zeta)$.

In particular, if $\mathcal{M}$ is reducing and MF denotes the multiplicity function of $M_z|_{\mathcal{M}}$, then we easily get that

$$MF(\cdot) = \mathsf{dim}(\mathsf{Ran}P_{\mathcal{M}}(\cdot))$$

μ-a.e.

2.3. Hilbert spaces of analytic functions. Let $n \in \mathbb{N}$ be given, let $\mathcal{B}$ be a Hilbert space whose elements are $\mathbb{C}^n$-valued analytic functions on Ω and assume that $\mathcal{B}$ is invariant under multiplication by polynomials. In particular, M_z is an operator on $\mathcal{B}$ by the obvious definition. Given $\lambda \in \Omega$ let $E_\lambda : \mathcal{B} \to \mathbb{C}^n$ denote the evaluation operator given by

$$E_\lambda(f) = f(\lambda).$$

Assume in addition that $\mathcal{B}$ is such that each E_λ is continuous and surjective, $\sigma(M_z) = \overline{\Omega}$ and $\mathsf{Ker}E_\lambda = \mathsf{Ran}(M_z - \lambda)$ for all $\lambda \in \Omega$. We will then refer to $\mathcal{B}$ as a Hilbert space of $\mathbb{C}^n$-valued analytic functions.

By the work of Cowen and Douglas [9] it follows that if T is an operator on a Hilbert space X such that $\sigma(T) = \overline{\Omega}$, $\cap_{\lambda\in\Omega}\mathsf{Ran}(T-\lambda) = \{0\}$, and $T-\lambda$ is bounded below with Fredholm index $-n$ for all $\lambda \in \Omega$, then there exists a unitary map U from X onto a Hilbert space of $\mathbb{C}^n$-valued analytic functions such that

$$UT = M_z U.$$

See [3] for a basic proof of the above result. The invariant subspaces of $\mathcal{B}$ are characterized in [4], under the additional assumption that $\mathsf{Ran}(M_z - \lambda)$ is dense in $\mathcal{B}$ for all $\lambda \in \partial\Omega$. To present this characterization, we first need some notation. For any $\lambda \in \Omega$ and $t \in \mathbb{N}$ we define $E^t_\lambda : \mathcal{B} \to (\mathbb{C}^n)^t$ by sending $f = \sum_{k=0}^\infty a_k z^k$ into

$$E^t_\lambda(f) = (a_k)^t_{k=0}.$$

Moreover, let $S : (\mathbb{C}^n)^{\{0,\ldots,t\}} \to (\mathbb{C}^n)^{\{0,\ldots,t\}}$ be the "shift-operator", i.e.,

$$S((a_k)^t_{k=0}) = (0, a_0, \ldots, a_{t-1}).$$

Example 2.6. Let $t, K \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_K \in \Omega$ be given. Moreover for each $1 \le k \le K$ let $\mathcal{N}_k$ be an S-invariant subspace of $(\mathbb{C}^n)^{\{0,\ldots,t\}}$. Set

$$\mathcal{M} = \Big\{ f \in \mathcal{H} : E^t_{\lambda_k}(f) \in \mathcal{N}_k,\ \forall k = 1 \ldots K \Big\}. \tag{2.4}$$

Then $\mathcal{M}$ is an invariant subspace with finite codimension.

Theorem 2.7. *Let $\mathcal{B}$ be a Hilbert space of $\mathbb{C}^n$-valued analytic functions. Then the subspace given in Example 2.6 has codimension*

$$\mathsf{codim}\mathcal{M} = \sum_{k=1}^K \mathsf{codim}\mathcal{N}_k.$$

Moreover, if $\mathsf{Ran}(M_z - \lambda)$ *is dense in* $\mathcal{B}$ *for all* $\lambda \in \partial\Omega$*, then any closed invariant subspace with finite codimension has the form* (2.4)*.*

3. Invariant subspaces of $L^2(\mu, l^2)$.

Assumption T. Throughout this section, Ω will be a domain that satisfies Assumption Ω, μ will be a finite measure with $\mathsf{supp}\,\mu \subset \partial\Omega$, and $\mathcal{H}$ will be an M_z-invariant subspace of $L^2(\mu, l^2)$ such that $T = M_z|_{\mathcal{H}}$ satisfies $\sigma(T) = \overline{\Omega}$.

