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ON AN INTERPOLATION PROCESS OF LAGRANGE–HERMITE TYPE

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Abstract. We consider a Lagrange–Hermite polynomial, interpolating a function at the Jacobi zeros and, with its first $(r-1)$ derivatives, at the points ± 1. We give necessary and sufficient conditions on the weights for the uniform boundedness of the related operator in certain suitable weighted L^p-spaces, $1<p<\infty$, proving a Marcinkiewicz inequality involving the derivative of the polynomial at ± 1. Moreover, we give optimal estimates for the error of this process also in the weighted uniform metric.

1. Introduction

Let us denote by $L_{m,r}(v^\alpha, f)$ the polynomial of Lagrange–Hermite type based on the Jacobi zeros $x_k = x_{m,k}(v^\alpha)$ related to the weight $v^\alpha(x) = (1-x^2)^\alpha$ and whose jth order derivatives at ± 1 are equal to $f^{(j)}(\pm 1)$, $j = 0, 1, \dots, r-1$, i.e.,

$$L_{m,r}(v^\alpha, f, x_k) = f(x_k), \quad k = 1, \dots, m,$$
$$L_{m,r}(v^\alpha, f)^{(j)}(\pm 1) = f^{(j)}(\pm 1), \quad j = 0, 1, \dots, r-1,$$

where $f^{(0)} \equiv f$.

This interpolation process is sometimes useful in the numerical treatment of differential equations with boundary conditions. The authors had already took into consideration a similar procedure obtaining some results that the reader can find in [**3**, pp. 260, 272]. In the present paper we are going to study the behaviour

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This paper is dedicated to Academician Anton Bilimović.

of the sequence $\{L_{m,r}(v^\alpha, f)\}_m$ in certain suitable weighted L^p-spaces and give necessary and sufficient conditions on the weights for the uniform boundedness of $\{L_m(v^\alpha)\}_m$. Optimal estimates of the error will be given and a Marcinkiewicz inequality involving the derivatives of the polynomial at ± 1 will be proved. The results of this paper cover the ones available in literature.

In Section 2 we will state our main results and in Section 3 we will prove them.

2. Main Results

In the following $\mathcal{C}$ denotes a positive constant which may have different values in different formulas. We will write $\mathcal{C} \neq \mathcal{C}(a, b, \ldots)$ to say that $\mathcal{C}$ is independent of the parameters $a, b, \ldots$ If $A, B > 0$ are quantities depending on some parameters, we write $A \sim B$, if there exists a positive constant $\mathcal{C}$ independent of the parameters of A and B, such that $B/\mathcal{C} \leqslant A \leqslant \mathcal{C}B$.

Now we introduce some function spaces, related to a Jacobi weight of the form

$$v^\gamma(x) = (1 - x^2)^\gamma, \quad \gamma \geqslant 0, \quad x \in (-1, 1). \tag{2.1}$$

Letting L^p, $1 \leqslant p < \infty$, denote the space of all measurable functions f with $\|f\|_p^p = \int_{-1}^1 |f|^p$, we say $f \in L^p_{v^\gamma}$ if $fv^\gamma \in L^p$, i.e., $\|f\|^p_{L^p_{v^\gamma}} = \int_{-1}^1 |fv^\gamma|^p < \infty$. For $p = \infty$ and $\gamma > 0$, we set $L^\infty_{v^\gamma} = C_{v^\gamma} = \big\{f \in C^0(-1, 1) : \lim_{|x|\to 1}(fv^\gamma)(x) = 0\big\}$ and $C_{v^0} \equiv C^0[-1, 1]$. Moreover, we set

$$C^0_r = \big\{f \in C^0(-1, 1) : f \text{ is } (r - 1) - \text{times differentiable at } \pm 1\big\},$$

where $r \geqslant 1$ is an integer number. Of course, $C^0_r \subset C_{v^\gamma}$, $\gamma \geqslant 0$, and $C^0_r \supset C^{r-1}[-1, 1]$, where $C^{r-1}[-1, 1]$ is the collection of all functions whose $(r - 1)$th derivative is continuous on $[-1, 1]$.

The Sobolev type spaces are defined as follows

$$W^s_p = W^s_p(v^\gamma) = \big\{f \in L^p_{v^\gamma} : f^{(s-1)} \in AC(-1, 1) \text{ and } \|f^{(s)}\varphi^s v^\gamma\|_p < \infty\big\},$$

where $\varphi(x) = \sqrt{1 - x^2}$, $AC(-1, 1)$ is the set of the absolutely continuous functions in every compact of $(-1, 1)$, $1 \leqslant p \leqslant \infty$ and $s \geqslant 1$ is an integer.

Let $v^\alpha(x) = (1 - x^2)^\alpha$, $\alpha > -1$, and $\{p_m(v^\alpha)\}_m$ be the corresponding sequence of orthonormal polynomials with positive leading coefficients. For every function $f \in C^0_r$, $r \geqslant 1$, an expression of the polynomial $L_{m,r}(v^\alpha, f, x)$, $\alpha > -1$, is given by

$$\begin{aligned} L_{m,r}(v^\alpha, f, x) &= \sum_{k=1}^m v^r(x) \frac{l_k(x)}{v^r(x_k)} f(x_k) \\ &\quad + (1 - x)^r p_m(v^\alpha, x) \sum_{i=0}^{r-1} \frac{(1 + x)^i}{i!} \left(\frac{f}{(1 - \cdot)^r p_m(v^\alpha)}\right)^{(i)} (-1) \\ &\quad + (1 + x)^r p_m(v^\alpha, x) \sum_{i=0}^{r-1} \frac{(1 - x)^i}{i!} \left(\frac{f}{(1 + \cdot)^r p_m(v^\alpha)}\right)^{(i)} (1) \\ &=: A(x) + B_1(x) + B_2(x), \end{aligned} \tag{2.2}$$

where x_k, $k = 1, \dots, m$, are zeros of $p_m(v^\alpha)$ and

$$l_k(x) = \frac{p_m(v^\alpha, x)}{p'_m(v^\alpha, x_k)(x - x_k)}, \quad k = 1, \dots, m.$$

We complete the definition of $L_{m,r}(v^\alpha, f)$ setting $L_{m,0}(v^\alpha, f) = L_m(v^\alpha, f)$.

