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# SYSTEMS OF ABSTRACT TIME-FRACTIONAL EQUATIONS

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Abstract. We analyze systems of abstract time-fractional equations in certain classes of sequentially complete locally convex spaces. We also consider arbitrary matrices of operators as generators of fractional regularized resolvent families, improving in such a way the results known for semigroups of operators.

## 1. Introduction and preliminaries

The fractional calculus is one of the active research fields in mathematical analysis, primarily from its importance in modeling of various problems in engineering, physics, chemistry and other sciences. Presumably the first systematic exposition on abstract time-fractional equations with Caputo fractional derivatives is that of Bazhlekova [**2**]. In this fundamental work, the abstract time-fractional equations with Caputo fractional derivatives have been studied by converting them into equivalent abstract Volterra equations [**17**].

The reading of paper [**7**] by Kisyński served as a starting point for the genesis of this paper. We shall prove a generalization of the assertion [**7**, Theorem 1, (a) $\Rightarrow$ (b)] for abstract time-fractional equations (Theorem 2.1, Remark 2.1). The second aim of the paper is to generalize [**3**, Theorem 14.1] to abstract time-fractional equations (Theorem 2.2), and to clarify some classes of sequentially complete locally convex spaces in which the above-mentioned result admits a reformulation (Theorem 2.3).

Throughout the paper, we assume that $E$ is a Hausdorff sequentially complete locally convex space, SCLCS for short, and that the abbreviation $\circledast$ stands for the fundamental system of seminorms which defines the topology of $E$. By $L(E)$ is denoted the space of all continuous linear mappings from $E$ into $E$. The domain, resolvent set, spectrum and range of a closed linear operator $A$ acting on $E$ are

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denoted by $D(A)$, $\rho(A)$, $\sigma(A)$ and $R(A)$, respectively. Put $D_\infty(A) := \bigcap_{n\in\mathbb{N}} D(A^n)$. Suppose $F$ is a linear subspace of $E$. Then the part of $A$ in $F$, denoted by $A_{|F}$, is a linear operator defined by $D(A_{|F}) := \{x \in D(A) \cap F : Ax \in F\}$ and $A_{|F}x := Ax$, $x \in D(A_{|F})$. If $B \in L(E)$ and $m \in \mathbb{N}$, then we define the operator $B_m \in L(E^m)$ (sometimes also denoted by $B$) by $B_m(x_1,\dots,x_m) := (Bx_1,\dots,Bx_m)$, $(x_1,\dots,x_m) \in E^m$. Given $s \in \mathbb{R}$ in advance, set $\lfloor s\rfloor := \sup\{l \in \mathbb{Z} : l \leqslant s\}$ and $\lceil s\rceil := \inf\{l \in \mathbb{Z} : s \leqslant l\}$. The Gamma function is denoted by $\Gamma(\cdot)$ and the principal branch is always used to take the powers. Set $0^\alpha := 0$ and $g_\alpha(t) := t^{\alpha-1}/\Gamma(\alpha)$ ($\alpha > 0$, $t > 0$). We refer the reader to [**15**, pp. 99–102] for the basic material concerning integration in SCLCSs, and to [**9**] for the definition and elementary properties of analytic functions with values in SCLCSs. The reader may consult [**16**] for the basic properties of distribution spaces used henceforward.

The following definition has been recently introduced in [**8, 9**].

Definition 1.1. Let $\alpha > 0$ and let $A$ be a closed linear operator on $E$. A strongly continuous family $(R_\alpha(t))_{t\geqslant 0}$ in $L(E)$ is said to be a (global) $(g_\alpha, C)$-*regularized resolvent family* having $A$ as a *subgenerator* iff the following holds:

(a) $R_\alpha(t)A \subseteq AR_\alpha(t),\ t \geqslant 0$, $R_\alpha(0) = C$ and $CA \subseteq AC$,
(b) $R_\alpha(t)C = CR_\alpha(t),\ t \geqslant 0$, and
(c) $R_\alpha(t)x = Cx + \int_0^t g_\alpha(t-s)AR_\alpha(s)x\,ds,\ t \geqslant 0,\ x \in D(A)$;

$(R_\alpha(t))_{t\geqslant 0}$ is said to be *exponentially equicontinuous* if there exists $\omega \in \mathbb{R}$ such that the family $\{e^{-\omega t}R_\alpha(t) : t \geqslant 0\}$ is equicontinuous.

The *integral generator* $\hat{A}$ of $(R_\alpha(t))_{t\geqslant 0}$ is defined by

$$\hat{A} := \left\{(x,y) \in E \times E : R_\alpha(t)x - Cx = \int_0^t g_\alpha(t-s)R_\alpha(s)y\,ds,\ t \geqslant 0\right\}.$$

Suppose that $(R_\alpha(t))_{t\geqslant 0}$ is exponentially equicontinuous and that the following equality holds:

$$R_\alpha(t)x - Cx = A\int_0^t g_\alpha(t-s)R_\alpha(s)x\,ds,\quad t \geqslant 0,\quad x \in E.$$

Then $\hat{A}$ is the maximal subgenerator of $(R_\alpha(t))_{t\geqslant 0}$ with respect to the set inclusion and $\hat{A} = C^{-1}AC$. Notice also that the above equality holds provided that $A$ is densely defined and that $(R_\alpha(t))_{t\geqslant 0}$ is locally equicontinuous [**9**].

Let $\alpha > 0$, $\beta \in \mathbb{R}$ and the *Mittag-Leffler function* $E_{\alpha,\beta}(z)$ be defined by $E_{\alpha,\beta}(z) := \sum_{n=0}^\infty z^n/\Gamma(\alpha n+\beta)$, $z \in \mathbb{C}$. In this place, we assume that $1/\Gamma(\alpha n+\beta) = 0$ if $\alpha n + \beta \in -\mathbb{N}_0$. Set, for short, $E_\alpha(z) := E_{\alpha,1}(z)$, $z \in \mathbb{C}$. Then it is well known that, for every $\alpha > 0$, there exists $c_\alpha > 0$ such that:

$$E_\alpha(t) \leqslant c_\alpha \exp\bigl(t^{1/\alpha}\bigr),\quad t \geqslant 0. \tag{1.1}$$

The asymptotic expansion of the entire function $E_{\alpha,\beta}(z)$ is given in the following lemma (cf. [**19**, Theorem 1.1]).

LEMMA 1.1. *Let $0 < \sigma < \frac{1}{2}\pi$. Then, for every $z \in \mathbb{C} \smallsetminus \{0\}$ and $m \in \mathbb{N} \smallsetminus \{1\}$:*

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha}\sum_s Z_s^{1-\beta}e^{Z_s} - \sum_{j=1}^{m-1}\frac{z^{-j}}{\Gamma(\beta-\alpha j)} + O(|z|^{-m}),\ |z| \to \infty,$$

*where $Z_s$ is defined by $Z_s := z^{1/\alpha}e^{2\pi i s/\alpha}$ and the first summation is taken over all those integers $s$ satisfying $|\arg z + 2\pi s| < \alpha(\frac{\pi}{2}+\sigma)$.*

Let $E$ be one of the spaces $L^p(\mathbb{R}^n)$ $(1 \leqslant p \leqslant \infty)$, $C_0(\mathbb{R}^n)$, $C_b(\mathbb{R}^n)$, $BUC(\mathbb{R}^n)$, $C^\sigma(\mathbb{R}^n)$ $(0 < \sigma < 1)$ and let $0 \leqslant l \leqslant n$. Put $\mathbb{N}_0^l := \{\eta \in \mathbb{N}_0^n : \eta_{l+1} = \cdots = \eta_n = 0\}$ and recall that the space $E_l$ is defined by $E_l := \{f \in E : f^{(\eta)} \in E \text{ for all } \eta \in \mathbb{N}_0^l\}$. The totality of seminorms $(q_\eta(f) := ||f^{(\eta)}||_E,\ f \in E_l;\ \eta \in \mathbb{N}_0^l)$ induces a Fréchet topology on $E_l$ (cf. the proof of [**20**, Lemma 5.6, p. 25]). Put $D^\eta\cdot := (-i)^{|\eta|}\cdot^{(\eta)}$ $(\eta \in \mathbb{N}_0^n)$.