The next two propositions show how to do reduce the study of $\mathcal{H}$ to the case when T is pure and $\mu = m$, as defined in Section 2.1. Let $\mu = \mu_a + \mu_s$ be the Lebesgue decomposition of μ in an absolutely continuous measure μ_a and singular measure μ_s, with respect to m. Moreover let $\mathsf{sing\text{-}supp}(\mu)$ be the singular support of μ, i.e., the (μ-a.e. unique) set such that $\mu_s = \chi_{\mathsf{sing\text{-}supp}(\mu)}\mu$.

Proposition 3.1. *Let* $\lambda_0 \in \Omega$ *be given. Then* $\mathcal{H}_0 = \bigcap_{k\in\mathbb{N}} \mathsf{Ran}(T-\lambda_0)^k$ *is the largest reducing subspace in* $\mathcal{H}$*. Moreover* $\mathcal{H}_1 = \mathcal{H} \ominus \mathcal{H}_0$ *is* T*-invariant,* $T|_{\mathcal{H}_1}$ *is pure and* μ_a *is mutually absolutely continuous with respect to* m*. Finally, for all* $f \in \mathcal{H}_1$ *we have* $f|_{\mathsf{sing\text{-}supp}(\mu)} = 0$ μ*-a.e. and* $f \neq 0$ m*-a.e.*

Proof. Fix $\lambda_0 \in \Omega$ and set

$$\mathcal{H}_0 = \bigcap_{k\in\mathbb{N}} \mathsf{Ran}(T-\lambda_0)^k, \quad T_0 = T|_{\mathcal{H}_0}.$$

Then $T_0 - \lambda_0$ is invertible and as Ω is connected we conclude by Fredholm theory that $\Omega \cap \sigma(T_0) = \emptyset$. It is also not hard to see that $\mathcal{H}_0$ is $(T-\lambda)^{-1}$-invariant for all $\lambda \in \mathbb{C}\setminus\overline{\Omega}$, and hence $\sigma(T_0) \subset \partial\Omega$. By Proposition 1.2 we get that $\sigma(T_0) \subset \mathsf{supp}\,\mu$ and Proposition 1.1 then implies that $\mathcal{H}_0$ is reducing. Thus $\mathcal{H}_1 = \mathcal{H} \ominus \mathcal{H}_0$ is T-invariant, so set $T_1 = T|_{\mathcal{H}_1}$. If $\mathcal{H}_0$ is not the largest reducing subspace then $\mathcal{H}_1$ has a reducing subspace $\mathcal{R}$. But this is impossible because then $(T_1 - \lambda_0)|_{\mathcal{R}}$ would be invertible by Proposition 2.4 which would contradict the obvious fact that

$$\bigcap_{k\in\mathbb{N}} \mathsf{Ran}(T_1-\lambda_0)^k = \{0\}.$$

We will now show that $f|_{\mathsf{sing\text{-}supp}(\mu)} = 0$ μ-a.e. for all $f \in \mathcal{H}_1$. Suppose not and let $f \in \mathcal{H}_1$ be a nonzero element such that there exists a set $E \in \mathsf{sing\text{-}supp}(\mu)$ with $\mu(E) \neq 0$ and $f(\zeta) \neq 0$ for all $\zeta \in E$. As μ is a finite Borel measure on $\mathbb{C}$ it follows from standard measure theory (see Proposition 1.5.6 in [6]) that we may assume that E is closed. Moreover we may clearly assume that E is a subset of one of the closed rectifiable curves that make up $\partial\Omega$. By Theorem 2.3 there exists a function $\phi \in A(\overline{\Omega})$ which peaks at E and moreover $\phi f \in \mathcal{H}_1$. Taking the limit of $\phi^k f$ as $k \to \infty$ we deduce that $\chi_E f \in \mathcal{H}_1$. Now, as E is closed and $m(E) = 0$, we infer that its complement is connected. By Runge's theorem it follows that the function $(z-\lambda_0)^{-k}$,

