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ON L^p-CONVERGENCE OF BERNSTEIN–DURRMEYER OPERATORS WITH RESPECT TO ARBITRARY MEASURE

Elena E. Berdysheva and Bing-Zheng Li

ABSTRACT. We consider Bernstein–Durrmeyer operators with respect to arbitrary measure on the simplex in the space $\mathbb{R}^d$. We obtain estimates for rate of convergence in the corresponding weighted L^p-spaces, $1 \leqslant p < \infty$.

1. Introduction

We consider Bernstein–Durrmeyer operators with respect to arbitrary measure. These are positive linear operators defined for functions on a d-dimensional simplex. We start with notation. Let

$$\mathbb{S}^d := \{x = (x_1, \dots, x_d) \in \mathbb{R}^d : \ 0 \leqslant x_1, \dots, x_d \leqslant 1, \quad 0 \leqslant x_1 + \dots + x_d \leqslant 1\}$$

denote the standard simplex in $\mathbb{R}^d$. We denote by $\partial\mathbb{S}^d$ the boundary of $\mathbb{S}^d$. We will also use barycentric coordinates on the simplex which we denote by the boldface symbol $\mathbf{x} = (x_0, x_1, \dots, x_d)$, $x_0 := 1 - x_1 - \dots - x_d$. We will use standard multiindex notation such as

$$\mathbf{x}^\alpha := x_0^{\alpha_0}\, x_1^{\alpha_1} \cdots x_d^{\alpha_d} \qquad \text{and} \qquad \frac{\alpha}{n} := \Big(\frac{\alpha_0}{n}, \frac{\alpha_1}{n}, \cdots, \frac{\alpha_d}{n}\Big)$$

for $\mathbf{x} = (x_0, x_1, \dots, x_d)$, $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_d) \in \mathbb{R}^{d+1}$, $n \in \mathbb{N}$. Functions defined on $\mathbb{S}^d$ are understood as functions of a point that can be given alternatively in cartesian or in barycentric coordinates.

The spaces $L^p(\mathbb{S}^d, \rho)$, $1 \leqslant p < \infty$, are defined in the standard way as spaces of (equivalence classes) of real-valued functions f for which $|f|^p$ is integrable with respect to a measure ρ with the norm

$$\|f\|_{L^p(\mathbb{S}^d,\rho)} := \bigg(\int_{\mathbb{S}^d} |f(x)|^p d\rho(x) \bigg)^{1/p}.$$

The space $L^\infty(\mathbb{S}^d, \rho)$ is the space of essentially bounded functions with the norm $\|f\|_{L^\infty(\mathbb{S}^d,\rho)} := \operatorname{ess\,sup}_{x\in\mathbb{S}^d} |f(x)|$. We will also consider the space $C(\mathbb{S}^d)$ of continuous bounded functions on $\mathbb{S}^d$ with the norm $\|f\|_{C(\mathbb{S}^d)} := \max_{x\in\mathbb{S}^d} |f(x)|$.

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The paper is dedicated to Giuseppe Mastroianni on the occasion of his retirement.

An important building stone of our construction are the *Bernstein basis polynomials* of degree $n \in \mathbb{N}$ on the simplex

$$B_\alpha(x) := \binom{n}{\alpha} \mathbf{x}^\alpha = \frac{n!}{\alpha_0! \alpha_1! \cdots \alpha_d!} (1 - x_1 - \cdots - x_d)^{\alpha_0} x_1^{\alpha_1} \cdots x_d^{\alpha_d},$$

with $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d)$, where $\alpha_0, \alpha_1, \ldots, \alpha_d$ are nonnegative integers and $|\alpha| := \alpha_0 + \alpha_1 + \cdots + \alpha_d = n$. Here, and in similar expressions later, 0^0 means 1. The Bernstein basis polynomials are nonnegative on $\mathbb{S}^d$, and

$$\sum_{|\alpha|=n} B_\alpha(x) = 1.$$

The polynomials $\{B_\alpha\}_{|\alpha|=n}$ constitute a basis of the space of real algebraic polynomials in d variables of total degree at most n.

Definition 1.1. Let ρ be a nonnegative bounded Borel measure on $\mathbb{S}^d$ such that

$$\operatorname{supp} \rho \setminus \partial \mathbb{S}^d \neq \emptyset. \tag{1.1}$$

The Bernstein–Durrmeyer operator with respect to the measure ρ is defined for $f \in C(\mathbb{S}^d)$ or $f \in L^p(\mathbb{S}^d, \rho)$, $1 \leqslant p \leqslant \infty$, by

$$\mathbf{M}_{n,\rho} f := \sum_{|\alpha|=n} \frac{\int_{\mathbb{S}^d} f B_\alpha d\rho}{\int_{\mathbb{S}^d} B_\alpha d\rho} B_\alpha, \qquad n \in \mathbb{N}. \tag{1.2}$$

Note that ρ is regular (being a nonnegative bounded Borel measure on a metric space), and thus polynomials are dense in the spaces $L^p(\mathbb{S}^d, \rho)$, $1 \leqslant p \leqslant \infty$. Condition (1.1) guarantees that $\int_{\mathbb{S}^d} B_\alpha d\rho > 0$ for all Bernstein basis polynomials B_α.