In the proofs of our main results, we will make use of the functional calculus for commuting generators of bounded $C_0$-groups ([**3**]). Denote by $\mathcal{F}$ and $\mathcal{F}^{-1}$ the $n$-dimensional Fourier transform and its inverse transform, respectively. That is

$$(\mathcal{F}f)(\xi) := \int_{\mathbb{R}^n} e^{i(x,\xi)}f(x)\,dx,\ \xi \in \mathbb{R}^n \text{ and } \mathcal{F}^{-1} := (2\pi)^{-n}\hat{\mathcal{F}},$$

where $\hat{}$ denotes the reflection in 0. Let $(E, \|\cdot\|)$ be a complex Banach space, $n \in \mathbb{N}$ and $iA_j,\ 1 \leqslant j \leqslant n$ be commuting generators of bounded $C_0$-groups on $E$. Set $A := (A_1, \ldots, A_n)$ and $A^\eta := A_1^{\eta_1}\cdots A_n^{\eta_n}$ for any $\eta = (\eta_1, \ldots, \eta_n) \in \mathbb{N}_0^n$. If $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ and $u \in \mathcal{A} := \{f \in C_0(\mathbb{R}^n) : \mathcal{F}f \in L^1(\mathbb{R}^n)\}$, put $|\xi| := (\sum_{j=1}^n \xi_j^2)^{1/2}$, $(\xi, A) := \sum_{j=1}^n \xi_j A_j$ and

$$u(A)x := (2\pi)^{-n}\int_{\mathbb{R}^n}\mathcal{F}u(\xi)e^{-i(\xi,A)}x\,d\xi,\ x \in E.$$

Then $u(A) \in L(E)$, $u \in \mathcal{A}$ and there exists a constant $M < \infty$ such that $\|u(A)\| \leqslant M\|\mathcal{F}u\|_{L^1(\mathbb{R}^n)}$, $u \in \mathcal{A}$.

Put $Z_n := \mathcal{F}\mathcal{D}(\mathbb{R}^n)$ and assume that $Z_n$ is equipped with the topology transported by $\mathcal{F}$ from $\mathcal{D}(\mathbb{R}^n)$. By $Z_n'$ we denote the strong dual of $Z_n$. It is clear that $Z_n = \mathcal{F}^{-1}\mathcal{D}(\mathbb{R}^n)$ and that the dual mapping of $\mathcal{F}_{|Z_n} : Z_n \to \mathcal{D}(\mathbb{R}^n)$ is an isomorphism of $\mathcal{D}'(\mathbb{R}^n)$ onto $Z_n'$. We have the following equality:

$$\langle \mathcal{F}T, \mathcal{F}\varphi\rangle = (2\pi)^n\langle T, \hat{\varphi}\rangle,\ \ T \in \mathcal{D}'(\mathbb{R}^n),\ \ \varphi \in \mathcal{D}(\mathbb{R}^n).$$

The operator $\partial/\partial x_j : Z_n' \to Z_n'$ is defined as the dual operator of $-\partial/\partial x_j : Z_n \to Z_n$, so that $\partial/\partial x_j\mathcal{F}T = \mathcal{F}(i\xi_j T)$, $T \in \mathcal{D}'(\mathbb{R}^n)$, $1 \leqslant j \leqslant n$ the actions of $\mathcal{F}$ on $(\mathcal{D}'(\mathbb{R}^n))^m$ and of $\mathcal{F}^{-1}$ on $(Z_n')^m$ are coordinatewise.

Let $m, n, d \in \mathbb{N}$ and let $M_m$ denote the ring of all complex matrices of format $m \times m$. Define $P(x) := \sum_{|\eta|\leqslant d} A_\eta x^\eta,\ x \in \mathbb{R}^n$ $(A_\eta \in M_m)$, $P(\partial/\partial x) := \sum_{|\eta|\leqslant d} A_\eta(\partial/\partial x)^\eta$, and $\tilde{P}(\xi) := \sum_{|\eta|\leqslant d} i^{|\eta|}A_\eta\xi^\eta$. Denote by $\lambda_1(\xi), \ldots, \lambda_m(\xi)$ the eigenvalues of $\tilde{P}(\xi)$ $(\xi \in \mathbb{R}^n)$.

In the sequel, we will always consider the case in which the space $E$ is barreled, so that every $(g_\alpha, C)$-regularized resolvent family $(R_\alpha(t))_{t\geqslant 0}$ in $E$ is locally equicontinuous (cf. [**9**, Remark 2.2]). The results obtained in this paper can be used in the analysis of the following system of abstract time-fractional equations with $\alpha > 0$:

$$\mathbf{D}_t^\alpha \vec{u}(t) = P(\partial/\partial x)_{|E}\vec{u}(t), \ \ t > 0; \ \ \vec{u}^{(k)}(0) = \vec{x_k}, \ \ k = 0, 1, \dots, \lceil\alpha\rceil - 1, \tag{1.2}$$

where $\mathbf{D}_t^\alpha$ denotes the *Caputo fractional derivative of order* $\alpha$ ([**2, 10**]). If $\vec{x_k} \in C(D(P(\partial/\partial x)_{|E}))$, $k = 0, 1, \dots, \lceil\alpha\rceil - 1$, then the unique solution of (1.2) is given by:

$$\vec{u}(t) = R_\alpha(t)C^{-1}\vec{x_0} + \sum_{j=1}^{\lceil\alpha\rceil}\int_0^t \frac{(t-s)^{j-1}}{(j-1)!}R_\alpha(s)C^{-1}\vec{x_{j-1}}\,ds, \ \ t \geqslant 0.$$

Let $\alpha > 0$. Then it is not difficult to prove that the formula $G_\alpha(t)\vec{f} := E_\alpha(t^\alpha\tilde{P}(\xi))\vec{f}$, $t \geqslant 0$, $\vec{f} \in (\mathcal{D}'(\mathbb{R}^n))^m$, determines a global $(g_\alpha, I)$-regularized resolvent family on $(\mathcal{D}'(\mathbb{R}^n))^m$, and that the integral generator of $(G_\alpha(t))_{t\geqslant 0}$ is the multiplication operator $\tilde{P}(\xi)_{|(\mathcal{D}'(\mathbb{R}^n))^m} \in L((\mathcal{D}'(\mathbb{R}^n))^m)$. Furthermore, the formula

$$R_\alpha(t)\vec{f} := \mathcal{F}E_\alpha\big(t^\alpha\tilde{P}(\xi)\big)\mathcal{F}^{-1}\vec{f}, \ \ t \geqslant 0, \ \ \vec{f} \in (Z_n')^m,$$

determines a global $(g_\alpha, I)$-regularized resolvent family $(R_\alpha(t))_{t\geqslant 0}$ on $(Z_n')^m$. The operator $P(\partial/\partial x)_{|(Z_n')^m} \in L((Z_n')^m)$ is the integral generator of $(R_\alpha(t))_{t\geqslant 0}$, and $(G_\alpha(t))_{t\geqslant 0}$ as well as $(R_\alpha(t))_{t\geqslant 0}$ can be extended to the whole complex plane. The following holds:

- (i) Let $\alpha \in (0,\infty) \smallsetminus \mathbb{N}$ and $\vec{f} \in (Z_n')^m$.
  - (i.1) The mapping $z \mapsto R_\alpha(z)\vec{f}$, $z \in \mathbb{C} \smallsetminus (-\infty, 0]$ is analytic.
  - (i.2) The mapping $t \mapsto R_\alpha(t)\vec{f}$, $t \geqslant 0$ belongs to the space $C^{\lfloor\alpha\rfloor}([0,\infty) : (Z_n')^m)$.
  - (i.3) For every compact set $K \subseteq \mathbb{C} \smallsetminus (-\infty, 0]$, the family $\{R_\alpha(z) : z \in K\} \subseteq L((Z_n')^m)$ is equicontinuous.
- (ii) Let $\alpha \in \mathbb{N}$ and $\vec{f} \in (Z_n')^m$.
  - (ii.1) The mapping $z \mapsto R_\alpha(z)\vec{f}$, $z \in \mathbb{C}$ is entire.
  - (ii.2) For every compact set $K \subseteq \mathbb{C}$, the family $\{R_\alpha(z) : z \in K\} \subseteq L((Z_n')^m)$ is equicontinuous.