$k \in \mathbb{N}$, can be approximated uniformly on E by polynomials, so we deduce that

$$(z-\lambda_0)^{-k}\chi_E f \in \mathcal{H}_1,$$

which finally implies the contradiction

$$\chi_E f \in \bigcap_{k\in\mathbb{N}} \mathsf{Ran}(T_1-\lambda_0)^k = \{0\}.$$

Let $f \in \mathcal{H}_1$ be arbitrary and let u be the function such that $\mu_a = um$. It remains to show that u and f are nonzero m-a.e. on $\partial\Omega$. Pick a $\phi \in \mathcal{H}^\perp$ such that $\langle (M_z-\lambda_0)^{-1}f, \phi\rangle \neq 0$ and note that

$$\Big\langle (M_z-\lambda)^{-1}f, \phi\Big\rangle = \int \frac{f\cdot\overline{\phi}(\zeta)}{\zeta-\lambda} d\mu(\zeta) = \int \frac{f\cdot\overline{\phi}(\zeta)}{\zeta-\lambda} u(\zeta)dm(\zeta). \tag{3.1}$$

(Recall that $f\cdot\overline{\phi}$ denotes the function $\zeta \mapsto \langle f(\zeta), \phi(\zeta)\rangle_{l^2}$.) Clearly, the function in (3.1) is 0 for $\lambda \notin \overline{\Omega}$, so by Theorem 2.1 we conclude that

$$\underset{\lambda\in\Omega;\lambda\to\xi}{\mathsf{nt\text{-}lim}} \int \frac{f\cdot\overline{\phi}(\zeta)}{\zeta-\lambda} u(\zeta)dm(\zeta) = 2\pi i (f\cdot\overline{\phi}(\xi))\frac{u(\xi)}{\alpha(\xi)} \tag{3.2}$$

for m-a.e. $\xi \in \partial\Omega$. The desired conclusion now follows by Privalov's theorem. $\square$

Due to the above proposition, we will in the remainder assume that $\mathcal{H}$ has no reducing subspaces, as the reducing subspaces of $L^2(\mu, l^2)$ are completely understood. In other words, we will assume that $T = M_z|_{\mathcal{H}}$ is pure. The next proposition shows that when this is the case we do not loose generality by assuming that $\mu = m$.

Proposition 3.2. *Let T be a pure operator as in Assumption T. Let u be such that $\mu_a = um$. Then $\mathcal{H}$ is unitarily equivalent to a subspace $\tilde{\mathcal{H}} \subset L^2(m, l^2)$, where the unitary operator $U : \mathcal{H} \to \tilde{\mathcal{H}}$ is given by $U(f)(\zeta) = \sqrt{u(\zeta)}f(\zeta)$.*

Proof. By Proposition 3.1 we get

$$\|f\|^2_{\mathcal{H}} = \int |f|^2 d\mu = \int |f|^2 d\mu_a = \int |\sqrt{u}f|^2 dm = \|Uf\|^2_{\tilde{\mathcal{H}}},$$

from which the proposition immediately follows. $\square$

The following result gives different conditions for T to be pure. It also enables us to use the results about Hilbert spaces of analytic functions, Section 2.3.

Proposition 3.3. *Let T be an operator as in Assumption T, let $\lambda_0 \in \Omega$ be arbitrary and let $\mathcal{O} \subset \Omega$ be any open set. Then*

$$\bigcap_{k\in\mathbb{N}} \mathsf{Ran}(T-\lambda_0)^k = \bigcap_{\lambda\in\Omega} \mathsf{Ran}(T-\lambda) = \bigcap_{\lambda\in\mathcal{O}} \mathsf{Ran}(T-\lambda).$$

Proof. Let L be the left inverse of $T-\lambda_0$ that annihilates $\mathcal{K}_{\lambda_0} = \mathcal{H} \ominus \mathsf{Ran}(T-\lambda_0)$. Then $\|L\| \leq \mathsf{dist}(\lambda_0, \partial\Omega)$. For $|\lambda-\lambda_0| < (\mathsf{dist}(\lambda_0,\partial\Omega))$ we define the operator L_λ by

