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AN UNEXPECTED PROPERTY OF ODD ORDER DERIVATIVES OF HARDY'S FUNCTION

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Abstract. Assuming the Riemann hypothesis, we show that the odd order derivatives of Hardy's function have, under some condition, an unexpected behavior for large values of t.

1. Introduction and main result

Let ζ be the Riemann zeta function, and Z the Hardy function defined by

$$Z(t) = e^{i\theta(t)}\zeta\left(\tfrac{1}{2} + it\right)$$

where

$$\theta(t) = \arg\bigl(\pi^{-i\frac{t}{2}}\,\Gamma\bigl(\tfrac{1}{4} + i\tfrac{t}{2}\bigr)\bigr)$$

and the argument is defined by continuous variation of t starting with the value 0 at $t = 0$. It can be shown [6] that

$$\theta(t) = \frac{t}{2}\log\frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + O\Bigl(\frac{1}{t}\Bigr).$$

The real zeros of Z coincide with the zeros of ζ located on the line of real part $\frac{1}{2}$. If the Riemann hypothesis is true, then the number of zeros of Z in the interval $]0, t]$ is given by [6]

$$N(t) = \frac{1}{\pi}\,\theta(t) + 1 + S(t) \tag{1.1}$$

where $S(t) = \frac{1}{\pi}\arg\zeta(\frac{1}{2} + it)$ if t is not a zero of Z and $\arg\zeta(\frac{1}{2} + it)$ is defined by continuous variation along the straight lines joining 2, $2 + it$ and $\frac{1}{2} + it$ starting

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Dedicated to Professor Jean Descloux on the occasion of his 80th birthday.

with the initial value $\arg\zeta(2) = 0$. If t is a zero of Z we set $S(t) = \lim_{\epsilon\to 0_+} S(t+\epsilon)$. Let us choose T, say $T = 500$, and let us plot the graphs of functions

$$f_{2k-1}(t) := (-1)^{k+1}\frac{Z^{(2k-1)}(t)}{\theta'(T)^{2k-1}}$$

for $k = 1, \dots, 5$ on the interval $[T-10, T+10]$. Observe that the term $1/\theta'(T)^{2k-1}$ is just a scaling factor. These graphs show that the functions $(-1)^{k+1}Z^{(2k-1)}$ have

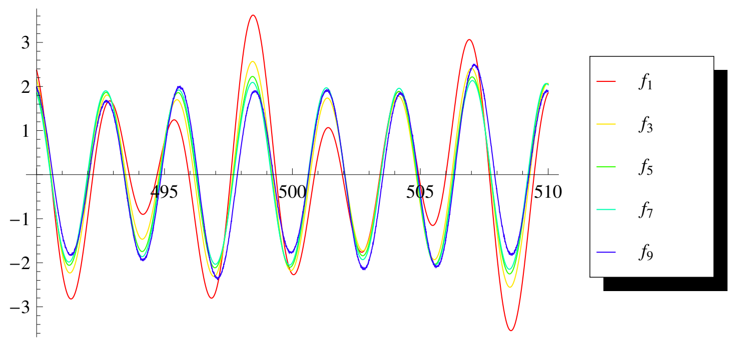

FIGURE 1. Graphs of f_{2k-1} for $k = 1, \dots, 5$

generally the same signs, at least for small values of k. This can be explained heuristically by a formula due to Lavrik [8], which asserts that for t sufficiently large and $1 \leqslant k \leqslant \frac{1}{4}\log t$, we have, uniformly in k,

$$Z^{(2k-1)}(t) = 2(-1)^k \sum_{1\leqslant n\leqslant\sqrt{t/2\pi}} \frac{1}{\sqrt{n}}(\theta'(t) - \log n)^{2k-1}\sin(\theta(t) - t\log n) + O\big(t^{-\frac{1}{4}}(\tfrac{3}{2}\log t)^{2k}\big).$$

Let us denote by $M = M(T)$ the largest integer r, possibly infinite, such that

$$\operatorname{sign}\big((-1)^{k+1}Z^{(2k-1)}(T)\big) = \operatorname{sign}(Z'(T)) \quad \text{for } k = 1, 2, \dots, r.$$

For some T the values of M are surprisingly large. Using Mathematica we compute Z with high precision and get for example $M(100) = 26$, $M(1000.4) = 138$ and $M(9999.5) = 402$.

Now let T be large enough such that $Z(T) > 0$ and let γ_k, where $k \neq 0$, be the zeros of Z ordered in increasing order, taking their multiplicities into account, and numbered so that $\cdots \leqslant \gamma_{-2} \leqslant \gamma_{-1} < T < \gamma_1 \leqslant \gamma_2 \leqslant \cdots$. Further, let $4(\log\log T)^{-1} \leqslant a \leqslant \sqrt{T}$ such that $T+a$ and $T-a$ are not zeros of Z and, finally, let $m, n \geqslant 1$ such that $\gamma_{-m-1} < T-a < \gamma_{-m}$ and $\gamma_n < T+a < \gamma_{n+1}$. Note that the existence of $m, n \geqslant 1$ is an immediate consequence of a result of Goldston and Gonek [4]. We assume that $(-1)^m Z'(T-a) \geqslant 0$ and $(-1)^n Z'(T+a) \leqslant 0$ and we denote by $K = K(T, a)$ the largest integer r, possibly infinite, such that

$$(-1)^{m+k+1} Z^{(2k-1)}(T-a) \geqslant 0 \quad \text{and} \quad (-1)^{n+k+1} Z^{(2k-1)}(T+a) \leqslant 0 \tag{1.2}$$

for $k = 1, 2, \ldots, r$. For some T and a, the values of K are also surprisingly large: $K(109.3, 9.4) = 21$, $K(1070.1, 8.5) = 108$ and $K(10025.5, 9.8) = 408$. The goal of this paper is to give a conditional upper bound for K.

It should be observed that if we replace $Z(t)$ by $\cos t$ and choose $T = 0$ and $a = l\pi + \frac{\pi}{4}$ where $l \in \mathbb{N}^*$, then $m = n = l$ and conditions (1.2) hold for every k and hence $K = \infty$.