The operator $\mathbf{M}_{n,\rho}$ is linear and positive, and it reproduces constant functions. It is a variant of the Bernstein polynomial operator $\mathbf{B}_n$ for integrable functions. The latter is defined as follows.

Definition 1.2. The Bernstein operator is defined for $f \in C(\mathbb{S}^d)$ by

$$\mathbf{B}_n f := \sum_{|\alpha|=n} f\left(\frac{\alpha}{n}\right) B_\alpha, \qquad n \in \mathbb{N}. \tag{1.3}$$

This is a linear positive operator that reproduces linear functions. The operator $\mathbf{B}_n$ was introduced by Bernstein [**7**] in the one-dimensional case in order to give a constructive proof of the Weierstrass Approximation Theorem. Many variants and generalizations of operator (1.3) were studied in hundreds of papers.

The operator $\mathbf{M}_{n,\rho}$ without weight (i.e., when ρ is the Lebesgue measure) was defined in [**12, 17**] and studied in [**8, 9**]. In the special case when ρ is the Jacobi weight, $\mathbf{M}_{n,\rho}$ was introduced in [**18, 6**]. It is very well understood; see, e.g., [**11**]. See also [**5**] for properties and further references.

Operators (1.2) in full generality were for the first time systematically studied in [**4**], to our knowledge. The motivation came from learning theory; Jetter and Zhou [**14**] used the univariate Bernstein–Durrmeyer operators of type (1.2) to obtain bias-variance estimates for support vector machine classifiers. In [**16**], the second named author applied multivariate operators (1.2) as a tool for proving learning rates of least-square regularized regression with polynomial kernels.

In this paper, we continue our investigations on convergence of operators (1.2). In [**2**], the first named author considered uniform convergence of operators $\mathbf{M}_{n,\rho}$. She proved that

$$\lim_{n\to\infty} \|f - \mathbf{M}_{n,\rho}\, f\|_{C(\mathbb{S}^d)} = 0 \quad \text{for every} \quad f \in C(\mathbb{S}^d)$$

if and only if ρ is strictly positive on $\mathbb{S}^d$ (i.e., $\operatorname{supp}\rho = \mathbb{S}^d$). In [**3**], she considered pointwise convergence on the support of the measure. She showed that $(\mathbf{M}_{n,\rho}\, f)(x) \to f(x)$ as $n \to \infty$ at each point $x \in \operatorname{supp}\rho$ if f is bounded on $\operatorname{supp}\rho$ and continuous at x. Moreover, the convergence is uniform on any compact set in the interior of $\operatorname{supp}\rho$. Her method does not lead to estimates for rates of convergence.

The second named author studied the weighted L^p-convergence of operators (1.2). In [**16**], she proved that

$$\lim_{n\to\infty} \|f - \mathbf{M}_{n,\rho}\, f\|_{L^p(\mathbb{S}^d,\rho)} = 0$$

for every $f \in L^p(\mathbb{S}^d, \rho)$, $1 \leqslant p < \infty$. Note that no additional assumptions on ρ are required. Moreover, she obtained estimates for the rate of convergence in the spaces $L^p(\mathbb{S}^d, \rho)$, $1 \leqslant p < \infty$, in terms of the following K-functional. Let $C^1(\mathbb{S}^d)$ be the space of functions $g \in C(\mathbb{S}^d)$ with continuous partial derivatives $\partial_i g := \frac{\partial g}{\partial x_i}$, $i = 1, \dots, d$, endowed with the seminorm

$$\|\nabla g\|_{C(\mathbb{S}^d)} := \max_{i=1,\dots,d} \|\partial_i g\|_{C(\mathbb{S}^d)}.$$

The K-functional used in [**16**] is defined by

$$\mathcal{K}(f,t)_p := \inf_{g\in C^1(\mathbb{S}^d)} \bigl\{ \|f-g\|_{L^p(\mathbb{S}^d,\rho)} + t\, \|\nabla g\|_{C(\mathbb{S}^d)} \bigr\}, \qquad 1 \leqslant p \leqslant \infty.$$