$$L_\lambda = L(I-(\lambda-\lambda_0)L)^{-1}.$$

Clearly $L_\lambda(T-\lambda) = I$ so $P_\lambda = (T-\lambda)L_\lambda$ defines a projection. Given any $f \in \mathcal{H}$ we have $P_\lambda f \in \mathsf{Ran}(T-\lambda)$ and $L(f-P_\lambda f) = 0$ so $f - P_\lambda f \in \mathcal{K}_{\lambda_0}$. Since $P_\lambda(\mathcal{K}_{\lambda_0}) = \{0\}$ we conclude that P_λ is the projection onto $\mathsf{Ran}(T-\lambda)$ parallel with $\mathcal{K}_{\lambda_0}$. Note that

$$(I-P_\lambda)f = \sum_{k=0}^{\infty} \big(L^k f - (T-\lambda_0)L^{k+1}f\big)(\lambda-\lambda_0)^k$$

and let $\epsilon < \mathsf{dist}(\lambda_0,\partial\Omega)$ be arbitrary. The following implications are now easily verified:

$$\begin{aligned}
&f \in \bigcap_{|\lambda-\lambda_0|<\epsilon} \mathsf{Ran}(T-\lambda)\\
&\iff f = P_\lambda f \text{ for all } |\lambda-\lambda_0| < \epsilon\\
&\iff L^k f = (T-\lambda_0)L^{k+1} f \text{ for all } k \in \mathbb{N}\\
&\iff f = (T-\lambda_0)^k L^k f \text{ for all } k \in \mathbb{N}\\
&\iff f \in \underset{k\in\mathbb{N}}{\cap} \mathsf{Ran}(T-\lambda_0)^k.
\end{aligned}$$

Let γ be any point such that $|\gamma-\lambda_0| < \mathsf{dist}(\lambda_0,\partial\Omega)/2$. As λ_0 is arbitrary and $\underset{k\in\mathbb{N}}{\cap} \mathsf{Ran}(T-\lambda_0)^k$ does not depend on ϵ, a short argument shows that

$$\bigcap_{|\lambda-\lambda_0|<\epsilon} \mathsf{Ran}(T-\lambda) = \bigcap_{|\lambda-\gamma|<\delta} \mathsf{Ran}(T-\lambda) \tag{3.3}$$

for all $\delta < \mathsf{dist}(\gamma,\partial\Omega)$. But if $\gamma \in \Omega$ is arbitrary then we can find a finite sequence $\lambda_1,\ldots,\lambda_N$ with $\lambda_N = \gamma$ and

$$|\lambda_{k-1}-\lambda_k| < \mathsf{dist}(\lambda_{k-1},\partial\Omega)/2$$

for all $k = 1,\ldots,N$. By repeated use of (3.3) it then follows that (3.3) actually holds for all $\gamma \in \Omega$ and $\delta < \mathsf{dist}(\gamma,\partial\Omega)$. By this the equalities

$$\underset{k\in\mathbb{N}}{\cap} \mathsf{Ran}(T-\lambda_0)^k = \underset{\lambda\in\Omega}{\cap} \mathsf{Ran}(T-\lambda) = \underset{\lambda\in\mathcal{O}}{\cap} \mathsf{Ran}(T-\lambda).$$

are now easily obtained. □

For future use, we record that as a consequence of the statements concerning P_λ, we have

$$\mathcal{K}_{\lambda_0} + \mathsf{Ran}(T-\lambda) = \mathcal{H}, \text{ whenever } |\lambda-\lambda_0| < \mathsf{dist}(\lambda_0,\partial\Omega). \tag{3.4}$$

We will now investigate the subspaces $\mathsf{Ran}(T-\lambda)$ for $\lambda \in \partial\Omega$. Part of the proof of the next proposition follows by a slight modification of the arguments in [2].

Proposition 3.4. *Let T be a pure operator as in Assumption T and let $\lambda \in \partial\Omega$ be arbitrary. Then*

$$\mathsf{cl}(\mathsf{Ran}(T-\lambda)) = \mathcal{H}.$$

Proof. Take $f_\lambda \in \mathsf{Ran}(M_z - \lambda)^\perp$. By Theorem 2.3 there is a $\phi \in A(\overline{\Omega})$ such that $\phi(\lambda) = 1$ and $|\phi(\zeta)| < 1$ for all $\zeta \in \overline{\Omega} \setminus \{\lambda\}$. With the same argument as in Lemma 2.1 in [2] we deduce that