Finally, letting $\mathbb{P}_m$ be the space of all polynomials of degree at most m, we denote by $E_m(f)_{v^\gamma,p} = \inf_{P_m \in \mathbb{P}_m} \|(f - P_m)v^\gamma\|_p$, $1 \leqslant p \leqslant \infty$, the error of best polynomial approximation in $L^p_{v^\gamma}$.

Now we are able to study the behaviour of the sequence $\{L_{m,r}(v^\alpha)\}_m$, $r \geqslant 1$, in the above introduced function spaces.

THEOREM 2.1. *Let v^α and v^γ be two Jacobi weight functions defined in* (2.1), *with $\gamma \geqslant 0$ and $\alpha > -1$. Then, for every $f \in C^0_r$, $r \geqslant 1$, we have*

$$\|L_{m,r}(v^\alpha, f)v^\gamma\|_\infty \leqslant \mathcal{C}\left\{\|fv^\gamma\|_\infty \log m + \frac{1}{m^{2\gamma}} \sum_{j=0}^{r-1} \frac{|f^{(j)}(-1)| + |f^{(j)}(1)|}{j!\, m^{2j}}\right\}, \tag{2.3}$$

where $\mathcal{C} \neq \mathcal{C}(m, f)$, if and only if

$$\frac{\alpha}{2} + \frac{1}{4} \leqslant \gamma + r \leqslant \frac{\alpha}{2} + \frac{5}{4}. \tag{2.4}$$

Moreover, if $f \in C^{r-1}[-1, 1]$, the condition (2.4) *implies*

$$\|[f - L_{m,r}(v^\alpha, f)]v^\gamma\|_\infty \leqslant \mathcal{C} E_{m+2r-1}(f)_{v^\gamma,\infty} \log m, \quad \mathcal{C} \neq \mathcal{C}(m, f). \tag{2.5}$$

The above theorem includes some special cases that are well-known in literature. For example, for $\gamma = 0$ and $r \geqslant 1$ we get Theorem 4.2.5 in [**3**, p. 260]. In the case $r = 0$ and $\gamma > 0$ we obtain Theorem 2.2 in [**4**] (see also [**3**, p. 272]), and for $\gamma = r = 0$ we get Theorem 14.4 in [**9**, p. 335].

THEOREM 2.2. *Let $1 \leqslant p < \infty$, $\gamma \geqslant 0$, and $\alpha > -1$. Then, for every function $f \in C^0_r$, $r \geqslant 1$, there exists a constant $\mathcal{C} \neq \mathcal{C}(m, f)$ such that*

$$\|L_{m,r}(v^\alpha, f)v^\gamma\|_p \leqslant \mathcal{C}\left\{\|fv^\gamma\|_\infty + \frac{1}{m^{2\gamma+2/p}} \sum_{j=0}^{r-1} \frac{|f^{(j)}(-1)| + |f^{(j)}(1)|}{j!\, m^{2j}}\right\} \tag{2.6}$$

if and only if

$$\frac{v^{\gamma+r}}{\sqrt{v^\alpha \varphi}} \in L^p \quad \textit{and} \quad \frac{\sqrt{v^\alpha \varphi}}{v^{\gamma+r}} \in L^1 \tag{2.7}$$

i.e.,

$$-\frac{1}{p} < \gamma + r - \frac{\alpha}{2} - \frac{1}{4} < 1.$$

Letting

$$\sigma_m(f) = \sum_{j=0}^{r-1} \frac{|f^{(j)}(-1)| + |f^{(j)}(1)|}{j! m^{2j}}, \tag{2.8}$$

it is easy to deduce from the proof of the above theorem that the only condition $v^{\gamma+r}/\sqrt{v^\alpha\varphi} \in L^p$ is necessary and sufficient to obtain the bound

$$\|L_{m,r}(v^\alpha, f)v^\gamma\|_p \leqslant \mathcal{C}\|f\|_\infty + \frac{\sigma_m(f)}{m^{2\gamma+2/p}}, \quad 1 \leqslant p < \infty,$$

which, for $r = 0$, follows from a well-know theorem of P. Nevai [**7**, p. 680].

The following theorem is a refinement of the previous one and implies some interesting consequences.

THEOREM 2.3. *Let $1 < p < \infty$, $\gamma \geqslant 0$, and $\alpha > -1$. Then, for every function $f \in C_r^0$, $r \geqslant 1$, there exists a constant $\mathcal{C} \neq \mathcal{C}(m, f)$ such that*

$$\|L_{m,r}(v^\alpha, f)v^\gamma\|_p \leqslant \mathcal{C}\Bigg\{\left(\sum_{k=1}^{m} \Delta x_k |fv^\gamma|^p(x_k)\right)^{1/p} \tag{2.9}$$

$$+ \frac{1}{m^{2\gamma+2/p}} \sum_{j=0}^{r-1} \frac{|f^{(j)}(-1)| + |f^{(j)}(1)|}{j!m^{2j}}\Bigg\} \tag{2.10}$$

if and only if

$$\frac{v^{\gamma+r}}{\sqrt{v^\alpha\varphi}} \in L^p \quad \textit{and} \quad \frac{\sqrt{v^\alpha\varphi}}{v^{\gamma+r}} \in L^q, \quad q^{-1} + p^{-1} = 1, \tag{2.11}$$

i.e.,

$$-\frac{1}{p} < \gamma + r - \frac{\alpha}{2} - \frac{1}{4} < \frac{1}{q}.$$

Now we state some estimates of the error $f - L_{m,r}(v^\alpha, f)$ for f varying in the previously introduced spaces.