We now define a quantity which appears in our main result. By Lavrik's formula [8], for t sufficiently large and $0 \leqslant k \leqslant \frac{1}{4}\log t$, we have, uniformly in k,

$$Z^{(2k)}(t) = 2(-1)^k \sum_{1 \leqslant n \leqslant \sqrt{t/2\pi}} \frac{1}{\sqrt{n}} (\theta'(t) - \log n)^{2k} \cos(\theta(t) - t\log n) + O\big(t^{-\frac{1}{4}} \big(\tfrac{3}{2}\log t\big)^{2k+1}\big)$$

and using $\theta'(t) = \frac{1}{2}\log\frac{t}{2\pi} + O\big(1/t^2\big)$ we get

$$|Z^{(2k)}(t)| = \theta'(t)^{2k} \bigg(2 \sum_{1 \leqslant n \leqslant \sqrt{t/2\pi}} \frac{1}{\sqrt{n}} \Big(1 - \frac{\log n}{\theta'(t)}\Big)^{2k} \cos(\theta(t) - t\log n)\bigg) + O\big(3^{2k} \theta'(t)^{2k} t^{-\frac{1}{4}} \log t\big). \tag{1.3}$$

As Ivić says in [5], it is difficult to get good uniform bounds for $Z^{(2k)}(t)$ from (1.3). Nevertheless, when $Z(T)$ is large, relation (1.3) suggests that

$$\int_{T-a}^{T+a} \big(Z^{(2k)}(t)\big)^2 dt = 2a\big(c_{2k}\theta'(T)^{2k} Z(T)\big)^2 \tag{1.4}$$

where c_{2k} is small. For the aforementioned computations, we used the results of Kotnik [7] and chose T in a neighborhood of 10^2, 10^3 and 10^4 and a approximately equal to 10 such that $Z(T)$ and $K(T, a)$ are large. We obtain $c_{42} = 0.56\ldots$ and $c_{216} = 0.34\ldots$ which show that c_{2k} can be small even for some $k > \frac{1}{4}\log T$ despite the fact that, for fixed T, the sequence c_{2k} is unbounded. Numerical experiments indicate that Lavrik's formula is probably true for larger values of k with a better error term.

THEOREM 1.1. *For T large enough, let $4(\log\log T)^{-1} \leqslant a \leqslant \sqrt{T}$, K be the number defined in the introduction, $\Delta S = S(T+a) - S(T-a)$ and further let $K^* = \frac{a}{\pi}\log T + \Delta S$. If the Riemann hypothesis is true and if $\Delta S \geqslant 1$, then*

$$K \leqslant \max\Big(\frac{a}{2\pi} A_{K,T} \frac{\log T}{\Delta S}\Big(1 + O\Big(\frac{\log\log\log T}{\log\log T}\Big)\Big), K^*\log K^*\Big) \tag{1.5}$$

where

$$A_{K,T} = \log c_{2K} + \Delta S \log 2 + \frac{1}{\sqrt{2}} \frac{\log T}{\log\log T}.$$

For T large enough such that $Z(T)$ is large in the sense of [7], numerical experiments show that the bound (1.5) is probably true without the term $\log c_{2K}$. If this is the case and if we neglect the big O in (1.5) and choose $T \leqslant 10^{50}$ and $a \leqslant 1$ such that $\Delta S = 1$, we get $K \leqslant 327$. Note that for the three values of $K(T, a)$ already given, we have $\Delta S < 0$. This suggests that the behaviour of K is different according to $\Delta S \leqslant 0$ or $\Delta S \geqslant 1$. This is unexpected.

This work stems from an observation of Ivić [5] about the values of the derivatives of Z in a neighborood of points where $|Z|$ attains a large value. Some of the material used in our proof has been published by the author in [1].

The notations used in this paper are standard: $\lfloor x \rfloor$ stand for the usual floor function and $\{x\} := x - \lfloor x \rfloor$. Bernoulli and Chebyshev polynomials of degree n are denoted by $B_n(x)$ and $T_n(x)$; they are defined by

$$\int_x^{x+1} B_n(t)\,dt = x^n \quad \text{and} \quad T_n(\cos\theta) = \cos n\theta.$$

The organization of this paper is as follows: In Section 2 we prove the key identity, a property of the derivatives of Bernoulli polynomials and preparatory lemmas. Section 3 is devoted to the proof of our main result.

2. Preliminary results

We first prove an identity which will be used later to establish a relation between the value of a function $f \in C^{2r}[-a, a]$ at 0, the zeros of f and the values of its derivatives of odd order on the boundaries of the interval.

Lemma 2.1. *Let $-a < x_{-m} < \cdots < x_{-1} < x_0 < x_1 < \cdots < x_n < a$ and for $l = 1, 2, \ldots$ let Ψ_{2l-1} be the function defined on $[-a, a]$ by*

$$\Psi_{2l-1}(x) = \frac{(4a)^{2l-1}}{(2l)!} \sum_{k=-m}^{n} \mu_k \left(B_{2l}\Big(\frac{1}{2} + \frac{x + x_k}{4a}\Big) + B_{2l}\Big(\Big\{\frac{x - x_k}{4a}\Big\}\Big)\right)$$

where $\sum_{k=-m}^{n} \mu_k = 0$. Then for $f \in C^{2r}[-a, a]$ where $r \geqslant 1$, we have the identity

$$\sum_{k=-m}^{n} \mu_k f(x_k) = \sum_{k=1}^{r} f^{(2k-1)}(a)\Psi_{2k-1}(a) - \sum_{k=1}^{r} f^{(2k-1)}(-a)\Psi_{2k-1}(-a) - \int_{-a}^{a} f^{(2r)}(x)\Psi_{2r-1}(x)\,dx. \tag{2.1}$$

Proof. By definition the function Ψ_{2r-1} is C^{2r-2}, piecewise polynomial and the relation $B_l'(x) = lB_{l-1}(x)$ for $l = 1, 2, \ldots$ leads to

$$\Psi_{2r-1}^{(j)}(x) = \frac{(4a)^{2r-j-1}}{(2r-j)!} \sum_{k=-m}^{n} \mu_k \left(B_{2r-j}\Big(\frac{1}{2} + \frac{x + x_k}{4a}\Big) + B_{2r-j}\Big(\Big\{\frac{x - x_k}{4a}\Big\}\Big)\right)$$

for $j = 1, \ldots, 2r-1$ and $x \neq x_{-m}, \ldots, x_n$ if $j = 2r-1$. This implies that

$$\Psi_{2r-1}^{(2r-2j)} = \Psi_{2j-1} \quad \text{for } j = 1, 2, \ldots, r-1 \tag{2.2}$$

and that

$$
\begin{aligned}
(2.3)\quad &\Psi_{2r-1}^{(2r-2j+1)}(\pm a)\\
&= \frac{(4a)^{2j-2}}{(2j-1)!}\sum_{k=-m}^{n}\mu_k\Big(B_{2j-1}\Big(\frac{1}{2}+\frac{\pm a+x_k}{4a}\Big)+B_{2j-1}\Big(\Big\{\frac{\pm a-x_k}{4a}\Big\}\Big)\Big)\\
&= \frac{(4a)^{2j-2}}{(2j-1)!}\sum_{k=-m}^{n}\mu_k\Big(B_{2j-1}\Big(\frac{1}{2}+\frac{\pm a+x_k}{4a}\Big)+B_{2j-1}\Big(\frac{1}{2}-\frac{\pm a+x_k}{4a}\Big)\Big)\\
&= 0 \quad \text{for } j=1,2,\dots,r.
\end{aligned}
$$