$$\langle \psi f_\lambda, f_\lambda \rangle = \psi(\lambda) \, \|f_\lambda\|^2$$

for all $\psi \in A(\overline{\Omega})$. But then we have

$$\langle \phi^k f_\lambda, f_\lambda \rangle = (\phi(\lambda))^k \, \|f_\lambda\|^2 = \|f_\lambda\|^2$$

for any $k \in \mathbb{N}$. Taking the limit as $k \to \infty$ we get, by the dominated convergence theorem, that $\|f_\lambda\|^2 = \int_{\{\lambda\}} |f_\lambda|^2 d\mu$, which by Proposition 3.1 equals 0, as desired. $\square$

4. Multicyclicity, index and fiber-dimensions on $\partial\Omega$

Following [1], we call T n-multicyclic if n is the smallest cardinality of a set S such that

$$[S]_{R(\overline{\Omega})} = \mathcal{H}.$$

Note that $[\mathcal{H}]_{L^\infty}$ is the smallest reducing subspace containing $\mathcal{H}$, so $M_z|_{[\mathcal{H}]_{L^\infty}}$ is the minimal unitary extension of T. We will from now on abbreviate the multiplicity function $MF_{M_z|_{[\mathcal{H}]_{L^\infty}}}$ of the operator $M_z|_{[\mathcal{H}]_{L^\infty}}$ with $MF_{\mathcal{H}}$. Intuitively, the number $MF_{\mathcal{H}}(\xi)$ is thus the dimension of the space $\{f(\xi) : f \in \mathcal{H}\}$.

Theorem 4.1. *Let T be as in Assumption T. Let $\lambda_0 \in \Omega$ be arbitrary and let $n \in \mathbb{N}$ be given. The following are equivalent:*

$$MF_{\mathcal{H}}(\cdot) = n \ m\text{-a.e. on } \partial\Omega. \tag{4.1}$$

$$\mathsf{ind}\,(T - \lambda_0) = -n. \tag{4.2}$$

$$T \text{ is } n\text{-multicyclic.} \tag{4.3}$$

Moreover, we have $\mathcal{H} \subset [\mathcal{K}_{\lambda_0}]_{L^\infty}$.

Proof. By Propositions 3.1 and 3.2 it follows that it is not a restriction to assume that $\mu = m$. Assume that there exists a subset $E \in \partial\Omega$ with $m(E) > 0$ such that $MF_{\mathcal{H}}(\cdot) = n$ m-a.e. on E, $n \in \mathbb{N} \cup \{\infty\}$. We will first show that $\dim \mathcal{K}_{\lambda_0} = n$. By the assumption that T is pure, Proposition 3.1 and Proposition 3.3 it follows that $\mathsf{Span}\{\mathcal{K}_\lambda : \lambda \in \Omega\}$ is dense in $\mathcal{H}$. Let $\tilde{n} \le n$ be finite. A short argument shows that we may pick $\psi_1, \dots, \psi_{\tilde{n}} \in \cup_{\lambda \in \Omega} \mathcal{K}_\lambda$ such that

$$\dim \mathsf{Span}\{\psi_j(\xi)\}_{j=1}^{\tilde{n}} = \tilde{n} \tag{4.4}$$

for each ξ in some set $E' \subset E$ of positive measure. Set $\phi_j = \overline{(z - \lambda_j)}\psi_j$, where λ_j is such that $\psi_j \in \mathcal{K}_{\lambda_j}$, put $\Phi = \mathsf{Span}\{\phi_j\}_{j=1}^{\tilde{n}}$ and let P_Φ denote the orthogonal projection in $L^2(m,l^2)$ onto Φ. Since $\Phi \subset \mathcal{H}^\perp$, equation (3.4) implies that

$$P_\Phi(M_z-\lambda)^{-1}(f) \in P_\Phi(M_z-\lambda)^{-1}(\mathcal{K}_{\lambda_0})$$

for all $f \in \mathcal{H}$ and all λ with $|\lambda-\lambda_0| < \mathsf{dist}(\lambda_0,\partial\Omega)$. Thus

$$P_\Phi(M_z-\lambda)^{-1}(\mathcal{H}) = P_\Phi(M_z-\lambda)^{-1}(\mathcal{K}_{\lambda_0}), \quad |\lambda-\lambda_0| < (\mathsf{dist}(\lambda_0,\partial\Omega)). \tag{4.5}$$

Claim 1. $\exists \lambda_1 \in \Omega$ such that $P_\Phi(M_z-\lambda_1)^{-1}(\mathcal{K}_{\lambda_0}) = \Phi$.