Observe also that the above assertions continue to hold for $(G_\alpha(t))_{t\geqslant 0}$ and $\vec{f} \in (\mathcal{D}'(\mathbb{R}^n))^m$, and that, for every $z \in \mathbb{C}$, $R_\alpha(z)Z_n^m \subseteq Z_n^m$ and $R_\alpha(z)(\mathcal{F}\mathcal{E}'(\mathbb{R}^n))^m \subseteq (\mathcal{F}\mathcal{E}'(\mathbb{R}^n))^m$. This implies that $(R_\alpha(t)_{|Z_n^m})_{t\geqslant 0}$, resp. $(R_\alpha(t)_{|(\mathcal{F}\mathcal{E}'(\mathbb{R}^n))^m})_{t\geqslant 0}$, is a locally equicontinuous $(g_\alpha, I)$-regularized resolvent family generated by $P(\partial/\partial x)_{|Z_n^m}$, resp. $P(\partial/\partial x)_{|(\mathcal{F}\mathcal{E}'(\mathbb{R}^n))^m}$.

## 2. Formulation and proof of main results

In the following theorem, we will transfer the assertion of [**7**, Theorem 1, (a) $\Rightarrow$ (b)] to abstract time-fractional equations.

THEOREM 2.1. *Suppose* $\omega \geqslant 0$, $0 < \alpha \leqslant 2$ *and*

$$\sup_{z\in\sigma(\tilde{P}(\xi))} \Re\big(z^{1/\alpha}\big) \leqslant \omega. \tag{2.1}$$

*Let $E$ be one of the spaces listed below:*

(i) *$E = (\mathcal{S}(\mathbb{R}^n))^m$ or $E = (\mathcal{S}'(\mathbb{R}^n))^m$.*

(ii) *$E = X_n$, where $X$ is $L^p(\mathbb{R}^n)$ $(1 \leqslant p \leqslant \infty)$, $C_0(\mathbb{R}^n)$, $C_b(\mathbb{R}^n)$, $BUC(\mathbb{R}^n)$ or $C^\sigma(\mathbb{R}^n)$ $(0 < \sigma < 1)$.*

(iii) *$E = \{\vec{f} \in (L^2(\mathbb{R}^n))^m : (P(\partial/\partial x))^l \vec{f} \in (L^2(\mathbb{R}^n))^m$ for all $l \in \mathbb{N}\}$, with the topology induced by the following family of seminorms:*

$$\|\vec{f}\|_l := \big\|\big(P(\partial/\partial x)\big)^l \vec{f}\big\|_{(L^2(\mathbb{R}^n))^m} \quad (\vec{f} \in E, \ l \in \mathbb{N}_0).$$

*Then the operator $P(\partial/\partial x)_{|E}$ is the integral generator of a global $(g_\alpha, I)$-regularized resolvent family $(S_\alpha(t))_{t \geqslant 0}$ on $E$ satisfying that, for every $p \in \circledast$ and $\epsilon > 0$, there exist $M \geqslant 1$ and $q \in \circledast$ such that:*

$$p\big(e^{-(\omega+\epsilon)t} S_\alpha(t)\big) \leqslant M q(x), \quad t \geqslant 0, \quad x \in E. \tag{2.2}$$

Proof. We will prove the theorem provided that $\alpha \in (0,2) \smallsetminus \{1\}$. Let $\epsilon > 0$ be fixed and let $\Gamma_\epsilon$ denote the boundary of the region $\{z \in \mathbb{C} : \Re(z^{1/\alpha}) \leqslant \omega + \epsilon\}$.

I. The case $1 < \alpha < 2$. Suppose a positively oriented curve $C_\xi$ encircles the spectrum of $\tilde{P}(\xi)$ and is a subset of $\{z \in \mathbb{C} : \Re(z^{1/\alpha}) \leqslant \omega + \epsilon/2\}$ $(\xi \in \mathbb{R}^n)$. Notice that, for every $\xi_0 \in \mathbb{R}^n$, there exists an open neighborhood $U_{\xi_0}$ of $\xi_0$ such that $C_{\xi_0}$ encircles the spectrum of $\tilde{P}(\xi)$ for all $\xi \in U_{\xi_0}$. Let $z_0 > (\omega + 2\epsilon)^\alpha$. Using [**7**, Theorem II, (iii)] and the Cauchy integral formula, we obtain that there exists $v \in \mathbb{N}$ such that:

$$\begin{aligned} E_\alpha\big(t^\alpha \tilde{P}(\xi)\big) &= \big(\tilde{P}(\xi) - z_0 I\big)^2 \frac{1}{2\pi i} \oint_{C_\xi} \frac{E_\alpha\big(t^\alpha z\big)}{\big(z - z_0\big)^2} \big(zI - \tilde{P}(\xi)\big)^{-1} dz \\ &= \big(\tilde{P}(\xi) - z_0 I\big)^2 \big[a_0(t,\xi) I + \cdots + a_{m-1}(t,\xi) \tilde{P}(\xi)^{m-1}\big], \quad t \geqslant 0, \ \xi \in \mathbb{R}^n, \end{aligned} \tag{2.3}$$

where $a_j(t,\xi)$ can be written as a finite sum, with coefficients independent of $t$ and $\xi$, of terms like

$$S_{i_1,\dots,i_m;l}(t,\xi) = \lambda_1^{i_1}(\xi) \cdots \lambda_m^{i_m}(\xi) \frac{1}{2\pi i} \oint_{C_\xi} \frac{E_\alpha\big(t^\alpha z\big)}{\big(z - z_0\big)^2} \frac{dz}{(z - \lambda_1(\xi)) \cdots (z - \lambda_l(\xi))},$$

where $t \geqslant 0$, $\xi \in \mathbb{R}^n$, $i_j \in \mathbb{N}_0$ for $1 \leqslant j \leqslant m$ and $i_1 + \cdots + i_m \leqslant v$. By Lemma 1.1 and (2.1) (cf. also the proof of [**11**, Theorem 2.1]), we get that there exists $M_\epsilon > 0$ such that $|E_\alpha(t^\alpha z)| \leqslant M_\epsilon e^{(\omega+\epsilon)t}$ for all $t \geqslant 0$ and $z \in \mathbb{C}$ with $\Re(z^{1/\alpha}) \leqslant \omega + \epsilon/2$. Since $\operatorname{dist}(\Gamma_{\epsilon/2}, \Gamma_\epsilon) := \kappa(\omega, \alpha, \epsilon) > 0$, the residue theorem implies that, for every $t \geqslant 0$ and $\xi \in \mathbb{R}^n$:

$$S_{i_1,\dots,i_m;l}(t,\xi) = \lambda_1^{i_1}(\xi) \cdots \lambda_m^{i_m}(\xi) \frac{1}{2\pi i} \int_{\Gamma_\epsilon} \frac{E_\alpha\big(t^\alpha z\big)}{\big(z - z_0\big)^2} \frac{dz}{(z - \lambda_1(\xi)) \cdots (z - \lambda_l(\xi))},$$

which yields the existence of a number $\sigma > 0$ such that:

$$\big|S_{i_1,\dots,i_m;l}(t,\xi)\big| \leqslant \frac{M_\epsilon}{2\pi} \big(1 + |\xi|\big)^\sigma e^{(\omega+\epsilon)t} \int_{\Gamma_\epsilon} \frac{d|z|}{\big|z - z_0\big|^2},$$

provided $i_j \in \mathbb{N}_0$ for $1 \leqslant j \leqslant m$ and $i_1 + \cdots + i_m \leqslant v$. In combination with (2.3), the above implies that there exist $N_\epsilon > 0$ and $\sigma_1 > 0$ such that:

$$\big\|E_\alpha\big(t^\alpha \tilde{P}(\xi)\big)\big\| \leqslant N_\epsilon \big(1 + |\xi|\big)^{\sigma_1} e^{(\omega+\epsilon)t}, \quad t \geqslant 0, \ \xi \in \mathbb{R}^n. \tag{2.4}$$

