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A NOTE ON DIFFERENCES OF POWER MEANS

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Abstract. We give some new inequalities concerning the differences of power means.

1. Introduction

Let $\tilde{x}_n = \{x_i\}_1^n$, $\tilde{p}_n = \{p_i\}_1^n$ denote two sequences of positive real numbers with $\sum_1^n p_i = 1$. From the Theory of Convex Means (cf. [1], [2], [3]), it is well known that for $t > 1$,

$$\sum_1^n p_i x_i^t \geqslant \left(\sum_1^n p_i x_i\right)^t, \tag{1}$$

and *vice versa* for $0 < t < 1$, The equality sign in (1) occurs if and only if all members of $\tilde{x}_n$ are equal (cf. [1]).

In this article we shall consider the difference

$$d_t = d_t^{(n)} = d_t^{(n)}(\tilde{x}_n, \tilde{p}_n) := \sum_1^n p_i x_i^t - \left(\sum_1^n p_i x_i\right)^t, \quad t > 1,$$

and thus generated sequence $d = \{d_m\}_{m \geqslant 2}$ of non-negative real numbers.

By the above, if all members of the sequence $\tilde{x}_n$ are equal, then all members of d are zero; hence this trivial case will be excluded in the sequel.

An interesting fact is that there exists an explicit constant c_m, independent of the sequences $\tilde{x}_n$ and $\tilde{p}_n$, such that $d_{m-1}d_{m+1} \geqslant c_m(d_m)^2$, $m \geqslant 3$.

On the contrary, we show that there is no constant C_m, depending only on m, such that $d_{m-1}d_{m+1} \leqslant C_m(d_m)^2$.

Nontrivial lower bound for d_m and corresponding integral inequalities will also be given.

Finally we posed an open problem concerning the above matter.

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2. Results

Denote by S_+ the space of all positive sequences. Our main result is

Theorem 1. *Let* $\tilde{p}_n, \tilde{x}_n \in S_+$ *and* $d_m = d_m^{(n)} := \sum_1^n p_i x_i^m - (\sum_1^n p_i x_i)^m$*;* $m \in \mathbb{N}$*. Then*

$$d_{m-1}d_{m+1} \geqslant c_m (d_m)^2, \quad m \geqslant 3, \tag{2}$$

with the best possible constant $c_m = 1 - \frac{2}{m(m-1)}$.

This inequality is very precise. For example

$$d_2^{(2)} d_4^{(2)} - \frac{2}{3}\big(d_3^{(2)}\big)^2 = \frac{1}{3}(p_1 p_2)^2 (1 + p_1 p_2)(x_1 - x_2)^6.$$

Non-trivial lower bound for d_m follows.

Theorem 2. *For* d_m *defined as above, we have*

$$d_m \geqslant \binom{m}{2} \frac{(d_3/3)^{m-2}}{(d_2)^{m-3}}, \quad m \geqslant 2.$$

Applying the standard procedure (cf. [1, p. 131]), we pass from finite sums to definite integrals and obtain

Theorem 3. *Let* $f(t), p(t)$ *be non-negative, continuous and integrable functions for* $t \in [a,b]$*, with* $\int_a^b p(t)\,dt = 1$*. Denote*

$$D_m = D_m(a, b\,; f, p) := \int_a^b p(t)\, f^m(t)\, dt - \left(\int_a^b p(t)\, f(t)\, dt \right)^m.$$

Then

(i) $D_{m-1}D_{m+1} \geqslant \big(1 - \frac{2}{m(m-1)}\big)(D_m)^2$*,* $m \geqslant 3$*;*

(ii) *If* $f(t) \neq C$*,* $t \in [a,b]$*, we have*

$$D_m \geqslant \binom{m}{2} \frac{(D_3/3)^{m-2}}{(D_2)^{m-3}}, \quad m \geqslant 2.$$

3. Proofs

We start with an interesting formula. For $\tilde{p}_n, \tilde{x}_n \in S_+$, making a shift $x_i \to x_i + t$, we obtain

$$d_m(t) := \sum_1^n p_i (x_i + t)^m - \left(\sum_1^n p_i (x_i + t) \right)^m = \sum_1^n p_i (t + x_i)^m - \left(t + \sum_1^n p_i x_i \right)^m.$$

Developing, we get

$$d_m(t) = \sum_2^n d_i \binom{m}{i} t^{m-i}. \tag{3}$$

Therefore $d_m(t)$ belongs to the class of Appell polynomials i.e., $d'_m(t) = m d_{m-1}(t)$ (cf [3], [4]).