PROPOSITION 2.1. *For any $f \in C^{r-1}[-1, 1]$, $r \geqslant 1$, under the assumptions* (2.7) *we have*

$$\big\|[f - L_{m,r}(v^\alpha, f)]v^\gamma\big\|_p \leqslant \mathcal{C}E_{m+2r-1}(f)_{v^\gamma,\infty}, \tag{2.12}$$

where $1 \leqslant p < \infty$ and $\mathcal{C} \neq \mathcal{C}(m, f)$. Moreover, if $f \in W_p^r, r \geqslant 1$, and $1 < p < \infty$, then the conditions (2.11) *imply*

$$\big\|[f - L_{m,r}(v^\alpha, f)]v^\gamma\big\|_p \leqslant \frac{\mathcal{C}}{m^r}\big\|f^{(r)}\varphi^r v^\gamma\big\|_p, \quad \mathcal{C} \neq \mathcal{C}(m, f). \tag{2.13}$$

Note that (2.13) shows that, if $f \in W_p^r$, with $r \geqslant 1$ and $1 < p < \infty$, the polynomial $L_{m,r}(v^\alpha, f)$ converges with the order of the best polynomial approximation in $L_{v^\gamma}^p$. Therefore, in the usual way, we can establish the next corollary that shows the uniform boundedness (with respect to m) of the operator $L_{m,r}(v^\alpha)$ in Sobolev spaces (under the assumptions (2.11)).

COROLLARY 2.1. *Under the conditions* (2.11)*, for every $f \in W_p^r$ with $r \geqslant 1$ and $1 < p < \infty$, we have*

$$\sup_m \big\|L_{m,r}(v^\alpha, f)\big\|_{W_p^r} \leqslant \mathcal{C}\big\|f\big\|_{W_p^r}, \quad \mathcal{C} \neq \mathcal{C}(f). \tag{2.14}$$

Coming back to Theorem 2.3, the estimate (2.9) with the notation (2.8) can be written as

$$\left\|L_{m,r}(v^\alpha, f)v^\gamma\right\|_p \leqslant \mathcal{C}\left\{\left(\sum_{k=1}^{m} \Delta x_k |fv^\gamma|^p(x_k)\right)^{1/p} + \frac{\sigma_m(f)}{m^{2\gamma+2/p}}\right\} =: \Gamma_m(f).$$

Of course, if f is a polynomial P of degree $m+2r-1$, the inequality

$$\Gamma_m(P) \geqslant \mathcal{C}\|L_{m,r}(v^\alpha, P)v^\gamma\|_p = \mathcal{C}\|Pv^\gamma\|_p, \quad 1 < p < \infty,$$

is equivalent to the conditions (2.11).

Moreover, it is easy to prove that for arbitrary $\alpha > -1$, $\gamma \geqslant 0$, and $r \geqslant 1$, the inverse inequality $\Gamma_m(P) \leqslant \mathcal{C}\|Pv^\gamma\|_p$ holds true for $1 \leqslant p < \infty$. In fact, the bound

$$\left(\sum_{k=1}^{m} \Delta x_k |Pv^\gamma|^p(x_k)\right)^{1/p} \leqslant \mathcal{C}\|Pv^\gamma\|_p \tag{2.15}$$

is well-known (see, for example, [**7**, p. 675]). In order to obtain

$$\frac{\sigma_m(P)}{m^{2\gamma+2/p}} \leqslant \mathcal{C}\|Pv^\gamma\|_p$$

it suffices to apply the inequalities of Markov, Schur and Nikol'skiĭ.

Therefore, we can state a new Marcinkiewicz inequality involving the derivatives of a polynomial at ± 1.

Corollary 2.2. *Let $x_k, k = 1, \ldots, m$, be the zeros of the mth Jacobi polynomial $p_m(v^\alpha)$ and let $1 < p < \infty$. Then, for every polynomial $P \in \mathbb{P}_{m+2r-1}$, the following equivalence*

$$\|Pv^\gamma\|_p \sim \left(\sum_{k=1}^{m} \Delta x_k |Pv^\gamma|^p(x_k)\right)^{1/p} + \frac{\sigma_m(P)}{m^{2\gamma+2/p}} \tag{2.16}$$

holds true, with the constants in "$\sim$" independent of m and P, if and only if

$$-\frac{1}{p} < r + \gamma - \frac{\alpha}{2} - \frac{1}{4} < \frac{1}{q} \quad (p^{-1} + q^{-1} = 1).$$

Finally, we want to observe that if we introduce the mth Christoffel function of the weight $v^{\gamma p}$,

$$\lambda_m(v^{\gamma p}, x) = \left[\sum_{k=0}^{m-1} p_k^2(v^{\gamma p}, x)\right]^{-1} \sim v^{\gamma p}(x)\frac{\sqrt{1-x^2}}{m},$$

then the sum in (2.16) can be replaced by

$$\left(\sum_{k=1}^{m} \lambda_m(v^{\gamma p}, x_k)|P(x_k)|^p\right)^{1/p}.$$

3. Proofs

In this section we will frequently use the Remez-type inequality in the following form

$$(3.1)\qquad (\forall P_m \in \mathbb{P}_m) \quad \|P_m v^\gamma\|_p \leqslant \mathcal{C}\|P_m v^\gamma\|_{L^p(A_m)},$$

where $A_m = \left[-1+am^{-2}, 1-am^{-2}\right]$, with $a>0$ fixed, and $\mathcal{C} \neq \mathcal{C}(m, P_m)$.

If I is a subinterval of $(-1,1)$, the Hilbert transform $H(f,t)$ is defined as follows

$$H(f,t) = \int_I \frac{f(x)}{x-t}dx, \quad t \in I,$$

where the integral is understood in the Cauchy principal value sense. For $1<p<\infty$, the following property is well-known:

$$\int_I gH(f) = -\int_I fH(g), \quad f \in L^p \text{ and } g \in L^q,\ \frac{1}{p}+\frac{1}{q}=1.$$

Moreover, with $v^\sigma(x) = (1-x^2)^\sigma$ and $1<p<\infty$, one has

$$\|(Hf)v^\sigma\|_p \leqslant \mathcal{C}\|fv^\sigma\|_p \quad \text{if and only if} \quad -\frac{1}{p} < \sigma < \frac{1}{q}.$$

Now, recalling (2.2) with the notation (2.8), we can state the following lemma:

Lemma 3.1. *With the notation* (2.2)*, we have*

$$\|(B_1+B_2)v^\gamma\|_\infty \leqslant \frac{\mathcal{C}}{m^{2\gamma}}\sigma_m(f), \quad \mathcal{C} \neq \mathcal{C}(f,m),$$

if and only if $\frac{\alpha}{2}+\frac{1}{4} \leqslant \gamma + r \leqslant \frac{\alpha}{2}+\frac{5}{4}$. *Moreover, for* $p \in [1,\infty)$*, we get*

$$\|(B_1+B_2)v^\gamma\|_p \leqslant \frac{\mathcal{C}}{m^{2\gamma+2/p}}\sigma_m(f), \quad \mathcal{C} \neq \mathcal{C}(f,m),$$

if and only if $-\frac{1}{p} < \gamma + r - \frac{\alpha}{2} - \frac{1}{4} < \frac{1}{q}$.