Now we will prove that, for every multi-index $\eta \in \mathbb{N}_0^n$ with $|\eta| > 0$, there exist $N_{\epsilon,\eta} > 0$ and $\sigma_\eta > 0$ such that:

$$\big\| D^\eta \big( E_\alpha \big( t^\alpha \tilde{P}(\xi) \big) \big) \big\| \leqslant N_{\epsilon,\eta} \big( 1 + |\xi| \big)^{\sigma_\eta} e^{(\omega+\epsilon)t}, ;\ t \geqslant 0,\ \ \xi \in \mathbb{R}^n. \tag{2.5}$$

Noticing that $D^{-1} = \operatorname{adj}(D)/\det(D)$ for every regular matrix $D \in M_m$, we obtain that there exist $l_\eta \in \mathbb{N}$ and polynomials $q_{ij}^\eta(\xi, z)$ in $(n+1)$ variables such that, for every $\xi \in \mathbb{R}^n$ and $z \in \rho(\tilde{P}(\xi))$:

$$D^\eta \big( zI - \tilde{P}(\xi) \big)^{-1} = \frac{[q_{ij}^\eta(\xi, z)]_{1 \leqslant i,j \leqslant m}}{\big( z - \lambda_1(\xi) \big)^{l_\eta} \cdots \big( z - \lambda_m(\xi) \big)^{l_\eta}}.$$

Set $N_\eta := \max\{ dg(q_{ij}^\eta(\xi, z)) : 1 \leqslant i, j \leqslant m \} + 2$. By the Cauchy integral formula, one has:

$$E_\alpha \big( t^\alpha \tilde{P}(\xi) \big) = \big( \tilde{P}(\xi) - z_0 I \big)^{N_\eta} \frac{1}{2\pi i} \oint_{C_\xi} \frac{E_\alpha \big( t^\alpha z \big)}{\big( z - z_0 \big)^{N_\eta}} \big( zI - \tilde{P}(\xi) \big)^{-1} dz,\ \ t \geqslant 0,\ \ \xi \in \mathbb{R}^n.$$

Further on,

$$\begin{aligned}
&D^\eta \frac{1}{2\pi i} \oint_{C_\xi} \frac{E_\alpha \big( t^\alpha z \big)}{\big( z - z_0 \big)^{N_\eta}} \big( zI - \tilde{P}(\xi) \big)^{-1} dz \\
&\qquad = \frac{1}{2\pi i} \oint_{C_\xi} \frac{E_\alpha \big( t^\alpha z \big)}{\big( z - z_0 \big)^{N_\eta}} D^\eta \big( zI - \tilde{P}(\xi) \big)^{-1} dz \\
&\qquad = \frac{1}{2\pi i} \oint_{C_\xi} \frac{E_\alpha \big( t^\alpha z \big)}{\big( z - z_0 \big)^{N_\eta}} \frac{\big[ q_{ij}^\eta(\xi, z) \big]_{1 \leqslant i,j \leqslant m} dz}{\big( z - \lambda_1(\xi) \big)^{l_\eta} \cdots \big( z - \lambda_m(\xi) \big)^{l_\eta}} \\
&\qquad = \frac{1}{2\pi i} \int_{\Gamma_\epsilon} \frac{E_\alpha \big( t^\alpha z \big)}{\big( z - z_0 \big)^{N_\eta}} \frac{\big[ q_{ij}^\eta(\xi, z) \big]_{1 \leqslant i,j \leqslant m} dz}{\big( z - \lambda_1(\xi) \big)^{l_\eta} \cdots \big( z - \lambda_m(\xi) \big)^{l_\eta}} &(2.6) \\
&\qquad \leqslant M \kappa(\omega, \alpha, \epsilon)^{-m l_\eta} \frac{M_\epsilon}{2\pi} e^{(\omega+\epsilon)t} \big( 1 + |\xi| \big)^{N_\eta - 2} \int_{\Gamma_\epsilon} \frac{\big( 1 + |z| \big)^{N_\eta - 2}}{\big| z - z_0 \big|^{N_\eta}} \, d|z|, &(2.7)
\end{aligned}$$

where (2.6) follows from the residue theorem. Using the matrix differentiation rules and (2.7), we immediately obtain (2.5).

(i): Let $E = (\mathcal{S}(\mathbb{R}^n))^m$. By the invariance of $E$ under the Fourier transform, it follows that $S_\alpha(t) := R_\alpha(t)_{|E} \in L(E)$ for all $t \geqslant 0$. Having in mind [**7**, Theorem B] and (2.4)–(2.5), we get that, for every $p \in \circledast$, there exist $M \geqslant 1$ and $q \in \circledast$ such that (2.2) holds. Let $\vec{x} \in E$, let $t \geqslant 0$ and let $\vec{x_n}$ be a sequence in $Z_n^m$ such that $\lim_{n \to \infty} \vec{x_n} = \vec{x}$ in $E$. Suppose that $p$ is a continuous seminorm on $E$. Let $M \geqslant 1$ and $q \in \circledast$ be such that (2.2) holds. Then $p_{|Z_n^m}$ is a continuous seminorm on $Z_n^m$, and the strong continuity of $(S_\alpha(t))_{t \geqslant 0}$ simply follows from the following estimate:

$$p \big( S_\alpha(t) \vec{x} - S_\alpha(s) \vec{x} \big) \leqslant M \big( e^{(\omega+\epsilon)t} + e^{(\omega+\epsilon)s} \big) + p_{|Z_n^m} \big( R_\alpha(t) \vec{x_n} - R_\alpha(s) \vec{x_n} \big).$$

Therefore, $(S_\alpha(t))_{t \geqslant 0}$ is an exponentially equicontinuous $(g_\alpha, I)$-regularized resolvent family generated by $P(\partial/\partial x)_{|E}$. The proof is quite similar in the case $E = (\mathcal{S}'(\mathbb{R}^n))^m$.

(ii): Suppose first $X \neq C^{\sigma}(\mathbb{R}^n)$. Then estimates (2.4)–(2.5), taken together with the product rule and the Bernstein's lemma [**1**, Lemma 8.2.1], imply that there exists a sufficiently large $v \in \mathbb{N}$ such that, for given $t \geqslant 0$ in advance, every entry of the matrix $f_t(\xi) \equiv [E_\alpha(t^\alpha \tilde{P}(\xi))(1 + |\xi|^2)^{-v}]$ belongs to $\mathcal{A}$. Then it is not difficult to prove that the expression $(W_\alpha(t) \equiv f_t(-i\partial/\partial x_1, \dots, -i\partial/\partial x_n))_{t\geqslant 0}$ (cf. the previous section for the definition of functional calculus) determines an exponentially bounded $(g_\alpha, (1-\Delta)^{-v})$-regularized resolvent family generated by $P(\partial/\partial x)_{|X}$. Furthermore, $\|W_\alpha(t)\|_X = O(e^{(\omega+\epsilon)t})$, $t \geqslant 0$. By the definition of topology of $E$, it follows that $(R_\alpha(t) \equiv W_\alpha(t)W_\alpha(0)^{-1})_{t\geqslant 0}$ is an exponentially equicontinuous $(g_\alpha, I)$-regularized resolvent family generated by $P(\partial/\partial x)_{|E}$, and that, for every $p \in \circledast$, there exist $M \geqslant 1$ and $q \in \circledast$ such that (2.2) holds. Keeping in mind the assertion [**6**, b), p. 374], a similar proof works in the case $X = C^{\sigma}(\mathbb{R}^n)$ $(0 < \sigma < 1)$.