The following estimates were proved in [**16**]. If $f \in L^p(\mathbb{S}^d, \rho)$, $1 \leqslant p < \infty$, then

$$\|f - \mathbf{M}_{n,\rho}\, f\|_{L^p(\mathbb{S}^d,\rho)} \leqslant 2d\mathcal{K}\bigl(f, n^{-1/2p}\bigl[\rho(\mathbb{S}^d)\bigr]^{1/p}\bigr)_p, \qquad 1 \leqslant p < 2, \tag{1.4}$$

$$\|f - \mathbf{M}_{n,\rho}\, f\|_{L^p(\mathbb{S}^d,\rho)} \leqslant 2d\mathcal{K}\bigl(f, n^{-1/p}\bigl[\rho(\mathbb{S}^d)\bigr]^{1/p}\bigr)_p, \qquad 2 \leqslant p < \infty. \tag{1.5}$$

In this paper, we improve the rates given in estimates (1.4) and (1.5). Namely, by a modification of the method of [**16**], we obtain the following result.

Theorem 1.1. *Let ρ be a nonnegative bounded Borel measure on $\mathbb{S}^d$ such that $\operatorname{supp}\rho \setminus \partial\mathbb{S}^d \neq \emptyset$, and let $f \in L^p(\mathbb{S}^d, \rho)$, $1 \leqslant p < \infty$. Then*

$$\|f - \mathbf{M}_{n,\rho}\, f\|_{L^p(\mathbb{S}^d,\rho)} \leqslant 2\mathcal{K}\bigl(f, C_p\, n^{-1/2} d\bigl[\rho(\mathbb{S}^d)\bigr]^{1/p}\bigr)_p, \qquad 1 \leqslant p < \infty,$$

where C_p is a constant that depends only on p. It holds $C_p \leqslant C_{\tilde p}$ for $p \leqslant \tilde p$. Moreover, one can take $C_p = \frac{1}{2}$ for $1 \leqslant p \leqslant 2$.

2. Proof of Theorem 1.1

Denote $\varphi_i(x) := x_i$, $i = 1, \dots, d$, and for $1 \leqslant p \leqslant \infty$

$$\Delta_{n,p} := \sum_{i=1}^{d} \big\|\mathbf{M}_{n,\rho}\big(|\varphi_i(x) - \varphi_i(\cdot)|\big)\big\|_{L^p(\mathbb{S}^d,\rho)}.$$

It is easy to see that

$$\|\mathbf{M}_{n,\rho} f - f\|_{L^p(\mathbb{S}^d,\rho)} \leqslant 2\mathcal{K}(f, \Delta_{n,p}/2)_p \tag{2.1}$$

for $f \in L^p(\mathbb{S}^d, \rho)$, $1 \leqslant p \leqslant \infty$ (see [**4**, Theorem 4.5] or [**16**, Theorem 2.1]). Thus, the key to proving estimates for the rate of convergence of the operator $\mathbf{M}_{n,\rho}$ is to study the behaviour of $\Delta_{n,p}$.

We were able to obtain estimates for $\Delta_{n,p}$ in case when $1 \leqslant p < \infty$. Theorem 1.1 is a direct consequence of the lemma given below.

LEMMA 2.1. *Let ρ be a nonnegative bounded Borel measure on $\mathbb{S}^d$ such that $\operatorname{supp}\rho \setminus \partial\mathbb{S}^d \neq \emptyset$, and let $f \in L^p(\mathbb{S}^d, \rho)$, $1 \leqslant p < \infty$. Then*

$$\|\mathbf{M}_{n,\rho}\big(|\varphi_i(x) - \varphi_i(\cdot)|\big)\|_{L^p(\mathbb{S}^d,\rho)} \leqslant c_p\, n^{-1/2}\big[\rho(\mathbb{S}^d)\big]^{1/p}, \qquad i = 1, \dots, d, \tag{2.2}$$

where c_p is a constant that depends only on p. It holds $c_p \leqslant c_{\tilde p}$ for $p \leqslant \tilde p$. Moreover, one can take $c_p = 1$ for $1 \leqslant p \leqslant 2$.

PROOF. Denote $\theta_\alpha := \int_{\mathbb{S}^d} B_\alpha\, d\rho$. Following [**16**], we write

$$\begin{aligned}
\mathbf{M}_{n,\rho}\big(|\varphi_i(x) - \varphi_i(\cdot)|\big)(x) &= \sum_{|\alpha|=n} \frac{1}{\theta_\alpha}\int_{\mathbb{S}^d} |\varphi_i(x) - \varphi_i(t)|\, B_\alpha(t)\, d\rho(t)\, B_\alpha(x) \\
&= \sum_{|\alpha|=n} \Big|\varphi_i(x) - \frac{\alpha_i}{n}\Big|\, B_\alpha(x) + \sum_{|\alpha|=n} \frac{1}{\theta_\alpha}\int_{\mathbb{S}^d} \Big|\frac{\alpha_i}{n} - \varphi_i(t)\Big|\, B_\alpha(t)\, d\rho(t)\, B_\alpha(x) \\
&= \mathbf{B}_n\big(|\varphi_i(x) - \varphi_i(\cdot)|\big)(x) + I(x),
\end{aligned}$$

where $\mathbf{B}_n$ is the Bernstein operator (1.3), and

$$I(x) := \sum_{|\alpha|=n} \frac{1}{\theta_\alpha}\int_{\mathbb{S}^d} \Big|\frac{\alpha_i}{n} - \varphi_i(t)\Big|\, B_\alpha(t)\, d\rho(t)\, B_\alpha(x).$$