If we assume that the claim is false, then for all $|\lambda-\lambda_0| < (\mathsf{dist}(\lambda_0,\partial\Omega))$ we have

$$\mathsf{det}\begin{pmatrix} \langle \frac{\psi_1}{z-\lambda},\phi_1\rangle & \dots & \langle \frac{\psi_{\tilde{n}}}{z-\lambda},\phi_1\rangle \\ \vdots & \ddots & \vdots \\ \langle \frac{\psi_1}{z-\lambda},\phi_{\tilde{n}}\rangle & \dots & \langle \frac{\psi_{\tilde{n}}}{z-\lambda},\phi_{\tilde{n}}\rangle \end{pmatrix} = 0,$$

by (4.5). The left hand side is an analytic function on Ω, which thus vanishes identically. By Theorem 2.1 we then get

$$0 = \left(\frac{2\pi i}{\alpha(\xi)}\right)^{\tilde{n}} \mathsf{det}\begin{pmatrix} \psi_1(\xi)\cdot\overline{\phi_1(\xi)} & \dots & \psi_{\tilde{n}}(\xi)\cdot\overline{\phi_1(\xi)} \\ \vdots & \ddots & \vdots \\ \psi_1(\xi)\cdot\overline{\phi_{\tilde{n}}(\xi)} & \dots & \psi_{\tilde{n}}(\xi)\cdot\overline{\phi_{\tilde{n}}(\xi)} \end{pmatrix} =$$

$$= \left(\frac{2\pi i}{\alpha(\xi)}\right)^{\tilde{n}} \left(\prod_{i=1}^{\tilde{n}}(\xi-\lambda_i)\right) \mathsf{det}\Big(\big(\psi_1(\xi),\dots,\psi_{\tilde{n}}(\xi)\big)^*\big(\psi_1(\xi),\dots,\psi_{\tilde{n}}(\xi)\big)\Big)$$

for a.e. $\xi \in E'$, (where $\big(\psi_1(\xi),\dots,\psi_{\tilde{n}}(\xi)\big)$ is treated as an $\infty\times\tilde{n}$-matrix and * means taking the transpose and conjugate). This clearly contradicts (4.4), and hence the claim is proven. In particular, we get $\mathsf{dim}\,\mathcal{K}_{\lambda_0} \geq n$.

Claim 2. $\mathsf{dim}\,\mathcal{K}_{\lambda_0} \leq n$.

If $n=\infty$ there is nothing to prove. Assume that $n<\infty$ and $\mathsf{dim}\,\mathcal{K}_{\lambda_0} > n$. Let $\{k_j\}_{j=1}^{n+1}$ be orthonormal vectors in $\mathcal{K}_{\lambda_0}$. Note that $\overline{(z-\lambda_0)}k_i \in \mathcal{H}^\perp$ and let $K(\lambda)$ be the matrix-valued analytic function given by

$$K(\lambda) = \mathsf{det}\begin{pmatrix} \langle \frac{k_1}{z-\lambda},\overline{(z-\lambda_0)}k_1\rangle & \dots & \langle \frac{k_{n+1}}{z-\lambda},\overline{(z-\lambda_0)}k_1\rangle \\ \vdots & \ddots & \vdots \\ \langle \frac{k_1}{z-\lambda},\overline{(z-\lambda_0)}k_{n+1}\rangle & \dots & \langle \frac{k_{n+1}}{z-\lambda},\overline{(z-\lambda_0)}k_{n+1}\rangle \end{pmatrix}.$$

Clearly $K(\lambda_0)=1$ and as above we get that

$$\begin{aligned} &\underset{\lambda\to\xi}{\mathsf{nt\text{-}lim}}\ K(\lambda) \\ &\quad = \Big(\frac{2\pi i(\xi-\lambda_0)}{\alpha(\xi)}\Big)^{n+1} \mathsf{det}\Big(\big(k_1(\xi),\dots,k_{n+1}(\xi)\big)^*\big(k_1(\xi),\dots,k_{n+1}(\xi)\big)\Big). \end{aligned}$$

