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THE QUASIASYMPTOTIC EXPANSION AND THE MOMENT EXPANSION OF TEMPERED DISTRIBUTIONS

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Abstract. We prove that an $f \in A'$, where A is one of spaces $\mathcal{E}, \mathcal{P}, \mathcal{O}_c, \mathcal{O}_m$, or $\mathcal{K}$, has the quasiasymptotic expansions of the first and second kind and that they are equal to the moment expansion of f. Also, Abelian-type results for the Stieltjes and the Laplace transforms of tempered distributions are given.

1. Introduction. We investigate the connection between the quasiasymptotic expansions of tempered distributions at infinity introduced by Zavialov [**9**] (see also [**7**]) and slightly modified and examined by Pilipović [**5**], and the moment expansions, introduced by Estrada and Kanwal [**1**], [**2**], of distributions that "decay very fast at infinity". These spaces A' are duals of the spaces $\mathcal{E}, \mathcal{P}, \mathcal{O}_c, \mathcal{O}_m$, or $\mathcal{K}$ which are denoted in [**1**] by A.

We shall show that every generalized function, supported by $[0, \infty)$, from A', has quasiasymptotic expansions of the first and the second kind at infinity and that those are equal.

For various generalized integral transforms, Estrada and Kanwal have derived tha moment expansions of corresponding kernels and have given the asymptotic expansions of integral transforms of elements of A. Contrary to this method, we have used the quasiasymptotic expansions of tempered distributions and for appropriate integral transforms, (Laplace transform [**6**], Stieltjes transform [**3**]) we have given the asymptotic expansions for their integral transforms.

We shall give the Abelian-type results for the Laplace and Stieltjes transforms of tempered distributions, wich have quasiasymptotic expansions of the first kind, and apply them on distributions with the moment expansions. Also, as an example, we shall use the moment expansion from [**1**], of $H(x)e^{i|x|^\rho}$, $\rho > 0$, where H is the Heaviside function, and give the behaviour of its Stieltjes and Laplace transforms.

2. Notions and known results. Following [**5**], we shall give the definitions of the quasiasymptotic expansions of the first and the second kind at infinity

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of tempered distributions. Denote by $\mathcal{S}$ the space of rapidly decreasing smooth functions, defined on the real line $\mathbf{R}$, supplied with the usual topology. Its dual is the space of tempered distributions $\mathcal{S}'$ and $\mathcal{S}'_+$ is its subspace with elements supported by $[0,\infty)$. As in [**8**], we denote by $\mathcal{S}_p$ the completion of $\mathcal{S}$ with respect to the norm

$$\|\varphi\|_p = \sup\left\{(1+|x|^2)^{p/2}|\varphi^{(\alpha)}(x)|;\ x\in\mathbf{R},\ \alpha\le p\right\}.$$

Recall that, $\mathcal{S}=\bigcap_{p\in N}\mathcal{S}_p$, $\mathcal{S}'=\bigcup_{p\in N}\mathcal{S}'_p$ have the topological meaning and that a sequence f_n from $\mathcal{S}'$ converges to $f\in\mathcal{S}'$ iff it belongs to some $\mathcal{S}'_{p_0}$, which, in the dual norm of $\mathcal{S}'_{p_0}$, converges to $f\in\mathcal{S}'_{p_0}$.

Denote by $c_m(\lambda),\ m\in\mathbf{N}$, a sequence of continuous positive functions defined on $(a_m,\infty),\ a_m>0$, such that $c_m(\lambda)=o(c_{m+1}(\lambda)),\ \lambda\to\infty,\ (m\in\mathbf{N})$ and by $u_m, m\in\mathbf{N}$, a sequence of $\mathcal{S}'_+$ such that $u_m\neq 0,\ m=1,\dots,p,\ p<\infty$, $u_m=0,\ m>p$ or $u_m\neq 0,\ m\in\mathbf{N}$. Denote by Λ the set of pairs of sequences $(c_m(\lambda),u_m)$.