Proof. We estimate only the norm $\|B_1 v^\gamma\|_p$, $1 \leqslant p \leqslant \infty$, since the estimate of $\|B_2 v^\gamma\|_p$ is similar. Using the Remez inequality (3.1) and letting

$$\bar{A}_i = \frac{1}{i!}\left(\frac{f}{(1-\cdot)^r p_m(v^\alpha)}\right)^{(i)}(-1),$$

we can write

$$\begin{aligned}
&\|B_1 v^\gamma\|_p \leqslant \mathcal{C}\|B_1 v^\gamma\|_{L^p(A_m)} \\
(3.2)\qquad &\leqslant \sum_{i=0}^{r-1} |\bar{A}_i| \|(1-x)^r(1+x)^i(1-x^2)^\gamma p_m(v^\alpha, x)\|_{L^p(A_m)} =: \sum_{i=0}^{r-1}|\bar{A}_i| b_i.
\end{aligned}$$

Of course, we have

$$\begin{aligned}
b_i &\leqslant \left\|(1-x)^r(1+x)^i(1-x^2)^\gamma p_m(v^\alpha,x)\right\|_{L^p(-1+a/m^2,\,0)} \\
&\quad + \left\|(1-x)^r(1+x)^i(1-x^2)^\gamma p_m(v^\alpha,x)\right\|_{L^p(0,1-a/m^2)} \\
&:= I_1 + I_2.
\end{aligned}$$

Moreover, by virtue of the estimate $|p_m(v^\alpha,x)| \leqslant \mathcal{C} v^{-\frac{\alpha}{2}-\frac{1}{4}}(x)$, $|x| \leqslant 1-a/m^2$, we have

$$I_1+I_2 \leqslant \mathcal{C}\Big\{\big\|(1+x)^{\gamma+i-\frac{\alpha}{2}-\frac{1}{4}}\big\|_{L^p(-1+a/m^2,\,0)} + \big\|(1-x)^{\gamma+r-\frac{\alpha}{2}-\frac{1}{4}}\big\|_{L^p(0,1-a/m^2)}\Big\}.$$

Now, under the assumptions on the parameters α, γ and r (and only in this case), the first summand is dominated by $\mathcal{C}\left(m^{-2}\right)^{\gamma+i-\frac{\alpha}{2}-\frac{1}{4}+\frac{1}{p}}$, with $\gamma+i-\frac{\alpha}{2}-\frac{1}{4}+\frac{1}{p} \leqslant 0$, $1 \leqslant p \leqslant \infty$, while the second summand is bounded. In particular, for $p<\infty$ we have $\gamma+i-\frac{\alpha}{2}-\frac{1}{4}+\frac{1}{p}<0$, while for $p=\infty$ we have $\gamma+i-\frac{\alpha}{2}-\frac{1}{4}=0$ only in the case $i=r-1$. In any case, since $|p_m(v^\alpha,\pm 1)| \sim m^{\alpha+1/2}$ (see for instance [**3**, p. 251, formula (4.2.10)]) we conclude that

$$I_1+I_2 \leqslant \frac{\mathcal{C}}{m^{2\gamma+2/p}} \frac{|p_m(v^\alpha,-1)|}{m^{2i}}, \quad 1 \leqslant p \leqslant \infty,$$

taking into account that $2\gamma+2i=\alpha+1/2$ for $i=r-1$ and $p=\infty$. Therefore, recalling (3.2), we have

$$\|B_1 v^\gamma\|_p \leqslant \mathcal{C}\frac{|p_m(v^\alpha,-1)|}{m^{2\gamma+2/p}} \sum_{i=0}^{r-1} \frac{\bar{A}_i}{m^{2i}}.$$

It remains to estimate $\bar{A}_i, i=0,1,\dots,r-1$. We have

$$|\bar{A}_i| \leqslant \frac{1}{i!}\sum_{j=0}^{i}\binom{i}{j}\big|f^{(j)}(-1)\big|\left|\left(\frac{1}{(1-x)^r}\frac{1}{p_m(v^\alpha,x)}\right)^{(i-j)}(-1)\right|$$

and, taking into account that [**7**, p. 674, formula (23)]

$$\left(\frac{1}{p_m(v^\alpha,x)}\right)^{(k)}(-1) \leqslant \mathcal{C}\frac{m^{2k}}{|p_m(v^\alpha,-1)|},$$

we obtain

$$\left|\left(\frac{1}{(1-x)^r}\frac{1}{p_m(v^\alpha,x)}\right)^{(i-j)}(-1)\right| \leqslant \mathcal{C}\frac{m^{2i-2j}}{|p_m(v^\alpha,-1)|}.$$