If the properties of this class of polynomials lead to the proof of Theorem 1 is left to the readers to examine. For example, by (1), $d_4(t)$ is non-negative for each $t \in \mathbb{R}$. Hence by (3),

$$d_4(t) = d_4 + 4d_3t + 6d_2t^2 \geqslant 0.$$

Putting $t = -\frac{1}{3}\frac{d_3}{d_2}$, we obtain (2) with $m = 3$.

In this article we turn the other way, noting that (2) can be rewritten in the form

$$\frac{d_{m-1}}{(m-1)(m-2)}\,\frac{d_{m+1}}{(m+1)m} \geqslant \Big(\frac{d_m}{m(m-1)}\Big)^2, \quad m \geqslant 3.$$

Hence, (2) is equivalent to the assertion that $\frac{d_m}{m(m-1)}$ is log-convex for $m \geqslant 3$.

Definition. A sequence of positive numbers $\{c_m\}$ is log-convex ($c_m \in LC$) if $c_{m-1}c_{m+1} \geqslant (c_m)^2$.

We quote here some useful lemmas from log-convex theory (cf [3]).

Lemma 3.1. *A positive sequence $\{c_m\}$ is log-convex if and only if the inequality $c_{m-1}u^2 + 2c_muv + c_{m+1}v^2 \geqslant 0$ holds for each real u, v.*

Lemma 3.2. *Let $a_m, b_m \in LC$ and A, B, C be arbitrary positive constants. Then:* (i) *$AC^{m+B}a_m \in LC$;* (ii) *$Aa_m + Bb_m \in LC$.*

Now we are able to produce a proof of Theorem 1 by induction on n.

Proof of Theorem 1. For $n = 2$ we have to prove that

$$\frac{p_1x_1^m + p_2x_2^m - (p_1x_1 + p_2x_2)^m}{m(m-1)} \in LC, \tag{4}$$

holds for each positive x_1, x_2, p_1, p_2 with $p_1 + p_2 = 1$. To this end, we need the following simple assertion

Lemma 3.3. *If $A \geqslant B > 0$, then $\frac{A^m - B^m}{m} \in LC$, holds for $m \geqslant 2$.*

Now, for fixed x_1, x_2, p_1, p_2 and arbitrary $\xi \geqslant 1$ put $A = \xi$, $B = p_1\xi + p_2$; note that $A \geqslant B$ since $p_1 + p_2 = 1$. By lemmas 1, 3 and 2(i), for arbitrary $u, v \in \mathbb{R}$, $m \geqslant 3$, we get

$$\begin{aligned} p_1x_2^{m-1}\Big(\frac{\xi^{m-2} - (p_1\xi + p_2)^{m-2}}{m-2}\Big)u^2 + 2p_1x_2^m\Big(\frac{\xi^{m-1} - (p_1\xi + p_2)^{m-1}}{m-1}\Big)uv \\ + p_1x_2^{m+1}\Big(\frac{\xi^m - (p_1\xi + p_2)^m}{m}\Big)v^2 \geqslant 0. \end{aligned} \tag{5}$$

Integrating (5) with respect to ξ over $\xi \in [1, x_1/x_2]$, we obtain

$$\begin{aligned} \frac{p_1x_1^{m-1} + p_2x_2^{m-1} - (p_1x_1 + p_2x_2)^{m-1}}{(m-1)(m-2)}u^2 + 2\frac{p_1x_1^m + p_2x_2^m - (p_1x_1 + p_2x_2)^m}{m(m-1)}uv \\ + \frac{p_1x_1^{m+1} + p_2x_2^{m+1} - (p_1x_1 + p_2x_2)^{m+1}}{(m+1)m}v^2 \geqslant 0. \end{aligned}$$

Therefore by Lemma 1 we conclude that (4) is true.