Further for $x \neq x_{-m},\dots,x_n$ we have

$$
\begin{aligned}
(2.4)\qquad \Psi_{2r-1}^{(2r-1)}(x) &= \sum_{k=-m}^{n}\mu_k\Big(B_1\Big(\frac{1}{2}+\frac{x+x_k}{4a}\Big)+B_1\Big(\Big\{\frac{x-x_k}{4a}\Big\}\Big)\Big)\\
&= \sum_{k=-m}^{n}\mu_k\Big(\frac{x+x_k}{4a}+\Big\{\frac{x-x_k}{4a}\Big\}-\frac{1}{2}\Big)
\end{aligned}
$$

and as $\sum_{k=-m}^{n}\mu_k = 0$ the function $\Psi_{2r-1}^{(2r-1)}$ is piecewise constant. Explicitly, for $x \in \,]x_j, x_{j+1}[$, we get

$$
\Psi_{2r-1}^{(2r-1)}(x) = \sum_{k=-m}^{j}\mu_k\Big(\frac{x}{2a}-\frac{1}{2}\Big)+\sum_{k=j+1}^{n}\mu_k\Big(\frac{x}{2a}+1-\frac{1}{2}\Big) = \sum_{k=j+1}^{n}\mu_k = -\sum_{k=-m}^{j}\mu_k
$$

which leads to

$$
\int_{x_j}^{x_{j+1}} f'(x)\Psi_{2r-1}^{(2r-1)}(x)\,dx = -\Big(\sum_{k=-m}^{j}\mu_k\Big)(f(x_{j+1})-f(x_j)) \quad \text{for } j=-m,\dots,n-1.
$$

Summing these equalities and using that $\Psi_{2r-1}^{(2r-1)} = 0$ on the intervals $[-a, x_{-m}[$ and $]x_n, a]$, which follows from (2.4), we have

$$
\sum_{k=-m}^{n}\mu_k f(x_k) = \int_{-a}^{a} f'(x)\,\Psi_{2r-1}^{(2r-1)}(x)\,dx
$$

and we complete the proof by integrating $2r-1$ times the right-hand side by parts taking into account relations (2.2) and (2.3). $\square$

For further use we recall some elementary facts concerning the divided differences.

LEMMA 2.2. *Let $I =]-T, T[$, $f \in C^{m+n}(I)$ and let g be the function defined for pairwise distinct numbers $t_{-m},\dots,t_n \in I$ by*

$$
g(t_{-m},\dots,t_n) = \sum_{k=-m}^{n}\frac{f(t_k)}{\prod_{\substack{-m\leqslant j\leqslant n\\ j\neq k}}(t_k-t_j)}.
$$

Then

a) *The function g has a continuous extension g^* defined for $t_{-m}, \dots, t_n \in I$ and there exists $\eta = \eta(t_{-m}, \dots, t_n) \in I$ such that*

$$g^*(t_{-m}, \dots, t_n) = \frac{f^{(m+n)}(\eta)}{(m+n)!}.$$

b) *Let $y_0, y_1, \dots, y_l$ be the distinct values of $t_{-m}, \dots, t_n$ considered as fixed and let r_k be the number of index j such that $t_j = y_k$. Then there exist $\alpha_{k,i}$ depending on $y_0, y_1, \dots, y_l$ such that*

$$g^*(t_{-m}, \dots, t_n) = \sum_{k=0}^{l} \sum_{i=0}^{r_k - 1} \alpha_{k,i} f^{(i)}(y_k).$$

Proof. Assertion a) is a consequence of the representation formula

$$\text{(2.5)} \quad g(t_{-m}, \dots, t_n) =$$
$$\int_0^1 d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{m+n-1}} f^{(m+n)}\Big(t_{-m} + \sum_{k=1}^{m+n} \tau_k (t_{-m+k} - t_{-m+k-1})\Big) d\tau_{m+n}.$$

A proof of b) is given in [9]. □

In the next lemma we indicate the choice of coefficients μ_k for which the identity of Lemma 2.1 is of practical use for large values of a. The main reason of this choice will appear in the proof of (2.11) in Lemma 2.9.

Lemma 2.3. *Let Ψ_{2l-1} be defined for pairwise distinct $x_{-m}, \dots, x_n \in]-a, a[$ by*

$$\Psi_{2l-1}(x_{-m}, \dots, x_n, x) = \frac{(4a)^{2l-1}}{(2l)!} \sum_{k=-m}^{n} \mu_k \Big(B_{2l}\Big(\frac{1}{2} + \frac{x + x_k}{4a}\Big) + B_{2l}\Big(\Big\{\frac{x - x_k}{4a}\Big\}\Big)\Big)$$

where

$$\mu_k = \frac{\alpha_k}{\alpha_0} \quad \textit{and} \quad \frac{1}{\alpha_k} = \prod_{\substack{-m \leqslant j \leqslant n \\ j \neq k}} \Big(\sin\Big(\pi \frac{x_k}{2a}\Big) - \sin\Big(\pi \frac{x_j}{2a}\Big)\Big) \quad \textit{for } k = -m, \dots, n.$$

Then

a) *For $l \geqslant 1$ the functions $\Psi_{2l-1}(\cdot, \dots, \cdot, \pm a)$ have continuous extensions $\Psi^*_{2l-1}(\cdot, \dots, \cdot, \pm a)$ defined for $x_{-m}, \dots, x_n \in]-a, a[$.*
b) *If $2l \geqslant m + n + 2$ the function Ψ_{2l-1} has a continuous extension Ψ^*_{2l-1} defined for $x_{-m}, \dots, x_n \in]-a, a[$ and $x \in [-a, a]$.*
c) *If $2r \geqslant m + n + 2$ and $f \in C^{2r}[-a, a]$ is defined on $[-a, a]$ and vanishes at x_k with $k \neq 0$ where $-a < x_{-m} \leqslant \dots \leqslant x_{-1} < x_0 < x_1 \leqslant \dots \leqslant x_n < a$ and the x_k are numbered taking into account their multiplicity, then we have the identity*

$$f(x_0) = \sum_{k=1}^{r} f^{(2k-1)}(a)\Psi^*_{2k-1}(a) - \sum_{k=1}^{r} f^{(2k-1)}(-a)\Psi^*_{2k-1}(-a) - \int_{-a}^{a} f^{(2r)}(x)\Psi^*_{2r-1}(x)\,dx \tag{2.6}$$

where for short $\Psi^*_{2k-1}(\pm a)$ and $\Psi^*_{2r-1}(x)$ stand for $\Psi^*_{2k-1}(\cdot,\dots,\cdot,\pm a)$ and $\Psi^*_{2r-1}(\cdot,\dots,\cdot,x)$.