(iii): Let $Q$ be the totality of indexes $q = (j_1, \dots, j_s)$ where $1 \leqslant s \leqslant m$ and $1 \leqslant j_1 < \cdots < j_s \leqslant m$. By [**4**, Lemma 3] (cf. also [**18**, Lemma 10.1]), we obtain that there exist absolute constants $\alpha^k_{p,q}$ $(0 \leqslant p \leqslant m-1,\ q \in Q,\ 0 \leqslant k \leqslant m-1)$ such that, for every $t \geqslant 0$ and $\xi \in \mathbb{R}^n$,

$$
\begin{aligned}
E_\alpha\big(t^\alpha \tilde{P}(\xi)\big) &= \big(\tilde{P}(\xi) - z_0 I\big)^{m+1} \\
&\quad \times \sum_{k=0}^{m-1} \Bigg( \sum_{0\leqslant p\leqslant m-1, q\in Q} \alpha^k_{p,q} \frac{1}{2\pi i} \oint_{C_\xi} \frac{z^p E_\alpha\big(t^\alpha z\big)}{\big(z - z_0\big)^{m+1} \prod_{j\in q}\big(z - \lambda_j(\xi)\big)}\, dz \Bigg) \tilde{P}(\xi)^k \\
&= \sum_{k=0}^{m-1} \Bigg( \sum_{0\leqslant p\leqslant m-1, q\in Q} \alpha^k_{p,q} \frac{1}{2\pi i} \oint_{C_\xi} \frac{z^p E_\alpha\big(t^\alpha z\big)}{\big(z - z_0\big)^{m+1} \prod_{j\in q}\big(z - \lambda_j(\xi)\big)}\, dz \Bigg) \\
&\quad \times \big(\tilde{P}(\xi) - z_0 I\big)^{m+1} \tilde{P}(\xi)^k.
\end{aligned}
$$

Then one gets the existence of a number $K_\epsilon > 0$ such that, for every $t \geqslant 0$, $\xi \in \mathbb{R}^n$, $0 \leqslant p \leqslant m-1$ and $q \in Q$,

$$
\begin{aligned}
&\Big\| \frac{1}{2\pi i} \oint_{C_\xi} \frac{z^p E_\alpha\big(t^\alpha z\big)}{\big(z - z_0\big)^{m+1} \prod_{j\in q}\big(z - \lambda_j(\xi)\big)}\, dz \Big\| \\
&\qquad = \Big\| \frac{1}{2\pi i} \int_{\Gamma_\epsilon} \frac{z^p E_\alpha\big(t^\alpha z\big)}{\big(z - z_0\big)^{m+1} \prod_{j\in q}\big(z - \lambda_j(\xi)\big)}\, dz \Big\| \\
&\qquad\qquad \leqslant K_\epsilon e^{(\omega+\epsilon)t} \int_{\Gamma_\epsilon} \frac{\big(1 + |z|\big)^{m-1}}{\big|z - z_0\big|^{m+1}}\, d|z|,
\end{aligned}
$$

and the proof of [**7**, Theorem 1(v)] can be repeated verbatim.

II. The case $0 < \alpha < 1$. Although technically complicated, the proof of the theorem in this case is almost the same as the proof of the theorem in the case I. The essential change is only the passing from the integration along the curve $C_\xi$, by using the residue theorem, to the integration along $\Gamma_\epsilon$. Put $k_0 := \lceil 1/2\alpha \rceil$ and suppose first that $\omega = 0$. Then $\Re(re^{i\theta}) \leqslant 0$ ($r > 0$, $\theta \in (-\pi, \pi]$) iff

$$\theta \in \Biggl(\bigcup_{k'=0}^{k_0-1} \pm\Bigl[\alpha\frac{(4k'+1)\pi}{2}, \alpha\frac{(4k'+3)\pi}{2}\Bigr]\Biggr) \cap (-\pi, \pi] =: S_\alpha.$$

Furthermore, the set $\Phi_\alpha := \{z \in \mathbb{C} \smallsetminus \{0\} : \arg(z) \in (-\pi, \pi] \smallsetminus S_\alpha\}$ has a finite number of connected components, which implies that $\Gamma_\epsilon = \{(\epsilon/\cos(\theta/\alpha))^\alpha e^{i\theta} : \theta \in (-\pi, \pi] \smallsetminus S_\alpha\}$ can be represented as a finite union of smooth curves. Set $\Gamma_{\epsilon,R} := \Gamma_\epsilon \cap \{z \in \mathbb{C} : |z| = R\}$ ($R > 0$). Then there exists $M_\epsilon > 0$ such that $|E_\alpha(t^\alpha z)| \leqslant M_\epsilon e^{\epsilon t}$, $t \geqslant 0$, $z \in \bigcup_{R>0}(\Gamma_{\epsilon,R})^\circ$. This implies that, for every $\xi \in \mathbb{R}^n$, there exists a sufficiently large $R_\xi > 0$ such that, for every $R \geqslant R_\xi$, the path of integration $C_\xi$, in any of the integrals considered in the case I, can be deformed into the curve $\Gamma_{\epsilon,R}$. Now the claimed assertion follows by observing that the distance between $\partial\Phi_\alpha$ and $\Gamma_\epsilon$ is positive, and that

$$\lim_{R\to\infty} \oint_{z\in\Gamma_{\epsilon,R},|z|=R} \frac{E_\alpha\bigl(t^\alpha z\bigr)}{\bigl(z - z_0\bigr)^2}\, dz = 0.$$

If $\omega > 0$, then $\Re(re^{i\theta}) \leqslant \omega$ ($r > 0$, $\theta \in (-\pi, \pi]$) is equivalent to

$$\theta \in S_\alpha \text{ or } \bigl(\theta \in (-\pi, \pi] \smallsetminus S_\alpha \text{ and } r \leqslant (\omega/(\cos(\theta/\alpha)))^\alpha\bigr),$$

so that the proof follows similarly as in the case $\omega = 0$. □

Remark 2.1. (i) Let $(E, \|\cdot\|)$ be a complex Banach space and let $iA_j$, $1 \leqslant j \leqslant n$ be commuting generators of bounded $C_0$-groups on $E$. For a polynomial matrix $P(x) = \sum_{|\eta|\leqslant d} P_\eta x^\eta$ ($P_\eta \in M_m$), we define $P(A) \equiv \sum_{|\eta|\leqslant d} P_\eta A^\eta$ with a maximal domain. Then it is well known that $P(A)$ is closable. Suppose $\omega \geqslant 0$ and

$$\sup_{x\in\mathbb{R}^n} \bigl\{\Re\bigl(\lambda(x)^{1/\alpha}\bigr) : \lambda(x) \in \sigma(P(x))\bigr\} \leqslant \omega.$$

Then the proof of Theorem 2.1(I) implies that there exists a sufficiently large $\sigma > 0$ such that $\overline{P(A)}$ is the integral generator of a global $(g_\alpha, (1 + |A|^2)^{-\sigma})$-regularized resolvent family $(S_\alpha(t))_{t\geqslant 0}$ on $E^m$ satisfying that, for every $\epsilon > 0$, there exists $M_\epsilon \geqslant 1$ such that $||S_\alpha(t)|| \leqslant M_\epsilon e^{(\omega+\epsilon)t}$, $t \geqslant 0$. Disappointingly, our method produces a completely imprecise estimate for the lower bound of $\sigma$; the additional difficulty is that the equality

$$D^\eta\bigl(E_\alpha\bigl(t^\alpha P(x)\bigr)\bigr) = \sum_{j=1}^{|\eta|} t^{\alpha j} E_\alpha^{(j)}\bigl(t^\alpha P(x)\bigr) Q_j(x), \quad t \geqslant 0, \quad x \in \mathbb{R}^n, \quad m = 1,$$

where $Q_j(x)$ are complex polynomials of degree $\leqslant Nj - |\eta|$ ($1 \leqslant j \leqslant |\eta|$), cannot be so easily interpreted in the matricial case $m > 1$. Using distributional techniques, the above generation result remains true, with suitable modifications, in the case that $E$ is $L^\infty(\mathbb{R}^n)$, $C_b(\mathbb{R}^n)$ or $C^\sigma(\mathbb{R}^n)$ ($0 < \sigma < 1$).