However, since $\mathsf{dim}\,\mathsf{Ran}P_{[\mathcal{H}]_{L^\infty}}(\xi) = n$ on E and $k_i(\xi) \in \mathsf{Ran}P_{[\mathcal{H}]_{L^\infty}}(\xi)$ for a.e. $\xi \in \partial\Omega$ and all $1 \leq i \leq n+1$, it is easily seen that the right-hand side is zero a.e. on E. By Privalov's theorem we obtain a contradiction and hence the claim is proven.

Thus we have shown that (4.1) and (4.2) are equivalent. For the remainder we let n be as in (4.1) and (4.2). To see that $\mathcal{H} \subset [\mathcal{K}_{\lambda_0}]_{L^\infty}$, note that the same argument as in the proof of Claim 2, but applied to $k_1, \ldots, k_n$ only, shows that $\mathsf{dim}(\mathsf{Span}\{k_1(\xi), \ldots, k_n(\xi)\}) = n$ m-a.e. on $\partial\Omega$. This implies that $P_{[\mathcal{H}]_{L^\infty}} = P_{[\mathcal{K}_{\lambda_0}]_{L^\infty}}$ a.e. on $\partial\Omega$, and hence $[\mathcal{H}]_{L^\infty} = [\mathcal{K}_{\lambda_0}]_{L^\infty}$.

Finally, let T be $\tilde{n}$-multicyclic for some $\tilde{n} \in \mathbb{N} \cup \{\infty\}$. Since clearly $MF_{\mathcal{H}}(\cdot) \le \tilde{n}$, we have $n \le \tilde{n}$. So it remains to exhibit n generating vectors for $\mathcal{H}$. We first note that $[\mathcal{K}_{\lambda_0}]_{R(\overline{\Omega})}$ has finite codimension in $\mathcal{H}$. To see this, set $\mathcal{A} = \mathcal{H} \ominus [\mathcal{K}_{\lambda_0}]_{R(\overline{\Omega})}$, let $k_1, \ldots, k_n$ be a basis for $\mathcal{K}_{\lambda_0}$ and consider the map $T : \mathcal{A} \to L^1(m, \mathbb{C}^n)$ given by

$$T(\phi) = \begin{pmatrix} \phi \cdot \overline{k_1} \\ \vdots \\ \phi \cdot \overline{k_n} \end{pmatrix}.$$

Since $\mathcal{A} \subset \mathsf{Ran}(T - \lambda_0)$ we have $r\phi \in \mathsf{Ran}(T - \lambda_0)$ for all $r \in R(\overline{\Omega})$ and $\phi \in \mathcal{A}$, i.e.,

$$\langle r\phi, k_i \rangle = \int_{\partial\Omega} r\phi \cdot \overline{k_i}\, dm = 0.$$

This combined with the obvious fact that $\langle \phi, rk_i \rangle = 0$ for all $r \in R(\overline{\Omega})$ shows that the measure $k_i \cdot \overline{\phi}\, dm$ annihilates $\mathsf{Re}R(\overline{\Omega})$ - the set of real parts of $R(\overline{\Omega})$. It is well known that the set of such measures is finite-dimensional, (see, e.g., Theorem 4.6, ch. VI in [13]). Thus $\mathsf{Ran}T$ is a finite-dimensional space. Moreover, by what we have already shown, $\phi(\xi) \in \mathsf{Span}\{k_i(\xi)\}_{i=1}^n$ for a.e. $\xi \in \partial\Omega$, and thus T is an injective map. It follows that $\mathcal{A}$ has finite dimension, as desired. To finish the proof, we prove the following claim.

Claim 3. Given $f_1, \ldots, f_n \in \mathcal{H}$ with $\mathsf{dim}\left(\mathcal{H} \ominus [f_1, \ldots, f_n]_{R(\overline{\Omega})}\right) = N$ for some $N \in \mathbb{N}$, there are $g_1, \ldots, g_n \in \mathcal{H}$ with $\mathsf{dim}\left(\mathcal{H} \ominus [g_1, \ldots, g_n]_{R(\overline{\Omega})}\right) < N$.