Hence we get

$$|\bar{A}_i| \leqslant \frac{\mathcal{C}}{i!\,|p_m(v^\alpha,-1)|}\sum_{j=0}^{i}\binom{i}{j}\big|f^{(j)}(-1)\big| m^{2i-2j}$$

and, for $1 \leqslant p \leqslant \infty$,

$$\begin{aligned}
\|B_1 v^\gamma\|_p &\leqslant \frac{\mathcal{C}}{m^{2\gamma+2/p}}\sum_{i=0}^{r-1}\frac{1}{i!}\sum_{j=0}^{i}\binom{i}{j}\frac{\big|f^{(j)}(-1)\big|}{m^{2j}} \\
&= \frac{\mathcal{C}}{m^{2\gamma+2/p}}\sum_{j=0}^{r-1}\frac{\big|f^{(j)}(-1)\big|}{m^{2j}}\sum_{i=j}^{r-1}\frac{1}{i!}\binom{i}{j} \\
&\leqslant \frac{\mathcal{C}}{m^{2\gamma+2/p}}\sum_{j=0}^{r-1}\frac{\big|f^{(j)}(-1)\big|}{j!\,m^{2j}},
\end{aligned}$$

which completes the proof. □

Proof of Theorem 2.1. In Lemma 3.1, we proved that

$$\|(B_1 + B_2)v^\gamma\|_\infty \leqslant \frac{\mathcal{C}}{m^{2\gamma}} \sigma_m(f).$$

Then, it remains to prove that $\|Av^\gamma\|_\infty \leqslant \mathcal{C}\|fv^\gamma\|_\infty \log m$. But, the latter inequality can be found in [**3**, p. 262] with a minor change. So, the proof of (2.3) is complete.

Concerning the estimate of the error (2.5), we refer to the proof of Proposition 2.1. □

We are going to prove Theorem 2.3 before Theorem 2.2.

Proof of Theorem 2.3. Taking into account Lemma 3.1, to prove the theorem, it suffices to show that the inequality

$$\|Av^\gamma\|_p^p \leqslant \mathcal{C} \sum_{k=1}^{m} \Delta x_k |fv^\gamma|^p(x_k), \quad 1 < p < \infty, \tag{3.3}$$

is equivalent to the conditions (2.11).

We first prove that (2.11) implies (3.3). To this end, using (3.1) and, letting $g = v^{\gamma(p-1)}|A|^{p-1} \operatorname{sgn} A$ in the interval A_m, we can write

$$\|Av^\gamma\|_{L^p(A_m)}^p = \int_{A_m} \sum_{k=1}^{m} v^{\gamma+r}(x) \frac{l_k(x) f(x_k)}{v^r(x_k)} g(x)\, dx,$$

where

$$l_k(x) = \frac{p_m(v^\alpha, x)}{p'_m(v^\alpha, x_k)(x - x_k)},$$

whence we deduce

$$\begin{aligned}\|Av^\gamma\|_{L^p(A_m)}^p &= \sum_{k=1}^{m} \frac{f(x_k) v^\gamma(x_k)}{p'_m(v^\alpha, x_k) v^{\gamma+r}(x_k)} \int_{A_m} \frac{v^{\gamma+r}(x) p_m(v^\alpha, x)}{x - x_k} g(x)\, dx \\ &\leqslant \mathcal{C} \sum_{k=1}^{m} \frac{\Delta x_k |fv^\gamma|(x_k)}{v^{\gamma+r-\frac{\alpha}{2}-\frac{1}{4}}(x_k)} \left| \int_{A_m} v^{\gamma+r}(x) p_m(v^\alpha, x) \frac{g(x)}{x - x_k}\, dx \right|,\end{aligned}$$

since, with $\Delta x_k = x_{k+1} - x_k$, $1/|p'_m(v^\alpha, x_k)| \sim \Delta x_k v^{\frac{\alpha}{2}+\frac{1}{4}}(x_k)$. Denoting by $G(x_k)$ the absolute value of the integral at the right-hand side and using the Hölder inequality, we get

$$\|Av^\gamma\|_{L^p(A_m)}^p \leqslant \left(\sum_{k=1}^{m} \Delta x_k |fv^\gamma|^p(x_k) \right)^{1/p} \left(\sum_{k=1}^{m} \Delta x_k \left[\frac{\sqrt{v^\alpha \varphi}}{v^{\gamma+r}}(x_k) G(x_k) \right]^q \right)^{1/q}.$$

It remains to prove that the L^q norm is bounded by $\mathcal{C}\|Av^\gamma\|_{L^p(A_m)}^{p-1}$. We note that for an arbitrary polynomial $Q \in \mathbb{P}_m$, we can write

$$G(x_k) = \left| \int_{A_m} \frac{p_m(v^\alpha, x) Q(x) - p_m(v^\alpha, x_k) Q(x_k)}{x - x_k} v^{\gamma+r}(x) \frac{g(x)}{Q(x)} dx \right|.$$

Therefore, $G(t)$ is a polynomial of degree $2m - 1$. Then, using the Marcinkiewicz inequality (2.15), the L^q norm is dominated by a positive constant $\mathcal{C}$ times the norm

$\|\frac{\sqrt{v^\alpha\varphi}}{v^{\gamma+r}}G\|_{L^q(A_m)}$ that is bounded under the assumption $\frac{\sqrt{v^\alpha\varphi}}{v^{\gamma+r}} \in L^q$. Moreover, denoting by H the Hilbert transform defined on the interval A_m, we can write

$$\left\|\frac{\sqrt{v^\alpha\varphi}}{v^{\gamma+r}}G\right\|_{L^q(A_m)} \leqslant \left\|\frac{\sqrt{v^\alpha\varphi}}{v^{\gamma+r}}H(p_m(v^\alpha)v^{\gamma+r}g)\right\|_{L^q(A_m)} + \left\|\frac{\sqrt{v^\alpha\varphi}}{v^{\gamma+r}}Qp_m(v^\alpha)H\Big(\frac{gv^{\gamma+r}}{Q}\Big)\right\|_{L^q(A_m)}.$$

Taking also into account the assumption $\frac{v^{\gamma+r}}{\sqrt{v^\alpha\varphi}} \in L^p$, the Hilbert transform is a bounded operator and the first norm is dominated by

$$\left\|\frac{\sqrt{v^\alpha\varphi}}{v^{\gamma+r}}p_m(v^\alpha)v^{\gamma+r}g\right\|_{L^q(A_m)} \leqslant \mathcal{C}\|g\|_{L^q(A_m)} = \|Av^\gamma\|^{p-1}_{L^p(A_m)},$$

since $|p_m(v^\alpha,x)\sqrt{(v^\alpha\varphi)(x)}| \leqslant \mathcal{C}$.