(ii) In contrast to [**7**, Theorem 1], Theorem 2.1(ii) covers the case $E = X_n$, where $X$ is $L^p(\mathbb{R}^n)$ ($p \in [1, \infty) \smallsetminus \{2\}$), $C_0(\mathbb{R}^n)$ or $C^\sigma(\mathbb{R}^n)$ ($0 < \sigma < 1$). Notice also that it is not clear how one can transfer the implication [**7**, Theorem 1, (b) ⇒ (a)] to abstract time-fractional equations.

Now we will prove the following extension of [**3**, Theorem 14.1].

THEOREM 2.2. *Let $(E, \|\cdot\|)$ be a complex Banach space and let $iA_j$, $1 \leqslant j \leqslant n$ be commuting generators of bounded $C_0$-groups on $E$. Suppose $\alpha > 0$ and $P(x) = \sum_{|\eta| \leqslant d} P_\eta x^\eta$ ($P_\eta \in M_m$, $x \in \mathbb{R}^n$) is a polynomial matrix. Then there exists an injective operator $L(E) \ni C$ with a dense range such that the operator $\overline{P(A)}$ is the integral generator of a global $(g_\alpha, C_m)$-regularized resolvent family $(W_\alpha(t))_{t \geqslant 0}$ on $E^m$. Furthermore, the mapping $t \mapsto W_\alpha(t)$, $t \geqslant 0$ can be extended to the entire complex plane and the following holds:*

(i) $R(W_\alpha(z)) \subseteq D_\infty(P(A))$, $z \in \mathbb{C}$ *and*

$$\overline{P(A)} \int_0^z g_\alpha(z-s) W_\alpha(s)\vec{x}\,ds = W_\alpha(z)\vec{x} - C_m\vec{x}, \quad z \in \mathbb{C}, \quad \vec{x} \in E^m. \tag{2.8}$$

(ii) *The mapping $z \mapsto W_\alpha(z)$, $z \in \mathbb{C} \smallsetminus (-\infty, 0]$ is analytic.*
(iii) *The mapping $z \mapsto W_\alpha(z)$, $z \in \mathbb{C}$ is entire, provided that $\alpha \in \mathbb{N}$.*

PROOF. Let $2|k$, $k > 1/\alpha$, $a > 0$ and $C := (e^{-a|x|^{kd}})(A)$. Then $C \in L(E)$, $C$ is injective and $D_\infty(A_1^2 + \cdots + A_n^2) \supseteq R(C)$ is dense in $E$ (cf. [**12**, p. 152]). Assume that $P(x)^l = [p_{ij;l}(x)]_{1 \leqslant i,j \leqslant m}$ for $l \in \mathbb{N}_0$ and $x \in \mathbb{R}^n$. Define

$$W_\alpha(z)\vec{x} := \sum_{l=0}^{\infty} \frac{z^{\alpha l}}{\Gamma(\alpha l + 1)} P(A)^l C_m \vec{x}, \quad z \in \mathbb{C}, \quad \vec{x} \in E^m.$$

Then it is checked at once that

$$W_\alpha(z) := \left[ \sum_{l=0}^{\infty} \frac{z^{\alpha l}}{\Gamma(\alpha l + 1)} \Big( p_{ij;l}(x) e^{-a|x|^{kd}} \Big)(A) \right]_{1 \leqslant i,j \leqslant m}, \quad z \in \mathbb{C}.$$

Let $\epsilon \in (0,1)$ be fixed. Then there exists a constant $M_1 < \infty$ such that, for every multi-index $\eta \in \mathbb{N}_0^n$ with $|\eta| \leqslant k_1 \equiv 1 + \lfloor n/2 \rfloor$,

$$\begin{gathered} \big| p_{ij;l}^{(\eta)}(x) \big| \leqslant M_1^l (1+|x|)^{ld}, \quad x \in \mathbb{R}^n, \quad 1 \leqslant i,j \leqslant m, \quad l \in \mathbb{N}_0 \text{ and} \\ \big| \big( e^{-a|x|^{kd}} \big)^{(\eta)}(x) \big| \leqslant M_1 e^{-\epsilon a |x|^{kd}}, \ x \in \mathbb{R}^n. \end{gathered} \tag{2.9}$$

The asymptotic formula for the Gamma function combined with the choice of $k$ implies that $\lim_{l \to \infty} \Gamma(\frac{2ld+n}{kd})^{1/2l} \Gamma(\alpha l + 1)^{(-1)/l} = 0$ and that the mapping $z \mapsto \sum_{l=0}^{\infty} \frac{z^l}{\Gamma(\alpha l + 1)} \Gamma(\frac{2ld+n}{kd})^{1/2}$, $z \in \mathbb{C}$ is entire. Furthermore, a direct computation shows that there exists a constant $M_3 < \infty$ such that

$$\left( \int_{\mathbb{R}^n} (1+|x|)^{2ld} e^{-2\epsilon a |x|^{kd}} dx \right)^{1/2} \leqslant M_3^l \Big[ 1 + \Gamma\Big( \frac{2ld+n}{kd} \Big)^{1/2} \Big], \ l \in \mathbb{N}_0. \tag{2.10}$$

Taking into account (1.1) and (2.9)–(2.10), an elementary calculus involving Bernstein's lemma, the dominated convergence theorem and the product rule, implies that there exist constants $M_2$, $M_4 < \infty$ such that, for $1 \leqslant i, j \leqslant m$ and $z \in \mathbb{C}$,

$$\begin{aligned}
&\Big\| \sum_{l=0}^{\infty} \frac{z^{\alpha l}}{\Gamma(\alpha l+1)} \Big( p_{ij;l}(x) e^{-a|x|^{kd}} \Big)(A) \Big\| \\
&\quad \leqslant M_2 \sum_{l=0}^{\infty} \frac{|z|^{\alpha l}}{\Gamma(\alpha l+1)} \Big\| \mathcal{F}\Big( p_{ij;l}(x) e^{-a|x|^{kd}} \Big) \Big\|_{L^1(\mathbb{R}^n)} \\
&\quad \leqslant M_2 \sum_{|\eta| \leqslant k_1} \sum_{l=0}^{\infty} \frac{|z|^{\alpha l}}{\Gamma(\alpha l+1)} \Big\| D^{\eta} \big( p_{ij;l}(x) e^{-a|x|^{kd}} \big) \Big\|_{L^2(\mathbb{R}^n)} \\
&\quad \leqslant M_1 M_2 M_4 \sum_{l=0}^{\infty} \frac{|z|^{\alpha l}}{\Gamma(\alpha l+1)} M_1^l \Big( \int_{\mathbb{R}^n} (1+|x|)^{2ld} e^{-2\epsilon a|x|^{kd}} \, dx \Big)^{1/2} \\
&\quad \leqslant M_1 M_2 M_4 \sum_{l=0}^{\infty} \frac{|z|^{\alpha l}}{\Gamma(\alpha l+1)} M_1^l M_3^l \Big[ 1 + \Gamma\Big( \frac{2ld+n}{kd} \Big)^{1/2} \Big] \\
&\quad \leqslant M_1 M_2 M_4 c_{\alpha} e^{|z| M_1^{1/\alpha} M_3^{1/\alpha}} \\
&\quad + M_1 M_2 M_4 \sum_{l=0}^{\infty} \frac{\big( |z|^{\alpha} M_1 M_3 \big)^l}{\Gamma(\alpha l+1)} \Gamma\Big( \frac{2ld+n}{kd} \Big)^{1/2} < \infty.
\end{aligned}$$