Proof. Introducing the function h defined by

$$h(t,x) = \frac{(4a)^{2l-1}}{(2l)!}\Big(B_{2l}\Big(\frac{1}{2}+\frac{x}{4a}+\frac{1}{2\pi}\operatorname{Arcsin} t\Big) + B_{2l}\Big(\Big\{\frac{x}{4a}-\frac{1}{2\pi}\operatorname{Arcsin} t\Big\}\Big)\Big)$$

we have

$$\Psi_{2l-1}(x_{-m},\dots,x_n,\pm a) = \frac{1}{\alpha_0}\sum_{k=-m}^{n}\alpha_k h\Big(\sin\Big(\pi\frac{x_k}{2a}\Big),\pm a\Big)$$

for pairwise distinct $x_{-m},\dots,x_n \in\,]-a,a[$ and assertion a) holds since the functions $h(\cdot,\pm a)$ belong to $C^\infty]-1,1[$.

By definition the function h belongs to $C^{2l-2}(\,]-1,1[\times[-a,a])$ and the assertion b) is a consequence of the representation formula (2.5) since we have $2l-2 \geqslant m+n$. For pairwise distinct $x_{-m},\dots,x_n \in\,]-a,a[$ the left-hand side of identity (2.1) reads

$$\frac{1}{\alpha_0}\sum_{k=-m}^{n}\alpha_k f\Big(\frac{2a}{\pi}\operatorname{Arcsin}\Big(\sin\Big(\pi\frac{x_k}{2a}\Big)\Big)\Big)$$

and thanks to Lemma 2.2 this expression, and hence the identity (2.1), extend to $x_{-m},\dots,x_n \in\,]-a,a[$. One completes the proof of c) by observing, thanks to Lemma 2.2, that the left-hand side reduces to $f(x_0)$ when the x_k are zeros of multiplicity r_k of f. □

The results stated in Lemma 2.4 play a central role in the proof of properties of functions $\Psi^*_{2l-1}(\cdot,\dots,\cdot,\pm a)$.

Lemma 2.4. *For all $m,k\in\mathbb{N}^*$ we have the inequality*

$$(-1)^{m+1}\frac{d^k}{dx^k}B_{2m}\Big(\frac{1}{2}+\frac{1}{\pi}\operatorname{Arcsin}\sqrt{x}\Big) > 0 \quad \text{for } x\in[0,1[\,.$$

The proof of Lemma 2.4 requires two technical results given in Lemmas 2.5 and 2.6.

Lemma 2.5. *For all $k\in\mathbb{N}$ we have the Taylor expansion*

$$(\operatorname{Arcsin} x)^{2k} = \sum_{l=0}^{\infty}\frac{(2k)!}{(2l)!}2^{2l-2k}b_{k,l}\,x^{2l} \quad \text{for } x\in[-1,1]$$

where $b_{k,l}$ are integers defined recursively by

$$b_{0,0} = 1 \textit{ and } b_{k,0} = b_{0,l} = 0 \quad \textit{for } k, l \geqslant 1$$
$$b_{k+1,l+1} = b_{k,l} + l^2 b_{k+1,l} \quad \textit{for } k, l \geqslant 0.$$

PROOF. We note first that the functions $f_{2k}(x) := (\operatorname{Arcsin} x)^{2k}$ satisfy

$$(1 - x^2) f''_{2k+2}(x) - x\, f'_{2k+2}(x) - (2k+2)(2k+1) f_{2k}(x) = 0 \quad \text{for } x \in \,]-1,1[\,.$$

From the definition of f_{2k} and the above equality it follows that numbers $c_{k,l}$ defined by $f_{2k}(x) = \sum_{l=0}^{\infty} c_{k,l}\, x^{2l}$ for $x \in [-1,1]$ are uniquely determined by the recurrence relations

$$c_{0,0} = 1 \text{ and } c_{k,0} = c_{0,l} = 0 \quad \text{for } k, l \geqslant 1$$
$$(2l+2)(2l+1) c_{k+1,l+1} - 4l^2 c_{k+1,l} - (2k+2)(2k+1) c_{k,l} = 0 \quad \text{for } k, l \geqslant 0.$$

A simple check shows that $c_{k,l} = \dfrac{(2k)!}{(2l)!} 2^{2l-2k} b_{k,l}$. □

LEMMA 2.6. *Let $b_{k,l}$ be the numbers defined in Lemma 2.5. Then*

$$\lim_{l \to \infty} \frac{b_{k,l}}{((l-1)!)^2} = \frac{\pi^{2k-2}}{(2k-1)!} \quad \textit{for all } k \geqslant 1. \tag{2.7}$$

PROOF. From the definition of numbers $b_{k,l}$ we infer that $b_{1,l} = ((l-1)!)^2$ for $l \geqslant 1$. Thus relation (2.7) is trivially true for $k = 1$. We then assume $k \geqslant 2$. As $b_{j,1} = 0$ for $j \geqslant 2$ the numbers $d_{j,l}$ defined for $j, l \geqslant 1$ by $d_{j,l} = \frac{b_{j,l}}{((l-1)!)^2}$ satisfy the recurrence relations

$$d_{j,1} = 0 \text{ and } d_{1,l} = 1 \quad \text{for } j \geqslant 2 \text{ and } l \geqslant 1,$$
$$d_{j+1,l+1} = \frac{1}{l^2} d_{j,l} + d_{j+1,l} \quad \text{for } j, l \geqslant 1.$$

Using the fact that $d_{j-1,l} = 0$ for $l = 1, \dots, j-2$ we get first for $j \geqslant 2$ the equality

$$d_{j,n_j} = \sum_{n_{j-1}=j-1}^{n_j - 1} \frac{1}{n_{j-1}^2} d_{j-1,n_{j-1}}$$

which we iterate to obtain

$$d_{k,l} = \sum_{n_{k-1}=k-1}^{l-1} \frac{1}{n_{k-1}^2} \sum_{n_{k-2}=k-2}^{n_{k-1}-1} \frac{1}{n_{k-2}^2} \cdots \sum_{n_2=2}^{n_3-1} \frac{1}{n_2^2} \sum_{n_1=1}^{n_2-1} \frac{1}{n_1^2}\,.$$

This leads to

$$\lim_{l \to \infty} d_{k,l} = \sum_{n_{k-1} > n_{k-2} > \cdots > n_2 > n_1 > 0} \prod_{j=1}^{k-1} \frac{1}{n_j^2}$$

and we recognize in the right-hand side the number $\zeta(\{2\}_{(k-1)})$ whose value, given in [2], is equal to the right-hand side of (2.7). □