In order to prove the estimate of the second norm at the right-hand side, we choose a polynomial Q such that $Q(x) \sim v^{\gamma+r}(x)$ for $x \in A_m$ (see [**2**]). Consequently, the second norm is dominated by

$$\left\|H\left(\frac{gv^{\gamma+r}}{Q}\right)\right\|_q \leqslant \mathcal{C}\left\|\frac{gv^{\gamma+r}}{Q}\right\|_q \leqslant \mathcal{C}\|g\|_q = \|Av^\gamma\|^{p-1}_{L^p(A_m)}.$$

Then (2.11) implies (3.3).

Now, we prove that (2.11) is a consequence of (3.3). To this end, for any $f \in C^0_r$, we consider a piecewise linear function F_m such that

$$\begin{cases} F_m^{(i)}(\pm1) = 0, & i = 0,1,\dots,r-1, \\ F_m(x_k) = 0, & \text{for } x_k \in [-a,a], \text{ with } a < \frac{1}{4} \text{ fixed}, \\ F_m(x_k) = |f(x_k)|\operatorname{sgn}\{-x_k p_m'(v^\alpha,x_k)\}, & \text{for } x_k \notin [-a,a]. \end{cases}$$

Taking into account that $\operatorname{sgn}(-x_k) = \operatorname{sgn}(x-x_k)$ for $x \in [-a,a]$ and $x_k \notin [-a,a]$, we get

$$\begin{aligned} |L_{m,r}(v^\alpha,F_m,x)v^\gamma(x)| &= |v^{\gamma+r}(x)p_m(v^\alpha,x)|\sum_{k=1}^m \frac{|F_mv^\gamma|(x_k)}{|p_m'(v^\alpha,x_k)v^{\gamma+r}(x_k)||x-x_k|} \\ &\geqslant \mathcal{C}\left|\frac{v^{\gamma+r}(x)}{2}p_m(v^\alpha,x)\right|\sum_{k=1}^m \Delta x_k v^{\alpha+\frac{1}{4}-\gamma-r}(x_k)|F_mv^\gamma|(x_k), \end{aligned}$$

since $|p_m'(v^\alpha,x_k)|^{-1} \sim \Delta x_k v^{\frac{\alpha}{2}+\frac{1}{4}}(x_k)$ and $|x-x_k| \leqslant 2$. Moreover, by virtue of a result in [**6**], we have

$$\liminf_{m\to\infty}\|v^{\gamma+r}\chi_a p_m(v^\alpha)\|_p \geqslant \mathcal{C}\left\|\frac{v^{\gamma+r}\chi_a}{\sqrt{v^\alpha\varphi}}\right\|_p \geqslant \mathcal{C},$$

being χ_a the characteristic function of $[-a,a]$.

Then, collecting the previous inequalities, by (3.3), we obtain

$$\begin{aligned}\sum_{k=1}^{m} \Delta x_k \frac{\sqrt{v^\alpha \varphi}}{v^{\gamma+r}}(x_k)|F_m v^\gamma|(x_k) &\leqslant \|L_{m,r}(v^\alpha, F_m)\chi_a v^\gamma\|_{L^p(A_m)} \\ &\leqslant \mathcal{C}\left(\sum_{k=1}^{m} \Delta x_k |F_m v^\gamma|^p(x_k)\right)^{1/p}. \end{aligned}\tag{3.4}$$

Now, letting

$$a_k = (\Delta x_k)^{1/q} \frac{\sqrt{v^\alpha \varphi}}{v^{\gamma+r}}(x_k), \quad c_k = (\Delta x_k)^{1/p}|F_m v^\gamma|(x_k), \quad \|\bar{c}\|_p^* = \left(\sum_{k=1}^{m} |c_k|^p\right)^{1/p},$$

where $\bar{c} = (c_1, c_2, \ldots, c_m)$, we can write (3.4) as $\sum_{k=1}^{m} a_k c_k \leqslant \mathcal{C}\|\bar{c}\|_p^*$ and, since $\mathcal{C} \neq \mathcal{C}(m, F_m)$,

$$\sup_m \sup_{\bar{c}} \sum_{k=1}^{m} a_k \frac{c_k}{\|\bar{c}\|_p^*} \leqslant \mathcal{C}.$$

Hence, we get $\sup_m \left(\sum_{k=1}^{m} |a_k|^q\right)^{1/q} \leqslant \mathcal{C}$, i.e.,

$$\sup_m \left(\sum_{k=1}^{m} \Delta x_k \left(\frac{\sqrt{v^\alpha \varphi}}{v^{\gamma+r}}(x_k)\right)^q\right)^{1/q} \leqslant \mathcal{C}.$$

The latter inequality is equivalent to $\left\|\frac{\sqrt{v^\alpha\varphi}}{v^{\gamma+r}}\right\|_q < \infty$ which is, therefore, a consequence of (3.3).

Finally, we prove that (3.3) implies also $\left\|\frac{v^{\gamma+r}}{\sqrt{v^\alpha\varphi}}\right\|_p < \infty$. To this end, since (3.3) holds true for every $f \in C^0(-1,1)$, letting $g(x) = f(x)v^r(x)$, we have

$$\|Av^\gamma\|_{L^p(A_m)} = \|L_m(v^\alpha, f)v^{\gamma+r}\|_{L^p(A_m)} \leqslant \mathcal{C}\|fv^{\gamma+r}\|_\infty,$$

i.e.,

$$\sup_m \sup_{\|fv^{\gamma+r}\|_\infty = 1} \|L_m(v^\alpha, f)v^{\gamma+r}\|_{L^p(A_m)} \leqslant \mathcal{C}, \quad \mathcal{C} \neq \mathcal{C}(m, f).$$

Therefore, using [**5**], we get $\sup_m \|p_m(v^\alpha)v^{\gamma+r}\|_p \leqslant \mathcal{C}$, i.e., $\frac{v^{\gamma+r}}{\sqrt{v^\alpha\varphi}} \in L^p$, and the proof is complete. □