By section 2.3, $\mathcal{H}$ is unitarily equivalent to a Hilbert space $\mathcal{B}$ of $\mathbb{C}^n$-valued analytic functions on Ω, such that T corresponds to M_z on $\mathcal{B}$. By Proposition 3.4 the invariant subspaces of finite codimension of $\mathcal{B}$ are described by Theorem 2.7. Denote by $\tilde{f}_1, \ldots, \tilde{f}_n$ the elements of $\mathcal{B}$ that correspond to $f_1, \ldots, f_n$, and set

$$\mathcal{M} = [\tilde{f}_1, \ldots, \tilde{f}_n]_{R(\overline{\Omega})}.$$

Let $t, K \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_K \in \Omega$ and $\mathcal{N}_k \subset (\mathbb{C}^n)^{\{0,\ldots,t\}}$ describe $\mathcal{M}$ via (2.4). Moreover, define $\mathcal{N}_k^0 \subset \mathbb{C}^n$ via

$$\mathcal{N}_k^0 = \mathsf{Ran}E_{\lambda_k}^0 = \{\tilde{f}(0) : \tilde{f} \in \mathcal{N}_k\}.$$

If $\mathcal{N}_k^0 = \mathbb{C}^n$ for some k, then by the shift invariance of $\mathcal{N}_k$ it is easy to see that $\mathcal{N}_k = (\mathbb{C}^n)^{\{0,\ldots,t\}}$, and hence we may assume that $\mathcal{N}_k^0 \neq \mathbb{C}^n$ for some k, because otherwise $\mathcal{M} = \mathcal{B}$ and we are done. The map

$$\mathsf{Span}\{\tilde{f}_1, \ldots, \tilde{f}_n\} \ni \tilde{f} \mapsto E^0_{\lambda_k}(\tilde{f}) \in \mathbb{C}^n$$

is then a linear map that is not surjective, and thus we can take new a basis $\{\tilde{h}_1, \ldots, \tilde{h}_n\}$ for $\mathsf{Span}\{\tilde{f}_1, \ldots, \tilde{f}_n\}$ such that $E^0_{\lambda_k}(\tilde{h}_1) = 0$. Note that

$$\mathcal{M} = [\tilde{f}_1, \ldots, \tilde{f}_n]_{R(\overline{\Omega})} = [\tilde{h}_1, \ldots, \tilde{h}_n]_{R(\overline{\Omega})}.$$

Since $\mathsf{Ker} E^0_{\lambda_k} = \mathsf{Ran}(M_z - \lambda_k)$ by Section 2.3, we can pick $g_1 \in \mathcal{H}$ such that

$$\tilde{g}_1 = \frac{\tilde{h}_1}{(z - \lambda_k)}.$$

Set $\tilde{g}_i = \tilde{h}_i$ for $1 < i \leq n$. Clearly

$$\mathcal{M} \subset [\tilde{g}_1, \ldots, \tilde{g}_n]_{R(\overline{\Omega})} \subset \mathcal{B}$$

and we are done if we show that the first inclusion is strict. If not, we have

$$\tilde{h}_1 \in (z - \lambda_k)\mathcal{M}, \tag{4.6}$$

which is impossible. One way to see this is to note that

$$\mathsf{dim}\,(\mathcal{M} \ominus (z - \lambda_k)\mathcal{M}) = n,$$

which is not hard to see using the material in Section 2.3. But if (4.6) holds then

$$\mathcal{M} = (z - \lambda_k)\mathcal{M} + \mathsf{Span}\{h_i\}_{i=2}^n. \qquad \square$$

As noted in the introduction, the above theorem has been obtained earlier in [1], but with completely different methods. We note that if one knows $\mathcal{K}_\lambda$ explicitly, then it is possible with the above proof to actually calculate the generating vectors $f_1, \ldots, f_n$, which might be one advantage of this approach. In the sequel, [5], we will go further in the case when Ω has only one hole, and prove that one can pick $f_1, \ldots, f_n$ such that

$$\mathcal{H} = [f_1]_{R(\overline{\Omega})} \oplus \ldots \oplus [f_n]_{R(\overline{\Omega})}.$$

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