Let $(c_m(\lambda),u_m)\in\Lambda$ and

$$\lim_{\lambda\to\infty}\left\langle\frac{u_m(\lambda x)}{c_m(\lambda)},\ \varphi(x)\right\rangle=\left\langle g_m(x),\varphi(x)\right\rangle,\varphi\in\mathcal{S},\ m\in\mathbf{N},\tag{1}$$

where $g_m(x)\neq 0$ if $u_m\neq 0,\ m\in\mathbf{N}$. In this case we can write $u_m\overset{q}{\sim}g_m$ at ∞, with respect to $c_m(\lambda)$.

It has been proven [**7**] that in this case

$$g_m=Cf_{\alpha_m+1},\ C\neq 0,\ \text{and}\ c_m(\lambda)=\lambda^{\alpha_m}L_m(\lambda),\ \lambda>\lambda_{0,m},\ \text{where,}$$

$$f_{p+1}(x)=\begin{cases}H(x)x^p/\Gamma(p+1), & p>-1;\\ D^kf_{p+k+1}(x), & p\le -1,\ p+k>-1,\ k\in\mathbf{N}\end{cases},\quad x\in\mathbf{R}$$

where D is the derivate in the sence of distributions, and L_m is the Karamata's slowly varying function, i.e. a positive measurable function such that $\lim_{\lambda\to\infty}L_m(\lambda x)/L_m(\lambda)=1$, uniformly for $x\in[a,b]\subset(0,\infty),\ m\in\mathbf{N}$.

Denote by Λ_1 a subset of Λ such that $(c_m(\lambda),u_m)\in\Lambda_1$ if (1) holds for every m for which $u_m\neq 0$ and $g_m\neq 0,\ m=1,\dots,p<\infty$ or $m\in\mathbf{N}$. Let $f\in\mathcal{S}'$ and $(c_m(\lambda),u_m)\in\Lambda_1$, such that

$$\lim_{\lambda\to\infty}\left\langle\left(f(\cdot)-\sum_{i=1}^{m}u_i(\cdot)\right)(\lambda x)/c_m(\lambda),\ \varphi(x)\right\rangle=0,\ \varphi\in\mathcal{S},$$

for $m=1,\dots,p<\infty$ or $m\in\mathbf{N}$. Then it is said that f has the quasiasymptotic expansion at infinity of the first kind with respect to $(c_m(\lambda),u_m)$, so that we can write

$$f(x)\overset{q\cdot e}{\approx}\sum_{i=1}^{p(\infty)}u_i(x)\qquad(c_i(\lambda))\ \text{at}\ \infty.$$

Let $f \in \mathcal{S}'_+$, $(c_m(\lambda), u_m) \in \Lambda_1$ with $u_m \in \mathcal{S}'_+$ $(m = 1, \dots, p < \infty$ or $m \in \mathbf{N})$ and

$$\lim_{\lambda\to\infty} \left\langle \left(f(\lambda x) - \sum_{i=1}^{m} u_i(x)c_i(\lambda)/c_m(\lambda),\ \varphi(x) \right) \right\rangle = 0,\ \varphi \in \mathcal{S},$$

for $m = 1, \dots, p < \infty$ or $m \in \mathbf{N}$. Then it is said that f has the quasiasymptotic expansion at infinity of the second kind with respect to $(c_m(\lambda), u_m)$, so that we can write

$$f(\lambda x) \overset{q\text{-}e}{\approx} \sum_{i=1}^{p(\infty)} u_i(x)c_i(\lambda) \qquad (c_i(\lambda)) \text{ at } \infty.$$

As we have noted, Estrada and Kanwal introduced the moment asymptotic expansion of generalized functions. They have considered several spaces of distributions that decay very fast at infinity, which are subspaces of the space of tempered distributions. The testing function space is denoted in [**1**] by $A(\mathbf{R}^n)$.