Proof of Theorem 2.2. We first show that (2.7) implies (2.6). Taking into account Lemma 3.1, it remains to estimate only the quantity $\|Av^\gamma\|_{L^p(A_m)}^p$, where A is given by (2.2). Using the same argument of the previous proof, we have

$$\|Av^\gamma\|_{L^p(A_m)}^p \leqslant \mathcal{C}\|fv^\gamma\|_\infty \sum_{k=1}^{m} \Delta x_k v^{-\gamma-r+\frac{\alpha}{2}+\frac{1}{4}}(x_k)|G(x_k)|$$

and

$$\begin{aligned} G(x_k) &= \int_{A_m} v^{\gamma+r}(x) \frac{p_m(v^\alpha, x)}{x - x_k} g(x)dx \\ &= \int_{A_m} \frac{p_m(v^\alpha, x)Q(x) - p_m(v^\alpha, x_k)Q(x_k)}{x - x_k} v^{\gamma+r}(x) \frac{g(x)}{Q(x)} dx, \end{aligned}$$

where, as in the proof of Theorem 2.3, $Q \in \mathbb{P}_m$ is equivalent to the weight $v^{\gamma+r}$ in the interval A_m. Then $G(t)$ is a polynomial of degree $2m-1$ and, using a Marcinkiewicz inequality, we get

$$\|Av^\gamma\|^p_{L^p(A_m)} \leqslant \mathcal{C}\|fv^\gamma\|_\infty \|v^{-\gamma-r+\frac{\alpha}{2}+\frac{1}{4}}G\|_{L^1(A_m)}$$

and the L^1 norm is bounded under our hypotheses. In fact, expressing G by means of the Hilbert transform, we have

$$\begin{aligned}\|v^{-\gamma-r+\frac{\alpha}{2}+\frac{1}{4}}G\|_{L^1(A_m)} &\leqslant \|v^{-\gamma-r+\frac{\alpha}{2}+\frac{1}{4}}H(p_m(v^\alpha)v^{\gamma+r}g)\|_{L^1(A_m)} \\ &\quad + \Big\|v^{-\gamma-r+\frac{\alpha}{2}+\frac{1}{4}}p_m(v^\alpha)QH\Big(\frac{v^{\gamma+r}g}{Q}\Big)\Big\|_{L^1(A_m)}.\end{aligned}$$

Concerning the second summand at the right-hand side, using the Hölder inequality, the boundedness of H and $Q \sim v^{\gamma+r}$, we get

$$\Big\|v^{-\gamma-r+\frac{\alpha}{2}+\frac{1}{4}}p_m(v^\alpha)QH\Big(\frac{v^{\gamma+r}g}{Q}\Big)\Big\|_{L^1(A_m)} \leqslant \Big\|H\Big(\frac{v^{\gamma+r}g}{Q}\Big)\Big\|_1 \leqslant \mathcal{C}\|g\|_q.$$

In order to estimate the first summand, we note that the function under the sign of the Hilbert transform is bounded and the one outside is $L(\log^+ L)$ (see [**8**]). Therefore, with $\Gamma = \operatorname{sgn} H(p_m(v^\alpha)v^{\gamma+r}g)$ and $\varrho = \gamma + r - \alpha/2 - 1/4$, we can write

$$\begin{aligned}\|v^{-\varrho}H(p_m(v^\alpha)v^{\gamma+r}g)\|_{L^1(A_m)} &\leqslant \|p_m(v^\alpha)v^{\gamma+r}gH(v^{-\varrho}\Gamma)\|_{L^1(A_m)} \\ &\leqslant \mathcal{C}\|v^\varrho gH(v^{-\varrho}\Gamma)\|_{L^1(A_m)} \\ &\leqslant \mathcal{C}\|g\|_q\|v^\varrho H(v^{-\varrho}\Gamma)\|_{L^p(A_m)} \\ &\leqslant \mathcal{C}\|g\|_q,\end{aligned}$$

since the L^p-norm is bounded (see, for example, [**7**, p. 676]). Therefore, (2.7) implies (2.6).

In order to prove that (2.6) is a consequence of (2.7) it suffices use the same arguments of the proof of Theorem 2.3 (the part dealing with the necessary condition (2.9) ⇒ (2.11)) replacing p by ∞ and q by 1. So, the theorem is completely proved. □

PROOF OF PROPOSITION 2.1. The proof is based on the following result due to Gopengauz [**1**]: "*For every function $f \in C^s$, $s \geqslant 0$, there exists a polynomial $q \in \mathbb{P}_{m+2s-1}$ such that, for $i = 0, 1, \dots, s$, one has $q^{(i)}(\pm 1) = f^{(i)}(\pm 1)$ and*

$$\big|\big(f^{(i)}(x) - q^{(i)}(x)\big)\big| \leqslant \mathcal{C}\left(\frac{\sqrt{1-x^2}}{m}\right)^{s-i}\omega\left(f^{(i)}, \frac{\sqrt{1-x^2}}{m}\right)_\infty, \quad |x| \leqslant 1,$$

where $\mathcal{C} \neq \mathcal{C}(m, f, x)$ and $\omega(\cdot,\cdot)_\infty$ is the ordinary modulus of smoothness" (in the uniform norm).

Then, if $f \in C^{r-1}$, we have

$$v^\gamma(x)\big|f(x) - L_{m,r}(v^\alpha, f, x)\big| = \big|f(x) - q(x)\big|v^\gamma(x) + \Big|v^{\gamma+r}(x)L_{m,r}\Big(v^\alpha, \frac{f-q}{v^r}, x\Big)\Big|,$$

whence, using (2.6),

$$\big\|[f - L_{m,r}(v^\alpha, f)]v^\gamma\big\|_p \leqslant \mathcal{C}\big\|(f-q)v^\gamma\big\|_\infty.$$

Therefore, for every polynomial P of degree $m+2r-1$, we get

$$\big\|[f-L_{m,r}(v^\alpha,f)]v^\gamma\big\|_p \leqslant \mathcal{C}\big\|[(f-q)-P]v^\gamma\big\|_\infty$$

and, assuming the infimum on P, the estimate (2.12) follows.