Hence, $W_\alpha(z) \in L(E)$, $z \in \mathbb{C}$. It is clear that $W_\alpha(0) = C_m$ and that the mapping $t \mapsto W_\alpha(t)$, $t \geqslant 0$ is strongly continuous. It is straightforward to prove that $C_m \overline{P(A)} \subseteq \overline{P(A)} C_m$, $W_\alpha(z) \overline{P(A)} \subseteq \overline{P(A)} W_\alpha(z)$, $z \in \mathbb{C}$ and $W_\alpha(z) C_m = C_m W_\alpha(z)$, $z \in \mathbb{C}$. Using the dominated convergence theorem and the closedness of $\overline{P(A)}$, we get that:

$$\begin{aligned}
\overline{P(A)} \int_0^z & g_\alpha(z-s) W_\alpha(s) \vec{x} \, ds \\
&= \overline{P(A)} \sum_{l=0}^{\infty} \int_0^z g_\alpha(z-s) g_{\alpha l+1}(s) \overline{P(A)}^{\,l} C_m \vec{x} \, ds \\
&= \overline{P(A)} \sum_{l=0}^{\infty} g_{\alpha(1+l)+1}(z) \overline{P(A)}^{\,l} C_m \vec{x} \\
&= \sum_{l=0}^{\infty} g_{\alpha(1+l)+1}(z) \overline{P(A)}^{\,l+1} C_m \vec{x} \\
&= W_\alpha(z) \vec{x} - C_m \vec{x}, \ z \in \mathbb{C}, \ \ \vec{x} \in E^m.
\end{aligned}$$

Therefore, $(W_\alpha(t))_{t \geqslant 0}$ is a global $(g_\alpha, C_m)$-regularized resolvent family which does have $\overline{P(A)}$ as a subgenerator. Furthermore, (2.8) holds and $C_m^{-1} \overline{P(A)} C_m = \overline{P(A)}$ [**21**], which implies that $\overline{P(A)}$ is, in fact, the integral generator of $(W_\alpha(t))_{t \geqslant 0}$. This completes the proof of (i). The proofs of (ii) and (iii) are left to the reader. $\square$

Remark 2.2. (i) If $m = 1$, $p_{11}(x) = \sum_{|\alpha| \leqslant d} a_\alpha x^\alpha$, $x \in \mathbb{R}^n$ ($a_\alpha \in \mathbb{C}$) and $E$ is a function space on which translations are uniformly bounded and strongly continuous (for example, $L^p(\mathbb{R}^n)$ with $p \in [1,\infty)$, $C_0(\mathbb{R}^n)$ or $BUC(\mathbb{R}^n)$; notice also that $E$ can be composed of functions defined on some bounded domain [**3, 12, 21**]), then the natural choice for $A_j$ is $i\partial/\partial x_j$ $(1 \leqslant j \leqslant n)$. In this case, $\overline{P(A)}$ is just the operator $\sum_{|\alpha|\leqslant d} a_\alpha i^{|\alpha|}(\partial/\partial x)^\alpha$ with its maximal distributional domain. By Theorem 2.2, we infer that for each $\alpha > 0$ there exists a dense subset $E_{0,\alpha}$ of $L^p(\mathbb{R}^n)$ such that the abstract Cauchy problem:

$$\mathbf{D}_t^\alpha u(t,x) = \sum_{|\alpha|\leqslant d} a_\alpha i^{|\alpha|}(\partial/\partial x)^\alpha u(t,x), \quad t > 0, \quad x \in \mathbb{R}^n;$$
$$\frac{\partial^l}{\partial t^l} u(t,x)_{|t=0} = f_l(x), \quad x \in \mathbb{R}^n, \quad l = 0,1,\ldots,\lceil\alpha\rceil - 1,$$

has a unique solution provided $f_l(\cdot) \in E_{0,\alpha}$, $l = 0,1,\ldots,\lceil\alpha\rceil - 1$. A similar assertion can be proved in the case that $E$ is $L^\infty(\mathbb{R}^n)$, $C_b(\mathbb{R}^n)$ or $C^\sigma(\mathbb{R}^n)$ $(0 < \sigma < 1)$.

(ii) The results stated in Remark 2.1(i), Theorem 2.2, and the first part of this remark, can be reformulated for (systems of) abstract time-fractional equations considered in $E_l$-type spaces.

We close the paper with the following theorem.

THEOREM 2.3. (i) *Suppose $\alpha > 0$ and $X$ is $\mathcal{S}(\mathbb{R}^n)$ or $\mathcal{S}'(\mathbb{R}^n)$. Then there exists an injective operator $C \in L(X)$ with dense range such that the operator $P(\partial/\partial x)_{|E}$ is the integral generator of a global $(g_\alpha, C_m)$-regularized resolvent family $(W_\alpha(t))_{t\geqslant 0}$ on $E \equiv X^m$. Furthermore, the mapping $t \mapsto W_\alpha(t)$, $t \geqslant 0$ can be extended to the entire complex plane and the properties* (i)–(ii) *stated in the first section remain true with $R_\alpha(\cdot)$ and $(Z_n')^m$ replaced by $W_\alpha(\cdot)$ and $E$, respectively.*

(ii) *Suppose $\alpha > 0$, $X$ is $L^2(\mathbb{R}^n)$ and $E = \{\vec{f} \in (L^2(\mathbb{R}^n))^m : (P(\partial/\partial x))^l \vec{f} \in (L^2(\mathbb{R}^n))^m$ for all $l \in \mathbb{N}\}$. Then there exists an injective operator $C \in L(X)$ such that the operator $P(\partial/\partial x)_{|E}$ is the integral generator of a global $(g_\alpha, C_{m|E})$-regularized resolvent family $(W_\alpha(t))_{t\geqslant 0}$ on $E$. Furthermore, $R(C_{m|E})$ is dense in $E$, the mapping $t \mapsto W_\alpha(t)$, $t \geqslant 0$ can be extended to the entire complex plane and the properties* (i)–(ii) *stated in the first section remain true with $R_\alpha(\cdot)$ and $(Z_n')^m$ replaced by $W_\alpha(\cdot)$ and $E$, respectively.*

PROOF. Suppose first that $E = (\mathcal{S}(\mathbb{R}^n))^m$. Let $a > 0$, $2|k$ and $k > 1/\alpha$. Define

$$W_\alpha(z) := \mathcal{F}E_\alpha\big(z^\alpha \tilde{P}(\xi)\big)e^{-a|\xi|^{kd}}\mathcal{F}^{-1}, \quad z \in \mathbb{C}$$

and $Cf := \mathcal{F}e^{-a|\xi|^{kd}}\mathcal{F}^{-1}f$, $f \in \mathcal{S}(\mathbb{R}^n)$. Let $\tilde{P}(\xi)^l = [p_{ij;l}(\xi)]_{1\leqslant i,j\leqslant m}$ $(l \in \mathbb{N}_0$, $\xi \in \mathbb{R}^n)$. Then it is obvious that:

$$W_\alpha(z)\vec{f} = \Bigg[\sum_{l=0}^\infty \frac{z^{\alpha l}}{\Gamma(\alpha l + 1)}\mathcal{F}\Big(p_{ij;l}(\xi)e^{-a|\xi|^{kd}}\Big)\mathcal{F}^{-1}\Bigg]_{1\leqslant i,j\leqslant m}\vec{f}, \quad z \in \mathbb{C}, \quad \vec{f} \in E.$$

Since $\mathcal{F}$ and $\mathcal{F}^{-1}$ are topological isomorphisms of $\mathcal{S}(\mathbb{R}^n)$, we immediately obtain that $L(X) \ni C$ is injective. Clearly, $R(C)$ is dense in $X$ and $L(E) \ni W_\alpha(0) = C_m$ is injective. In order to prove that $W_\alpha(z) \in L(E)$ for every $z \in \mathbb{C}$, it suffices to show (cf. [**16**, Theorem 8.19-8.21]) that, for every multi-index $\eta \in \mathbb{N}_0^n$, there exist $M_\eta \geqslant 1$ and $N_\eta \in \mathbb{N}$ such that:

$$\Big\| D^\eta \Big[ E_\alpha\big(z^\alpha \tilde{P}(\xi)\big) e^{-a|\xi|^{kd}} \Big] \Big\| \leqslant M_\eta (1+|\xi|)^{N_\eta},\ \xi \in \mathbb{R}^n. \tag{2.11}$$