THEOREM A. [**1**] *Let $A(\mathbf{R}^n)$ be any of the spaces $\mathcal{E}, \mathcal{P}, \mathcal{O}_c, \mathcal{O}_m$, or $\mathcal{K}$. If $f \in A'(\mathbf{R}^n)$, then*

$$f(\lambda x) \approx \sum_{|\alpha|=0}^{\infty} \frac{(-1)^{|\alpha|}\mu_\alpha D^\alpha \delta(x)}{\alpha!\lambda^{|\alpha|+n}},\ \text{as } \lambda \to \infty,$$

where $\mu_\alpha = \langle f(x), x^\alpha \rangle$, $\alpha \in \mathbf{N}_0^n$, are the moments of the generalized function f, in the sense that for every $\phi \in A(\mathbf{R}^n)$,

$$\langle f(\lambda x), \phi(x) \rangle = \sum_{|\alpha|=0}^{N} \frac{\mu_\alpha D^\alpha \phi(0)}{\alpha!\lambda^{|\alpha|+n}} + O\left\{ \frac{1}{\lambda^{N+n+1}} \right\},\ \text{as } \lambda \to \infty.$$

It has been noted that the moment asymptotic expansion does not generally hold for elements of the spaces such as $\mathcal{D}'$ or $\mathcal{S}'$.

3. Relations between asymptotic expansions. The quasiasymptotic expansion at infinity of the first kind is more natural than the quasiasymptotic expansion at infinity of the second kind. The following example can be used to show that.

Example 1. Assume $f(x) = x^4 ln^2|x| + x^5$, $c_1(\lambda) = \lambda^4 ln^2\lambda$, $c_2(\lambda) = \lambda^5$, $\lambda > 1$. Then,

$$f(x) \overset{q\text{-}e}{\approx} x^4 ln^2|x| + x^5, \text{ and}$$

$$f(\lambda x) \overset{q\text{-}e}{\approx} (x^4 ln^2|x|)\lambda^4 ln^2\lambda + (x^4 ln^2|x|\lambda^4 ln\lambda + (x^4 ln^2|x|)\lambda^4 + \lambda^5 x^5.$$

PROPOSITION 1. *Let $A(\mathbf{R})$ be any of the spaces $\mathcal{E}, \mathcal{P}, \mathcal{O}_c, \mathcal{O}_m$, or $\mathcal{K}$ ([**1**]). If $f \in A'(\mathbf{R})$, then f has the quasiasymptotic expansion of the first and second kind at infinity and they are equal.*

Proof. If $f \in A'(\mathbf{R})$, then, according to Theorem A, we obtain the following moment asymptotic expansion

$$f(\lambda x) \approx \sum_{i=0}^{\infty} \frac{(-1)^i \mu_i \delta^{(i)}(x)}{i!\lambda^{i+1}} \quad \text{as } \lambda \to \infty,$$

where μ_i are moments of the generalized function f, $\mu_i = \langle f(x), x^i \rangle$, $i \in \mathbf{N}_0$, in the sense that if $\phi \in A(\mathbf{R})$, then

$$\langle f(\lambda x), \phi(x) \rangle = \sum_{i=0}^{m} \frac{\mu_i \phi^{(i)}(0)}{i! \lambda^{i+1}} + O\left(\frac{1}{\lambda^{m+2}}\right) \text{ as } \lambda \to \infty.$$

Since $\mathcal{S} \subset A(\mathbf{R})$ and

$$\lim_{\lambda \to \infty} \lambda^{i+1} \langle \delta^{(i)}(\lambda x),\ \phi(x) \rangle = (-1)^i \phi^{(i)}(0) = \langle \delta^{(i)}(x),\ \phi(x) \rangle,\ \phi \in \mathcal{S},$$

we obtain

$$\lim_{\lambda \to \infty} \lambda^{m+1} \langle f(\lambda x) - \sum_{i=0}^{m} \frac{(-1)^i \mu_i \delta^{(i)}(\lambda x)}{i!},\ \phi(x) \rangle =$$