Let $T := \frac{1}{1-p_n}\sum_1^{n-1} p_i x_i$. Then

$$\frac{d_m^{(n)}}{m(m-1)} = (1-p_n)\frac{d_m^{(n-1)}}{m(m-1)} + \frac{(1-p_n)T^m + p_n x_n^m - ((1-p_n)T + p_n x_n)^m}{m(m-1)}.$$

Since $\frac{d_m^{(n-1)}}{m(m-1)} \in LC$ by induction hypothesis, by (4) and Lemma 2(ii), it follows that $\frac{d_m^{(n)}}{m(m-1)} \in LC$, and the proof is done. □

To see that the constant $c_m = 1 - \frac{2}{m(m-1)}$ is best possible, consider the representation (3). Since variable t is independent of the sequences $\tilde{p}_n, \tilde{x}_n$, we have $d_m(t) \sim d_2\binom{m}{2}t^{m-2}$ $(t \to \infty)$. Hence

$$\frac{d_{m-1}(t)d_{m+1}(t)}{(d_m(t))^2} \sim \frac{\binom{m-1}{2}t^{m-3}\binom{m+1}{2}t^{m-1}}{\left(\binom{m}{2}t^{m-2}\right)^2} = c_m \quad (t \to \infty).$$

Proof of Theorem 2. From (2) we get $d_{m+1}/d_m \geqslant c_m(d_m/d_{m-1})$, $m \geqslant 3$. Hence

$$\prod_3^m \left(\frac{d_{k+1}}{d_k}\right) \geqslant \prod_3^m \frac{(k+1)(k-2)}{k(k-1)} \prod_3^m \left(\frac{d_k}{d_{k-1}}\right),$$

i.e.,

$$\frac{d_{m+1}}{d_m} \geqslant \left(\frac{m+1}{3(m-1)}\right)\left(\frac{d_3}{d_2}\right), \quad m \geqslant 2.$$

Therefore, the conclusion follows from

$$\frac{d_m}{d_2} = \prod_2^{m-1}\left(\frac{d_{k+1}}{d_k}\right) \geqslant \prod_2^{m-1}\left(\frac{k+1}{k-1}\right)\prod_2^{m-1}\left(\frac{d_3}{3d_2}\right) = \binom{m}{2}\left(\frac{d_3}{3d_2}\right)^{m-2}. \qquad \square$$

Proof of Theorem 3. Write $d_m^{(n)}$ in the form

$$d_m^{(n)} = \frac{\sum_1^n p_{ni}x_{ni}^m}{\sum_1^n p_{ni}} - \left(\frac{\sum_1^n p_{ni}x_{ni}}{\sum_1^n p_{ni}}\right)^m,$$

with $p_{ni} := p\big(a + i\frac{b-a}{n}\big)$, $x_{ni} := f\big(a + i\frac{b-a}{n}\big)$. Passing to the limit, we obtain $\lim_{n\to\infty} d_m^{(n)} = D_m$ and from Theorems 1, 2 the assertions of Theorem 3 follow. □

There remains a problem of inverse inequality for the sequence d.

Question 1. Is there a constant C_m, independent of $\tilde{p}_n$, $\tilde{x}_n \in S_+$, such that $d_{m-1}d_{m+1} \leqslant C_m(d_m)^2$, $m \geqslant 2$.

The answer to this question is negative.

Proof. We apply a special choice of the sequences $\tilde{p}_n$, $\tilde{x}_n \in S_+$. Namely, for fixed $n \geqslant 2$ let $p_i := \binom{n-1}{i-1}/2^{n-1}$; $x_i := (1-t)^{i-1}(1+t)^{n-i}$, $-1 < t < 1$. We obtain a sequence $d^* = \{d_m^*(t)\}$ with

$$d_m^*(t) = \left(\frac{(1-t)^m + (1+t)^m}{2}\right)^{n-1} - 1.$$

For sufficiently large n, we have

$$d_2^*(1/\sqrt{2}) \sim (3/2)^{n-1}; \quad d_4^*(1/\sqrt{2}) \sim (17/4)^{n-1}; \quad d_3^*(1/\sqrt{2}) \sim (5/2)^{n-1}.$$

Hence $C_3 \geqslant (51/50)^{n-1} \to \infty$ $(n \to \infty)$. $\square$

Therefore, we have to reformulate the problem.

QUESTION 2. Is there a constant $C_{m,n}$ such that $d_{m-1}^{(n)} d_{m+1}^{(n)} \leqslant C_{m,n}(d_m^{(n)})^2$, for each $m, n \geqslant 2$, independently of sequences $\tilde{p}_n, \tilde{x}_n \in S_+$?

The best possible constant (if exists) is given by

$$C_{m,n} = \sup\left\{ \frac{d_{m-1}^{(n)} d_{m+1}^{(n)}}{(d_m^{(n)})^2} \mid \tilde{p}_n, \tilde{x}_n \in S_+ \right\}$$

Examining the sequence d^*, we conclude that $C_{m,n} \geqslant (1 + C/m^2)^{n-1}$, where C is an absolute constant.

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