It can be easily seen that there exists $M_1 \geqslant 1$ such that, for every $\eta \in \mathbb{N}_0^n$, one has $|p_{ij;l}^{(\eta)}(\xi)| \leqslant M_1^l (1+|\xi|)^{ld}$, $\xi \in \mathbb{R}^n$, $1 \leqslant i, j \leqslant m$, $l \in \mathbb{N}_0$, which implies by (1.1) that, for every $z \in \mathbb{C}$, $\xi \in \mathbb{R}^n$ and $l \in \mathbb{N}_0$:

$$\big\| E_\alpha\big(z^\alpha \tilde{P}(\xi)\big) \big\| \leqslant E_\alpha\big(M_1 |z|^\alpha \big(1+|\xi|\big)^d\big) \leqslant c_\alpha e^{M_1^{1/\alpha}|z|(1+|\xi|)^{d/\alpha}}.$$

Further on,

$$\begin{aligned}
&\Big\| \frac{\partial}{\partial \xi_j} E_\alpha\big(z^\alpha \tilde{P}(\xi)\big) \Big\| \\
&\qquad = \Big\| \sum_{l=1}^{\infty} \frac{z^{\alpha l}}{\Gamma(\alpha l+1)} \Big[ \tilde{P}(\xi)^{l-1} \Big( \frac{\partial}{\partial \xi_j} \tilde{P}(\xi) \Big) + \cdots + \Big( \frac{\partial}{\partial \xi_j} \tilde{P}(\xi) \Big) \tilde{P}(\xi)^{l-1} \Big] \Big\| \\
&\qquad \leqslant \sum_{l=1}^{\infty} \frac{|z|^{\alpha l}}{\Gamma(\alpha l+1)} l M_1^{l-1} \big(1+|\xi|\big)^{d(l-1)} \\
&\qquad \leqslant \sum_{l=1}^{\infty} \frac{|z|^{\alpha l}}{\Gamma(\alpha l+1)} \big(2M_1\big)^l \big(1+|\xi|\big)^{dl} \\
&\qquad \leqslant c_\alpha e^{|z|(2M_1)^{1/\alpha}(1+|\xi|)^{d/\alpha}}, \quad z \in \mathbb{C}, \quad \xi \in \mathbb{R}^n, \quad 1 \leqslant j \leqslant n.
\end{aligned}$$

Continuing in this way, we obtain that, for every $\eta \in \mathbb{N}_0^n$, there exists $b_\eta \geqslant 1$ such that:

$$\begin{aligned}
\big\| D^\eta E_\alpha\big(z^\alpha \tilde{P}(\xi)\big) \big\| &\leqslant \sum_{l=1}^{\infty} \frac{|z|^{\alpha l}}{\Gamma(\alpha l+1)} \big(b_\eta M_1\big)^l \big(1+|\xi|\big)^{dl} \\
&\leqslant c_\alpha e^{|z| b_\eta^{1/\alpha} M_1^{1/\alpha}(1+|\xi|)^{d/\alpha}}, \quad z \in \mathbb{C}, \quad \xi \in \mathbb{R}^n.
\end{aligned}$$

Taken together with the product rule and (2.9), the last estimate immediately implies (2.11). The strong continuity of $(W_\alpha(t))_{t\geqslant 0}$ follows form the estimate $|t^{\alpha l} - s^{\alpha l}| \leqslant l|t^\alpha - s^\alpha| \max(t,s)^{l-1}$, $t, s \geqslant 0$ and the previously given arguments. Applying twice the Darboux inequality, one yields that, for every $z \in \mathbb{C} \smallsetminus (-\infty, 0]$, there exists a constant $\kappa(z) \geqslant 1$ such that:

$$\Big| \frac{(z+h)^{\alpha l} - z^{\alpha l}}{h} - \alpha l z^{\alpha l-1} \Big| \leqslant |h| \kappa(z)^{\alpha l}, \tag{2.12}$$

for any $h \in \mathbb{C}$ with $|h| < \operatorname{dist}(z, (-\infty, 0])$. Using (2.12), it readily follows that the mapping $z \mapsto W_\alpha(z)\vec{f}$, $z \in \mathbb{C} \smallsetminus (-\infty, 0]$ is analytic for every fixed $\vec{f} \in E$ and that, for every $z \in \mathbb{C} \smallsetminus (-\infty, 0]$ and $\vec{f} \in E$:

$$\frac{d}{dz} W_\alpha(z)\vec{f} = \Big[ \sum_{l=1}^{\infty} \frac{\alpha l z^{\alpha l-1}}{\Gamma(\alpha l+1)} \mathcal{F}\Big( p_{ij;l}(\xi) e^{-a|\xi|^{kd}} \Big) \mathcal{F}^{-1} \Big]_{1\leqslant i,j\leqslant m} \vec{f}.$$

One obtains similarly that, for every $\vec{f} \in E$, the mapping $z \mapsto W_\alpha(z)\vec{f}$, $z \in \mathbb{C}$ is entire, provided $\alpha \in \mathbb{N}$, and that the mapping $t \mapsto W_\alpha(t)\vec{f}$, $t \geqslant 0$ is in $C^{\lfloor \alpha \rfloor}([0, \infty) : E)$. The rest of the proof in the case $E = (\mathcal{S}(\mathbb{R}^n))^m$ is simple. The proof of the theorem in the case $E = (\mathcal{S}'(\mathbb{R}^n))^m$ is similar and therefore omitted.

To prove (ii), set $W_\alpha(z)\vec{f} := \mathcal{F}E_\alpha(z^\alpha \tilde{P}(\xi))e^{-a|\xi|^{kd}}\mathcal{F}^{-1}\vec{f}$, $z \in \mathbb{C}$, $\vec{f} \in (L^2(\mathbb{R}^n))^m$ and $Cf := \mathcal{F}e^{-a|\xi|^{kd}}\mathcal{F}^{-1}f$, $f \in L^2(\mathbb{R}^n)$, where $a > 0$, $2|k$ and $k > 1/\alpha$. Certainly, $R(C_{m|E})$ is dense in $E$ and there exists $c > 0$ such that:

$$\max_{1 \leqslant j \leqslant m} \left|\lambda_j(\xi)\right| \leqslant c\left(1 + |\xi|\right)^d, \quad \xi \in \mathbb{R}^n.$$

Keeping in mind the proofs of [**7**, Theorem 1(v)] and Theorem 2.1(iii), it suffices to show that there exists a non-negative function $t \mapsto f(t)$, $t \geqslant 0$ such that, for every $p \in \{0, \dots, m-1\}$, $q \in Q$, $t \geqslant 0$ and $\xi \in \mathbb{R}^n$:

$$\left\| \frac{1}{2\pi i} \oint_{C_\xi} \frac{z^p E_\alpha\left(t^\alpha z\right)}{\left(z - 3c(1+|\xi|)^d\right)^{m+1} \prod_{j \in q}\left(z - \lambda_j(\xi)\right)} e^{-a|\xi|^{kd}} dz \right\| \leqslant f(t). \tag{2.13}$$

Using the Cauchy theorem and (1.1), we easily infer that there exists $\mu > 0$ such that:

$$\begin{aligned}
&\left\| \frac{1}{2\pi i} \oint_{C_\xi} \frac{z^p E_\alpha\left(t^\alpha z\right)}{\left(z - 3c(1+|\xi|)^d\right)^{m+1} \prod_{j \in q}\left(z - \lambda_j(\xi)\right)} e^{-a|\xi|^{kd}} dz \right\| \\
&\quad = \left\| \frac{1}{2\pi i} \oint_{|z| = 2c(1+|\xi|)^d} \frac{z^p E_\alpha\left(t^\alpha z\right)}{\left(z - 3c(1+|\xi|)^d\right)^{m+1} \prod_{j \in q}\left(z - \lambda_j(\xi)\right)} e^{-a|\xi|^{kd}} dz \right\| \\
&\quad \leqslant \mu e^{\mu t} + e^{\mu t |\xi|^{d/\alpha} - a|\xi|^{kd}}, \quad t \geqslant 0, \quad \xi \in \mathbb{R}^n.
\end{aligned}$$

Now the existence of function $t \mapsto f(t)$, $t \geqslant 0$ satisfying (2.13) follows from the choice of $k$. The proof of the theorem is completed. □

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