PROOF OF LEMMA 2.4. It suffices to prove that the numbers $e_{m,\,l}$ defined by

$$(-1)^{m+1}B_{2m}\Big(\frac{1}{2}+\frac{1}{\pi}\,\mathrm{Arcsin}\,x\Big)=\sum_{l=0}^{\infty}e_{m,\,l}\,x^{2l} \tag{2.8}$$

satisfy $e_{m,\,l}>0$ for all $m,l\in\mathbb{N}^*$. Using Taylor's formula and the evenness of function $B_{2m}(\frac{1}{2}+\frac{t}{\pi})$, we have

$$B_{2m}\Big(\frac{1}{2}+\frac{t}{\pi}\Big)=\sum_{k=0}^{m}\frac{1}{(2k)!}\,B_{2m}^{(2k)}\Big(\frac{1}{2}\Big)\Big(\frac{t}{\pi}\Big)^{2k}=\sum_{k=0}^{m}\binom{2m}{2k}B_{2m-2k}\Big(\frac{1}{2}\Big)\Big(\frac{t}{\pi}\Big)^{2k}$$

and the Taylor expansion of $(\mathrm{Arcsin}\,x)^{2k}$ given in Lemma 2.5 leads to

$$\begin{aligned}B_{2m}\Big(\frac{1}{2}+\frac{1}{\pi}\,\mathrm{Arcsin}\,x\Big)&=\sum_{k=0}^{m}\left(\binom{2m}{2k}B_{2m-2k}\Big(\frac{1}{2}\Big)\pi^{-2k}\sum_{l=0}^{\infty}\frac{(2k)!}{(2l)!}2^{2l-2k}b_{k,\,l}x^{2l}\right)\\&=\frac{(2m)!}{(2\pi)^{2m}}\sum_{k=0}^{m}\left(\frac{(2\pi)^{2m-2k}}{(2m-2k)!}\,B_{2m-2k}\Big(\frac{1}{2}\Big)\sum_{l=0}^{\infty}\frac{2^{2l}}{(2l)!}b_{k,\,l}x^{2l}\right).\end{aligned}$$

We then change the order of summation to get

$$(-1)^{m+1}B_{2m}\Big(\frac{1}{2}+\frac{1}{\pi}\,\mathrm{Arcsin}\,x\Big)=\frac{(2m)!}{(2\pi)^{2m}}\sum_{l=0}^{\infty}\frac{2^{2l}}{(2l)!}f_{m,\,l}x^{2l} \tag{2.9}$$

where

$$f_{m,\,l}=(-1)^{m+1}\sum_{k=0}^{m}\frac{(2\pi)^{2m-2k}}{(2m-2k)!}B_{2m-2k}\Big(\frac{1}{2}\Big)b_{k,\,l}.$$

We prove by recurrence over m that $f_{m,\,l}>0$ for $m,\,l\geqslant 1$. To this end we set $g_{m,l}=\frac{f_{m,l}}{((l-1)!)^2}$ for $m,l\geqslant 1$ and since $b_{0,\,l}=0$ for $l\geqslant 1$ we have

$$\begin{aligned}g_{m+1,\,l+1}&=\frac{(-1)^{m+2}}{(l!)^2}\sum_{k=1}^{m+1}\frac{(2\pi)^{2m+2-2k}}{(2m+2-2k)!}B_{2m+2-2k}\Big(\frac{1}{2}\Big)\,b_{k,\,l+1}\\&=\frac{(-1)^{m+2}}{(l!)^2}\sum_{k=1}^{m+1}\frac{(2\pi)^{2m+2-2k}}{(2m+2-2k)!}B_{2m+2-2k}\Big(\frac{1}{2}\Big)\,\Big(b_{k-1,\,l}+l^2\,b_{k,\,l}\Big)\\&=\frac{(-1)^{m+2}}{(l!)^2}\sum_{k=1}^{m+1}\frac{(2\pi)^{2m+2-2k}}{(2m+2-2k)!}B_{2m+2-2k}\Big(\frac{1}{2}\Big)\,b_{k-1,\,l}+g_{m+1,\,l}\\&=-\frac{(-1)^{m+1}}{(l!)^2}\sum_{k=0}^{m}\frac{(2\pi)^{2m-2k}}{(2m-2k)!}B_{2m-2k}\Big(\frac{1}{2}\Big)\,b_{k,\,l}+g_{m+1,\,l}\\&=-\frac{1}{l^2}g_{m,\,l}+g_{m+1,\,l}\end{aligned}$$

and this implies that

$$g_{m+1,\,l+1}+\frac{1}{l^2}g_{m,\,l}=g_{m+1,\,l}\quad\text{for } l\geqslant 1.$$

We have $g_{1,l} = f_{1,l} = 1$ for all $l \geqslant 1$. Let us suppose that $g_{m,l} > 0$ for all $l \geqslant 1$. Then $g_{m+1,l+1} < g_{m+1,l}$ and it follows that $g_{m+1,l} > \lim_{l\to\infty} g_{m+1,l}$. Thanks to Lemma 2.6 we have

$$\begin{aligned} \lim_{l\to\infty} g_{m+1,l} &= (-1)^{m+2} \sum_{k=1}^{m+1} \frac{(2\pi)^{2m+2-2k}}{(2m+2-2k)!} B_{2m+2-2k}\Big(\frac{1}{2}\Big) \frac{\pi^{2k-2}}{(2k-1)!} \\ &= (-1)^{m+2}\pi^{2m} \sum_{k=1}^{m+1} \frac{2^{2m+2-2k}}{(2m+2-2k)!(2k-1)!} B_{2m+2-2k}\Big(\frac{1}{2}\Big) \end{aligned} \tag{2.10}$$

and using $B_j(\frac{1}{2}) = 0$ for all odd j and the formula

$$B_n(x+y) = \sum_{j=0}^{n} \binom{n}{j} B_j(x) y^{n-j}$$

we check that the sum which appears in (2.10) is equal to

$$\begin{aligned} \sum_{j=0}^{2m+1} \frac{2^j}{j!(2m+1-j)!} B_j\Big(\frac{1}{2}\Big) &= \frac{2^{2m+1}}{(2m+1)!} \sum_{j=0}^{2m+1} \binom{2m+1}{j} B_j\Big(\frac{1}{2}\Big)\Big(\frac{1}{2}\Big)^{2m+1-j} \\ &= \frac{2^{2m+1}}{(2m+1)!} B_{2m+1}(1) = 0. \end{aligned}$$

Hence $g_{m,l} > 0$ for $m, l \geqslant 1$ and this implies, thanks to (2.9), that the numbers $e_{m,l}$ defined by (2.8) are positive for $m, l \geqslant 1$. □

We are now in position to prove main properties of functions $\Psi^*_{2l-1}(\cdot, \dots, \cdot, \pm a)$.

Lemma 2.7. *Let* $\Psi^*_{2l-1}(\cdot, \dots, \cdot, \pm a)$ *be the functions defined in Lemma* 2.3. *Then*

a) $(-1)^{n+l+1}\Psi^*_{2l-1}(x_{-m}, \dots, x_n, a) > 0$ *for* $x_{-m}, \dots, x_n \in]-a, a[$.
b) $(-1)^{m+l+1}\Psi^*_{2l-1}(x_{-m}, \dots, x_n, -a) > 0$ *for* $x_{-m}, \dots, x_n \in]-a, a[$.