$$\lim_{\lambda \to \infty} \lambda^{m+1} \left[\sum_{i=0}^{m} \frac{\mu_i \phi^{(i)}(0)}{i! \lambda^{i+1}} + O\left(\frac{1}{\lambda^{m+2}}\right) - \sum_{i=0}^{m} \frac{(-1)^i \lambda^{i+1} \mu_i}{i! \lambda^{i+1}} \langle \delta^{(i)}(\lambda x),\ \phi(x) \rangle \right] =$$

$$\lim_{\lambda \to \infty} O\left(\frac{1}{\lambda}\right) = 0,\ \phi \in \mathcal{S}.$$

With $u_i(x) = (-1)^i \mu_i \delta^{(i)}(x)/i!$ and $c_i(\lambda) = \lambda^{-i-1}$, $i \in \mathbf{N}_0$; the statement has been proven.

4. Application on Stieltjes and the Laplace transforms. Assume $f \in \mathcal{S}'_+$. By following Lavoine and Misra we say that $f \in J'(r)$, if there exists an $m \in \mathbf{N}_0$ and a locally integrable function F, supp $F \subset [0, \infty)$, such that

$$\text{a) } f = F^{(m)}, \quad \text{b) } \int_{\mathbf{R}} \frac{|F(x)|}{(x+\beta)^{r+m+1}} dx < \infty \quad \text{for } \beta > 0.$$

The Stieltjes transform S_r of index r, $r \in \mathbf{R} \setminus (-\mathbf{N}_0)$ of a distribution $f \in J'(r)$, with properties a) and b) is a complex-valued function given by

$$\begin{aligned}(S_r f)(z) &= (r+1)_m \int_0^\infty F(x)(x+z)^{-r-m-1} dx \\ &= (r+1)_m \langle F(x), (x+z)^{-r-m-1} \rangle \quad z \in \mathbf{C} \setminus (-\infty, 0],\end{aligned}$$

where $(r)_k = r(r+1)\dots(r+k-1)$, $k > 0$ and $(r)_0 = 1$.

It is easy to show that $(S_r f)(z)$ is a holomorphic function of a complex variable $z \in \mathbf{C} \setminus (-\infty, 0]$.

Let us recall that

$$(S_r f_{\alpha+1})(s) = \frac{s^{\alpha-r}\Gamma(r-\alpha)}{\Gamma(r+1)},\ s > 0, r > \alpha. \tag{2}$$

The Laplace transform of $f \in \mathcal{S}'_+$ is defined by $\mathcal{L}(f)(z) = \langle f(x), e^{zx} \rangle$, $z = u + iv \in \mathbf{R}_+ + i\mathbf{R} = T_+$, where $\mathcal{L}(f)$ is an analytic function.

Here,

$$\mathcal{L}(f_{\alpha+1})(z) = (-iz)^{-\alpha-1}, \ \ z \in T_+, \ \ \alpha \in \mathbf{R}, (i = \sqrt{-1}). \tag{3}$$

From previous expresions and Proposition 1, we derive the next proposition.

PROPOSITION 2. *Let L_k, $k \in \mathbf{N}$, be a sequence of slowly varying functions at infinity; α_k, $k \in \mathbf{N}$, be a strictly decreasing sequence of real numbers; A_k, $k \in \mathbf{N}$, a sequence of real numbers $\neq 0$; and $r \in \mathbf{R} \setminus (-\mathbf{N})$, $r > \alpha_1$. Assume that $f \in \mathcal{S}'_+$ has the quasiasymptotic expansion at infinity of the first kind with respect to $(\lambda^{\alpha_m} L_m(\lambda), A_m f_{\alpha_m+1})$, $m \in \mathbf{N}$. Then, in the sense of ordinary asymptotic expansions, we have*

$$(S_r f)(\lambda) \approx \sum_{k=1}^{\infty} A_k \frac{\lambda^{\alpha_k - r}}{\Gamma(r+1)} \Gamma(r - \alpha_k), \ \ \lambda \to \infty, \tag{4}$$