Now we prove (2.13). Using the polynomial q of Gopengauz and (2.9), we get

$$\big\|[f-L_{m,r}(v^\alpha,f)]v^\gamma\big\|_p \leqslant \mathcal{C}\big\|(f-q)v^\gamma\big\|_p + \mathcal{C}\biggl(\sum_{k=1}^m \Delta x_k\Bigl[\omega\Bigl(f,\frac{\varphi(x_k)}{m}\Bigr)_\infty v^\gamma(x_k)\Bigr]^p\biggr)^{1/p}. \tag{3.5}$$

Now, we have

$$\begin{aligned}|f(x)-q(x)|v^\gamma(x) &\leqslant \mathcal{C}v^\gamma(x)\,\omega\Bigl(f,\frac{\varphi(x)}{m}\Bigr)_\infty\\ &\leqslant \mathcal{C}v^\gamma(x)\int_{x-\frac{\varphi(x)}{m}}^{x+\frac{\varphi(x)}{m}}|f'(t)|\,dt \leqslant \frac{\mathcal{C}}{m}\frac{m}{\varphi(x)}\int_{x-\frac{\varphi(x)}{m}}^{x+\frac{\varphi(x)}{m}}|f'\varphi v^\gamma|(t)\,dt,\end{aligned}$$

since $1\pm x\sim 1\pm t$ if $|x-t|\leqslant \mathcal{C}\frac{\varphi(x)}{m}$ for $x,t\in[x_1,x_m]$.

Then, using the maximal function of $f'\varphi v^\gamma$, the first summand in (3.5) is dominated by

$$\frac{\mathcal{C}}{m}\biggl(\int_{-1}^1\biggl(\frac{m}{\varphi(x)}\int_{x-\frac{\varphi(x)}{m}}^{x+\frac{\varphi(x)}{m}}|f'\varphi v^\gamma|(t)\,dt\biggr)^p dx\biggr)^{1/p} \leqslant \frac{\mathcal{C}}{m}\|f'\varphi v^\gamma\|_p.$$

Concerning the sum in (3.5), for a sufficiently large s, we have

$$\begin{aligned}\omega\Bigl(f,\frac{\varphi(x_k)}{m}\Bigr)_\infty v^\gamma(x_k) &\leqslant s\;\omega\Bigl(f,\frac{\varphi(x_k)}{sm}\Bigr)_\infty v^\gamma(x_k)\\ &\leqslant \mathcal{C}\;s\int_{x_k-\frac{\varphi(x_k)}{sm}}^{x_+\frac{\varphi(x_k)}{sm}}|f'(t)|v^\gamma(t)\,dt \leqslant \mathcal{C}\int_{x_{k-1}}^{x_{k+1}}|f'v^\gamma|(t)\,dt.\end{aligned}$$

Then, using the Hölder inequality in the latter integral, we get

$$\Delta x_k\Bigl[\omega\Bigl(f,\frac{\varphi(x_k)}{m}\Bigr)_\infty\Bigr]^p \leqslant \mathcal{C}(\Delta x_k)^p\int_{x_{k-1}}^{x_{k+1}}|f'v^\gamma|^p(t)\,dt \leqslant \frac{\mathcal{C}}{m^p}\int_{x_{k-1}}^{x_{k+1}}|f'\varphi v^\gamma|^p(t)\,dt,$$

for $k=1,\dots,m$ and $x_0=-1$. Adding up on k, we obtain that also the second term in (3.5) is dominated by $\frac{\mathcal{C}}{m}\|f'\varphi v^\gamma\|_p$. Consequently, in a usual way, we deduce

$$\|[f-L_{m,r}(v^\alpha,f)]v^\gamma\|_p \leqslant \frac{\mathcal{C}}{m}E_{m+2r-2}(f')_{v^\gamma\varphi,p}\,.$$

Iterating the latter relation, (2.13) follows. □

Proof of Corollary 2.1. The bound (2.14) is a consequance of (2.13) and the well-known estimate

$$\|(f-L_{m,r}(v^\alpha,f))^{(r)}v^\gamma\varphi^r\|_p \leqslant \mathcal{C}m^r\|(f-L_{m,r}(v^\alpha,f))v^\gamma\|_p + \|f^{(r)}\varphi^r v^\gamma\|_p$$

which holds for any $f\in W_p^r$, $1<p<\infty$, and $r\geqslant 1$. We omit the details. □

References

1. I. E. Gopengauz, *On a theorem of A. F. Timan on the approximation of functions by polynomials on a finite interval*, Math. Notes **1**(2) (1967), 110–116. [Translated from Mat. Zametki **1**(2) (1967), 163–172]
2. G. Mastroianni, *Some weighted polynomial inequalities*, Proceedings of the International Conference on Orthogonality, Moment Problems and Continued Fractions (Delft, 1994), J. Comput. Appl. Math. **65**(1–3) (1995), 279–292.
3. G. Mastroianni and G. V. Milovanović, *Interpolation Processes – Basic Theory and Applications*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2008.
4. G. Mastroianni and M. G. Russo, *Lagrange Interpolation in some weighted uniform spaces*, Facta Univ. Ser. Math. Inform. **12** (1997), 185–201.
5. G. Mastroianni and P. Vértesi, *Mean convergence of Lagrange interpolation on arbitrary system of nodes*, Acta Sci. Math. (Szeged) **57**(1–4) (1993), 429–441.
6. A. Máté, P. Nevai and V. Totik, *Necessary conditions for weighted mean convergence of Fourier series in orthogonal polynomials*, J. Approx. Theory **46**(2) (1986), 314–322.
7. P. G. Nevai, *Mean convergence of Lagrange interpolation. III*, Trans. Amer. Math. Soc. **282**(2) (1984), 669–698.
8. P. Nevai, *Hilbert transforms and Lagrange interpolation*, J. Approx. Theory **60**(3) (1990), 360–363.
9. G. Szegő, *Orthogonal Polynomials*, Fourth edition, Colloquium Publications 23, American Mathematical Society, Providence, R.I., 1975.

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