Proof. For pairwise distinct $x_{-m}, \dots, x_n \in]-a, a[$ we have

$$\Psi^*_{2l-1}(x_{-m}, \dots, x_n, \pm a) = 2\frac{(4a)^{2l-1}}{(2l)!} \sum_{k=-m}^{n} \mu_k\, B_{2l}\Big(\frac{1}{2} + \frac{\pm a + x_k}{4a}\Big)$$

since the function $B_{2m}(\frac{1}{2} + t)$ is even and then

$$\begin{aligned} &(-1)^{n+l+1}\, \Psi^*_{2l-1}(x_{-m}, \dots, x_n, \pm a) \\ &\qquad = 2\frac{(4a)^{2l-1}}{(2l)!} \Big(\frac{(-1)^n}{\alpha_0}\Big) \sum_{k=-m}^{n} \alpha_k (-1)^{l+1} B_{2l}\Big(\frac{1}{2} + \frac{\pm a + x_k}{4a}\Big). \end{aligned}$$

The first two terms of the right-hand side are positive and the third term reads $\sum_{k=-m}^{n} \alpha_k h_\pm(\sin(\pi \frac{x_k}{2a}))$ where

$$h_\pm(t) = (-1)^{l+1} B_{2l}\Big(\frac{3}{4} \pm \frac{1}{2\pi} \operatorname{Arcsin} t\Big) \quad \text{for } t \in [-1,1].$$

The identities

$$\frac{3}{4} \pm \frac{1}{2\pi} \operatorname{Arcsin} t = \frac{1}{2} + \frac{1}{\pi} \operatorname{Arcsin} \sqrt{\frac{1 \pm t}{2}} \quad \text{for } t \in [-1,1]$$

together with Lemma 2.4 show that $h_+^{(m+n)}$ and $(-1)^{m+n} h_-^{(m+n)}$ are positive on $]-1,1[$ and the conclusion holds by Lemma 2.2. □

The last point is to bound the integral which appears in the right-hand side of the identity (2.6). This is the content of Lemma 2.9, whose proof needs the following result.

LEMMA 2.8. *Let $b_{r,s}$ the numbers defined for integers $r \geqslant 4$ and $s \geqslant 0$ by*

$$b_{r,s} = \Big(\frac{r}{r+s}\Big)^{2r\log r - 1} \binom{2r+s-1}{s}.$$

Then $\sum_{s=0}^{\infty} b_{r,s}^2 = 1 + o(1)$ as $r \to \infty$.

PROOF. We have $b_{r,s} = O_r(s^{-2r\log r + 2r}) = O_r(s^{-1})$ and hence $\sum_{s=0}^{\infty} b_{4,s}^2$ is convergent. We now prove that $b_{r,s} \leqslant b_{4,s}$ for $r \geqslant 4$. We have $\log b_{r,s} = g(r,s)$ where the function g is defined for $(x,y) \in [4,\infty[\times [0,\infty[$ by

$$g(x,y) = (2x\log x - 1)\log\Big(\frac{x}{x+y}\Big) + \log\Gamma(2x+y) - \log\Gamma(y+1) - \log\Gamma(2x).$$

Straightforward computations lead to

$$\frac{\partial g}{\partial x}(x,y) = (2\log x + 2)\log\Big(\frac{x}{x+y}\Big) + (2x\log x - 1)\frac{y}{x(x+y)} + 2\Psi(2x+y) - 2\Psi(2x)$$

and

$$\frac{\partial^2 g}{\partial y \partial x}(x,y) = -\frac{1+2x+2y+2y\log x}{(x+y)^2} + 2\Psi'(2x+y)$$

where Ψ is the derivative of $\log\Gamma$. We have $\frac{\partial g}{\partial x}(x,0) = 0$ and moreover since $\Psi'(z) = \sum_{k=0}^{\infty} \frac{1}{(z+k)^2}$, we get $\Psi'(z) \leqslant \frac{1}{z} + \frac{1}{z^2}$ for $z > 0$ and therefore

$$\begin{aligned}\frac{\partial^2 g}{\partial y \partial x}(x,y) &\leqslant -\frac{1+2x+2y+2y\log x}{(x+y)^2} + \frac{2}{2x+y} + \frac{2}{(2x+y)^2} \\ &= -\Big(\frac{4x^3 + 2(1+3y)x^2 + (2x-1)y^2}{(x+y)^2(2x+y)^2} + \frac{2y\log x}{(x+y)^2}\Big) \leqslant 0.\end{aligned}$$

Hence $\frac{\partial g}{\partial x}(x,y) \leqslant \frac{\partial g}{\partial x}(x,0) = 0$ and this implies that $g(x,y) \leqslant g(4,y)$ and hence $b_{r,s} \leqslant b_{4,s}$ for $r \geqslant 4$.

Let $\epsilon > 0$ and s_0 be such that $\sum_{s=s_0}^{\infty} b_{4,s}^2 \leqslant \frac{\epsilon}{2}$. Since $b_{r,s} \to 0$ as $r \to \infty$ for $s \geqslant 1$, there exists r_0 such that $\sum_{s=1}^{s_0-1} b_{r,s}^2 \leqslant \frac{\epsilon}{2}$ for $r \geqslant r_0$. Hence

$$1 \leqslant \sum_{s=0}^{\infty} b_{r,s}^2 \leqslant b_{r,0}^2 + \sum_{s=1}^{s_0-1} b_{r,s}^2 + \sum_{s=s_0}^{\infty} b_{4,s}^2 \leqslant 1+\epsilon \quad \text{for } r \geqslant r_0.$$

The proof is complete. □

LEMMA 2.9. *For positive integers m, n and l which satisfy $m+n \geqslant 4$ and $l \geqslant (m+n)\log(m+n)$ and for $x_{-m}, \dots, x_n \in]-a, a[$, let Ψ_{2l-1}^* be the function defined in Lemma* 2.3. *Then*

$$\|\Psi_{2l-1}^*\|_2 = \frac{2^{m+n-1}}{|\alpha_0|\sqrt{a}} \left(\frac{2a}{(m+n)\pi}\right)^{2l} (1+o(1)) \quad \text{as } m+n \to \infty$$

where

$$\|\Psi_{2l-1}^*\|_2^2 = \int_{-a}^{a} \left(\Psi_{2l-1}^*(x_{-m}, \dots, x_n, x)\right)^2 dx.$$

PROOF. The use of the Fourier series expansion

$$B_{2l}(x) = (-1)^{l+1} 2((2l)!) \sum_{j=1}^{\infty} \frac{1}{(2j\pi)^{2l}} \cos(2j\pi x) \quad \text{for } x \in [0,1]$$

and the identity $\cos\alpha + \cos\beta = 2\cos(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2})$ lead, for pairwise distinct $x_{-m}, \dots, x_n \in]-a, a[$, to the expression

$$\begin{aligned} &\Psi_{2l-1}^*(x_{-m}, \dots, x_n, x) \\ &= (-1)^{l+1} 2\, \frac{(2a)^{2l-1}}{\alpha_0 \pi^{2l}} \sum_{j=1}^{\infty} \frac{1}{j^{2l}} \Bigg(\sum_{k=-m}^{n} \alpha_k \cos\Big(j\pi\Big(\frac{1}{2} + \frac{x_k}{2a}\Big)\Big)\Bigg) \cos\Big(j\pi\Big(\frac{1}{2} + \frac{x}{2a}\Big)\Big). \end{aligned}$$