with respect to $\lambda^{\alpha_m} L_m(\lambda)$, $m \in \mathbf{N}$. Also, we have

$$\mathcal{L}(f)(\varepsilon) \approx \sum_{k=1}^{\infty} A_k (-i\varepsilon)^{-\alpha_k - 1}, \ \ \varepsilon \to 0^+, \tag{5}$$

with respect to $\varepsilon^{-(\alpha_m+1)} L_m(1/\varepsilon)$, $m \in \mathbf{N}$. Particularly, if $f \in A'(\mathbf{R})$ and $r > 0$, then,

$$(S_r f)(\lambda) \approx \sum_{k=0}^{\infty} \frac{(-1)^k \mu_k \Gamma(r+k)}{k! \Gamma(r+1) \lambda^{k+r+1}}, \ \ \lambda \to \infty,$$

with respect to λ^{-m-1}, $m \in \mathbf{N}_0$, and

$$\mathcal{L}(f)(\varepsilon) \approx \sum_{k=0}^{\infty} \frac{i^k \mu_k \varepsilon^k}{k!}, \ \ \varepsilon \to 0^+,$$

with respect to ε^m, $m \in \mathbf{N}_0$.

Proof. Since for every $m \in \mathbf{N}$

$$\frac{\left(f - \sum_{k=1}^{m} A_k f_{\alpha_k+1}\right)(\lambda x)}{\lambda^{\alpha_m} L_m(\lambda)} \to 0, \ \lambda \to \infty, \text{ in } \mathcal{S}',$$

using the well-known argumets for the quasiasymptotic behaviour [**7**, the structural theorem], it follows that there exists a p_0 such that, for every $p \geq p_0$, there is a continuous function F_p supported by $[0, \infty)$ such that $f = F_p^{(p)}$ and

$$\frac{F_p(\lambda x) - \sum_{k=1}^{m} A_k f_{\alpha_k+p+1}(\lambda x)}{\lambda^{\alpha_m+p} L_m(\lambda)} \to 0, \ \lambda \to \infty.$$

First, assume that $r - 1 \leq \alpha_m < r$. Then, as in [**4**, Ch. 4], with $\eta \in \mathbf{C}^\infty$, $\eta = 1$,

$x > -\varepsilon$, $\eta = 0$, $x < -1 - \varepsilon$, we have

$$\begin{aligned}
&\frac{\left(S_{r+1}f - \sum_{k=1}^{m} A_k f_{\alpha_k+1}\right)(\lambda s)}{\lambda^{\alpha_m - r - 1} L_m(\lambda)} \\
&= \frac{(r+2)_p}{\lambda^{\alpha_m - r - 1}} \Big\langle (F_p - \sum_{k=1}^{m} A_k f_{\alpha_k+p+1})(x), \frac{\eta(x)}{(\lambda s + x)^{r+p+2}} \Big\rangle \\
&= (r+2)_p \Big\langle \frac{F_p(\lambda x) - \sum_{k=1}^{m} A_k f_{\alpha_k+p+1}(\lambda x)}{\lambda^{\alpha_m+p} L_m(\lambda)}, \frac{\eta(x)}{(s+x)^{r+p+2}} \Big\rangle,
\end{aligned}$$

where $\langle .\,,\, . \rangle$ is the dual pairing of $\mathcal{S}'_{r+p+1}$ and $\mathcal{S}_{r+p+1}$, which, because of $\alpha_m \geq r-1$, implies that $\eta(x)/(s+x)^{r+p+2} \in \mathcal{S}_{r+p+1}$ and, hence, for $s > 0$

$$\frac{F_p(\lambda s) - \sum_{k=1}^{m} A_k f_{\alpha_k+p+1}(\lambda s)}{\lambda^{\alpha_m+p} L_m(\lambda)}$$

converges to 0 in $\mathcal{S}'_{r+p+1}$ when $\lambda \to \infty$. This also implies that for every $s > 0$,

$$\frac{\left(S_{r+1}f - \sum_{k=1}^{m} A_k f_{\alpha_k+1}\right)(\lambda s)}{\lambda^{\alpha_m - r - 1} L_m(\lambda)} \to 0,\ \lambda \to \infty.$$