Using the identity $\cos(j\pi(\frac{1}{2}+y)) = (-1)^j T_j(\sin(\pi y))$ and introducing the numbers $a_{j,k} = (-1)^j T_j(\sin(\pi \frac{x_k}{2a}))$ we have $\sum_{k=-m}^{n} \alpha_k\, a_{j,k} = 0$ for $j = 1, \dots, m+n-1$, this is crucial, and, therefore

$$\begin{aligned} \text{(2.11)} \quad &\Psi_{2l-1}^*(x_{-m}, \dots, x_n, x) \\ &= (-1)^{l+1} 2\, \frac{(2a)^{2l-1}}{\alpha_0 \pi^{2l}} \sum_{j=m+n}^{\infty} \frac{1}{j^{2l}} \Bigg(\sum_{k=-m}^{n} \alpha_k a_{j,k} \Bigg) \cos\Big(j\pi\Big(\frac{1}{2} + \frac{x}{2a}\Big)\Big). \end{aligned}$$

Using Lemma 2.2, squaring (2.11) and integrating on the interval $[-a, a]$, we get

$$\text{(2.12)} \qquad \|\Psi_{2l-1}^*\|_2^2 = \left(2\, \frac{(2a)^{2l-1}}{\alpha_0 \pi^{2l}}\right)^2 \sum_{j=m+n}^{\infty} \frac{1}{j^{4l}} \left(\frac{T_j^{(m+n)}(\tau_j)}{(m+n)!}\right)^2 a$$

for some $\tau_j \in]-1, 1[$. It is well known [10] that

$$\max_{-1 \leqslant x \leqslant 1} \left|T_j^{(m+n)}(x)\right| = T_j^{(m+n)}(1) = 2^{m+n-1}\,(m+n-1)!\, j \binom{m+n+j-1}{j-m-n}$$

for $j = m+n, m+n+1, \dots$ and then

$$\|\Psi^*_{2l-1}\|_2^2 \leqslant \left(\frac{2^{m+n-1}}{\alpha_0\sqrt{a}}\Big(\frac{2a}{(m+n)\pi}\Big)^{2l}\right)^2 \sum_{j=m+n}^{\infty}\Big(\frac{m+n}{j}\Big)^{4l-2}\binom{m+n+j-1}{j-m-n}^2.$$

We set $j = m+n+s$ and since $l \geqslant (m+n)\log(m+n)$ we have

$$\begin{aligned}\|\Psi^*_{2l-1}\|_2^2 &\leqslant \left(\frac{2^{m+n-1}}{\alpha_0\sqrt{a}}\Big(\frac{2a}{(m+n)\pi}\Big)^{2l}\right)^2 \sum_{s=0}^{\infty}\left(\Big(\frac{m+n}{m+n+s}\Big)^{2l-1}\binom{2(m+n)+s-1}{s}\right)^2\\ &\leqslant \left(\frac{2^{m+n-1}}{\alpha_0\sqrt{a}}\Big(\frac{2a}{(m+n)\pi}\Big)^{2l}\right)^2 \sum_{s=0}^{\infty} b^2_{m+n,s}\\ &= \left(\frac{2^{m+n-1}}{\alpha_0\sqrt{a}}\Big(\frac{2a}{(m+n)\pi}\Big)^{2l}\right)^2 (1+o(1))\end{aligned}$$

as $m+n\to\infty$, thanks to Lemma 2.8. To complete the proof we compute a lower bound for $\|\Psi^*_{2l-1}\|_2^2$ using the first term in the series which appears in the right-hand side of (2.12). ☐

3. Proof of Theorem

In this section we assume that the Riemann hypothesis is true. Our result is a consequence of identity (2.6) which for $Z(T)$ gives

$$\begin{aligned}Z(T) = \sum_{k=1}^{K} Z^{(2k-1)}(T+a)\Psi^*_{2k-1}(a) &- \sum_{k=1}^{K} Z^{(2k-1)}(T-a)\Psi^*_{2k-1}(-a)\\ &- \int_{-a}^{a} Z^{(2K)}(T+x)\Psi^*_{2K-1}(x)\,dx\end{aligned} \tag{3.1}$$

where $2K \geqslant m+n+2$ and for short $\Psi^*_{2k-1}(\pm a)$ and $\Psi^*_{2m-1}(x)$ stand respectively for $\Psi^*_{2k-1}(x_{-m},\dots,x_n,\pm a)$ and $\Psi^*_{2m-1}(x_{-m},\dots,x_n,x)$ and $x_k = \gamma_k - T$. The main step in the proof is to bound the integral which appears in the right-hand side of (3.1). In our proof we use the bound

$$\left|\int_T^{T+h} S(u)\,du\right| \leqslant \frac{\pi}{16}\frac{\log T}{(\log\log T)^2} + O\left(\frac{\log T\log\log\log T}{(\log\log T)^3}\right) \tag{3.2}$$

for $0 < h \leqslant \sqrt{T}$, due to Carneiro et al. [3].

LEMMA 3.1. *Let T be sufficiently large, $4(\log\log T)^{-1} \leqslant a \leqslant \sqrt{T}$, m and n be the integers defined in the introduction and let α_0 be the coefficient defined in Lemma 2.3 where $x_0 = 0$ and $x_k = \gamma_k - T$. Then*

$$-\log|\alpha_0| \leqslant -\frac{2a}{\pi}\theta'(T)\log 2 + \frac{1}{\sqrt{2}}\frac{\log T}{\log\log T}\left(1 + O\left(\frac{\log\log\log T}{\log\log T}\right)\right). \tag{3.3}$$

Further, if $S(T+a)-S(T-a)\geqslant 1$, *then*

$$m+n=\frac{2a}{\pi}\theta'(T)+\Delta S\Big(1+O\Big(\frac{a^3}{T^2}\Big)\Big) \tag{3.4}$$

where $\Delta S=S(T+a)-S(T-a)$.

Proof. By definition

$$\Big|\frac{1}{\alpha_0}\Big|=\prod_{\substack{-m\leqslant j\leqslant n\\ j\neq 0}}\Big|\sin\Big(\pi\frac{x_j}{2a}\Big)\Big|$$

and using (1.1) and Stieltjes integral we have

$$-\log|\alpha_0|=\int_{T-a}^{T+a}\log\Big|\sin\Big(\pi\frac{t-T}{2a}\Big)\Big|\,d\Big(\frac{1}{\pi}\theta(t)+1+S(t)\Big)$$

and an integration by parts leads to

$$\begin{aligned}-\log|\alpha_0|=&\frac{1}{\pi}\int_{T-a}^{T+a}\theta'(t)\log\Big|\sin\Big(\pi\frac{t-T}{2a}\Big)\Big|\,dt\\ &-\frac{\pi}{2a}\int_{T-a}^{T+a}\cot\Big(\pi\frac{t-T}{2a}\Big)(S(t)-S(T))\,dt.\end{aligned}$$

Now for $t\in[T-a,T+a]$ we have

$$\theta'(t)=\theta'(T)+\theta''(T)(t-T)+\frac{1}{2}\theta'''(\tau)(t-T)^2\quad\text{for some }\ \tau\in[T-a,T+a]$$

and using $\theta'''(t)=O(1/t^2)$ together with

$$\int_{T-a}^{T+a}\log\Big|\sin\Big(\pi\frac{t-T}{2a}\Big)\Big|\,dt=\frac{4a}{\pi}\int_0^{\pi/2}\log\sin\tau\,d\tau=-2a\log 2$$

we get

$$\begin{aligned}-\log|\alpha_0|=&-\frac{2a}{\pi}\theta'(T)\log 2\\ &-\frac{\pi}{2a}\int_{T-a}^{T+a}\cot\Big(\pi\frac{t-T}{2a}\Big)(S(t)-S(T))\,dt+O\Big(\frac{a^3}{T^2}\Big).\end{aligned} \tag{3.5}$$

Further, for $\eta\in\,]0,a]$ we have

$$\begin{aligned}&\int_{T-a}^{T+a}\cot\Big(\pi\frac{t-T}{2a}\Big)(S(t)-S(T))\,dt\\ &\quad=\int_{T-a}^{T-\eta}\cot\Big(\pi\frac{t-T}{2a}\Big)S(t)\,dt+\int_{T-\eta}^{T}\cot\Big(\pi\frac{t-T}{2a}\Big)(S(t)-S(T))\,dt\\ &\qquad+\int_{T}^{T+\eta}\cot\Big(\pi\frac{t-T}{2a}\Big)(S(t)-S(T))\,dt+\int_{T+\eta}^{T+a}\cot\Big(\pi\frac{t-T}{2a}\Big)S(t)\,dt\end{aligned}$$

and we bound from below the third term using

$$S(t) - S(T) \geqslant -\frac{1}{\pi}(\theta(t) - \theta(T)) \geqslant -\frac{1}{\pi}\theta'(T+a)(t-T) \geqslant -\frac{1}{2\pi}\log T\,(t-T)$$

which, together with the inequality $\cot x < \frac{1}{x}$ for $x \in]0, \frac{\pi}{2}]$, implies

$$\int_T^{T+\eta} \cot\Big(\pi\frac{t-T}{2a}\Big)(S(t) - S(T))\,dt \geqslant -\frac{\eta a}{\pi^2}\log T.$$

Applied to the fourth term, the second mean-value theorem gives

$$\begin{aligned}\int_{T+\eta}^{T+a} \cot\Big(\pi\frac{t-T}{2a}\Big)S(t)\,dt &= \cot\Big(\pi\frac{\eta}{2a}\Big)\int_{T+\eta}^{\tau} S(t)\,dt\\ &\geqslant -\frac{2a}{\pi\eta}\bigg(\frac{\pi}{16}\frac{\log T}{(\log\log T)^2} + O\Big(\frac{\log T\log\log\log T}{(\log\log T)^3}\Big)\bigg)\end{aligned}$$

where $\tau \in [T+\eta, T+a]$ and (3.2) has been used. Proceeding in the same way to bound from below the first and second term we finally get

$$\begin{aligned}&-\frac{\pi}{2a}\int_{T-a}^{T+a} \cot\Big(\pi\frac{t-T}{2a}\Big)(S(t) - S(T))\,dt\\ &\qquad\leqslant \frac{\eta}{\pi}\log T + \frac{1}{\eta}\bigg(\frac{\pi}{8}\frac{\log T}{(\log\log T)^2} + O\Big(\frac{\log T\log\log\log T}{(\log\log T)^3}\Big)\bigg).\end{aligned}$$

We choose $\eta = \pi\big(2\sqrt{2}\log\log T\big)^{-1}$ and we complete the proof of (3.3) using (3.5). Since $m + n = N(T+a) - N(T-a)$, the Taylor formula leads to

$$m + n = \frac{1}{\pi}\big(\theta(T+a) - \theta(T-a)\big) + \Delta S = \frac{2a}{\pi}\theta'(T) + \Delta S\Big(1 + O\Big(\frac{a^3}{T^2}\Big)\Big).$$

which proves (3.4). □

Proof of Theorem 1.1. Either $K \leqslant K^*\log K^*$ and there is nothing to prove or $K > K^*\log K^*$ and this implies that $K \geqslant (m+n)\log(m+n)$. Assuming T large enough we have also $m + n \geqslant 4$. By definition of K and thanks to Lemma 2.7, the first sum in the right-hand side of identity (3.1) is nonpositive and the second one is nonnegative. Using Cauchy–Schwarz inequality, (1.4) and Lemma 2.9, we get

$$Z(T) \leqslant \|Z^{(2K)}\|_2\,\|\Psi^*_{2K-1}\|_2 \leqslant C_{2K}Z(T)$$

where

$$\begin{aligned}C_{2K} &= c_{2K}2^{m+n-\frac{1}{2}}|\alpha_0|^{-1}\Big(\frac{2a\theta'(T)}{(m+n)\pi}\Big)^{2K}(1+o(1))\\ &\leqslant c_{2K}2^{m+n}|\alpha_0|^{-1}\bigg(1 + \pi\frac{\Delta S}{2a\theta'(T)}\Big(1 + O\Big(\frac{a^3}{T^2}\Big)\Big)\bigg)^{-2K}\end{aligned}$$

for T sufficiently large. Since $C_{2K} \geqslant 1$ we have $\log C_{2K} \geqslant 0$ and this implies that

$$\log c_{2K} + (m+n)\log 2 - \log|\alpha_0| - 2K\log\left(1 + \pi\frac{\Delta S}{2a\theta'(T)}\Big(1 + O\Big(\frac{a^3}{T^2}\Big)\Big)\right) \geqslant 0.$$

Finally, as $\Delta S = O\big(\log T(\log\log T)^{-1}\big)$ and thanks to Lemma 3.1, we get

$$\begin{aligned} &2K\pi\,\frac{\Delta S}{2a\theta'(T)}\Big(1 + O\Big(\frac{1}{\log\log T}\Big)\Big) \\ &\qquad\qquad \leqslant \Big(\log c_{2K} + \Delta S\log 2 + \frac{1}{\sqrt{2}}\frac{\log T}{\log\log T}\Big)\Big(1 + O\Big(\frac{\log\log\log T}{\log\log T}\Big)\Big) \end{aligned}$$

and the proof is complete. $\square$

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