Since

$$\begin{aligned}
\left(S_r f - \sum_{k=1}^{m} A_k f_{\alpha_k+1}\right)(\lambda) &= (r+1) \int_{\lambda}^{\infty} \left(S_{r+1}f - \sum_{k=1}^{m} A_k f_{\alpha_k+1}\right)(u)du \\
&= \lambda(r+1) \int_{1}^{\infty} \left(S_{r+1}f - \sum_{k=1}^{m} A_k f_{\alpha_k+1}\right)(\lambda s)ds,
\end{aligned}$$

by Lebesgue's theorem we have assertion (4) for $\alpha_m \geq r-1$. If $\alpha_m < r-1$, then we directly consider the S_r-transform of $f - \sum_{k=1}^{m} A_k f_{\alpha_k+1}$ and note that $\eta(x)/(s+x)^{r+p+1} \in \mathcal{S}_{r+p}$, and

$$\frac{F_p(\lambda s) - \sum_{k=1}^{m} A_k f_{\alpha_k+p+1}(\lambda s)}{\lambda^{\alpha_m+p} L_m(\lambda)},\ s > 0,$$

converge to 0 in $\mathcal{S}'_{r+p}$ as $\lambda \to \infty$. The proof can be completed by using (2).

Similar arguments as in (4) imply in (5). One has to use (3) and the fact that $\delta^{(k)} = f_{-k},\ k \in \mathbf{N}_0$.

As we have mentioned in the introduction, Estrada and Kanwal in [**1**], [**2**] give the asymptotic expansions for $S_r\phi$ and $\mathcal{L}(\phi),\ \phi \in A(\mathbf{R})$.

From [**1**] we recall the following moment expansion:

Example 2. If $\rho > 0$, then the function $f(x) = H(\pm x)\exp(\pm i|x|^{\rho})$, as well as its combinations, belong to $\mathcal{O}'_c$. Using values

$$\int_0^{\infty} x^{\alpha}\exp(ix^{\rho})dx = \frac{1}{\rho}\Gamma\left(\frac{\alpha+1}{\rho}\right)\exp\left(\frac{\pi i(\alpha+1)}{2\rho}\right),\ \alpha \neq -\rho, -2\rho, -3\rho, \dots\ ,$$

one obtains

$$H(x)\exp(\pm i(\lambda x)^{\rho}) \approx \frac{1}{\rho}\sum_{n=0}^{\infty} \frac{(-1)^n \Gamma\left(\frac{n+1}{\rho}\right) \exp\left(\frac{\pm\pi i(n+1)}{2\rho}\right) \delta^{(n)}(x)}{n!\lambda^{n+1}}, \quad \lambda \to \infty,$$

and, consequently, by Proposition 2, for $r > -1$, $z \in \mathbf{C} \setminus (-\infty, 0]$, one obtains

$$(S_r f)(\lambda z) \approx \frac{1}{\rho}\sum_{m=0}^{\infty} \frac{(-1)^n \Gamma\left(\frac{m+1}{\rho}\right) \exp\left(\frac{\pm\pi i(m+1)}{2\rho}\right) \Gamma(r+n+1)}{(m)!(\lambda z)^{m+r+1}\Gamma(r+1)}, \quad \lambda \to \infty,$$

$$\mathcal{L}(f)(\varepsilon) \approx \frac{1}{\rho}\sum_{m=0}^{\infty} \frac{\Gamma\left(\frac{m+1}{\rho}\right) \exp\left(\frac{\pm\pi i(m+1)}{2\rho}\right) \varepsilon^m}{(m)!}, \quad \varepsilon \to 0.$$

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