By Cauchy–Schwarz inequality for positive operators (e.g., [**13**]), we have

$$\mathbf{B}_n\big(|\varphi_i(x) - \varphi_i(\cdot)|\big)(x) \leqslant \big(\mathbf{B}_n\left([\varphi_i(x) - \varphi_i(\cdot)]^2\right)(x)\big)^{1/2}\, \left(\mathbf{B}_n(1)(x)\right)^{1/2}.$$

It is well known and easy to prove that

$$\mathbf{B}_n\left([\varphi_i(x) - \varphi_i(\cdot)]^2\right)(x) = \frac{\varphi_i(x)\,(1 - \varphi_i(x))}{n} \leqslant \frac{1}{4n}, \tag{2.3}$$

and $\mathbf{B}_n(1) = 1$. Thus, $\mathbf{B}_n\big(|\varphi_i(x) - \varphi_i(\cdot)|\big)(x) \leqslant \frac{1}{2\sqrt{n}}$, and

$$\|\mathbf{B}_n\big(|\varphi_i(x) - \varphi_i(\cdot)|\big)\|_{L^p(\mathbb{S}^d,\rho)} \leqslant \frac{1}{2\sqrt{n}}\big[\rho(\mathbb{S}^d)\big]^{1/p}. \tag{2.4}$$

Next we obtain an estimate for $\|I\|_{L^p(\mathbb{S}^d,\rho)}$. Take q such that $\frac{1}{p}+\frac{1}{q}=1$. Applying the Hölder inequality two times, we obtain

$$
\begin{aligned}
\|I\|^p_{L^p(\mathbb{S}^d,\rho)} &= \int_{\mathbb{S}^d}\Big\{\sum_{|\alpha|=n}\frac{1}{\theta_\alpha}\int_{\mathbb{S}^d}\Big|\frac{\alpha_i}{n}-\varphi_i(t)\Big|B_\alpha(t)\,d\rho(t)\,B_\alpha^{\frac{1}{p}+\frac{1}{q}}(x)\Big\}^p d\rho(x)\\
&\leqslant \int_{\mathbb{S}^d}\sum_{|\alpha|=n}\frac{1}{\theta_\alpha^p}\Big(\int_{\mathbb{S}^d}\Big|\frac{\alpha_i}{n}-\varphi_i(t)\Big|B_\alpha(t)\,d\rho(t)\Big)^p B_\alpha(x)\,d\rho(x)\\
&= \sum_{|\alpha|=n}\frac{1}{\theta_\alpha^{p-1}}\Big(\int_{\mathbb{S}^d}\Big|\frac{\alpha_i}{n}-\varphi_i(t)\Big|B_\alpha^{\frac{1}{p}+\frac{1}{q}}(t)\,d\rho(t)\Big)^p\\
&\leqslant \sum_{|\alpha|=n}\frac{1}{\theta_\alpha^{p-1}}\int_{\mathbb{S}^d}\Big|\frac{\alpha_i}{n}-\varphi_i(t)\Big|^p B_\alpha(t)\,d\rho(t)\,\theta_\alpha^{p/q}\\
&= \int_{\mathbb{S}^d}\sum_{|\alpha|=n}\Big|\frac{\alpha_i}{n}-\varphi_i(t)\Big|^p B_\alpha(t)\,d\rho(t)\\
&= \big\|\mathbf{B}_n\big(|\varphi_i(x)-\varphi_i(\cdot)|^p\big)\big\|_{L^1(\mathbb{S}^d,\rho)}.
\end{aligned}
$$

First suppose that $p\geqslant 1$ is an even integer. In this case, the expression in the last line of the previous formula is the $L^1(\mathbb{S}^d,\rho)$-norm of a moment of the Bernstein operator (1.3), namely, of $\mathbf{B}_n\big([\varphi_i(x)-\varphi_i(\cdot)]^p\big)(x)$. First we note that the value of this moment is independent of the dimension d. To see this, consider without loss of generality $i=1$. Then

$$
\begin{aligned}
\mathbf{B}_n\big([\varphi_1(x)-\varphi_1(\cdot)]^p\big)(x) &= \sum_{|\alpha|=n}\Big(x_1-\frac{\alpha_1}{n}\Big)^p B_\alpha(x)\\
&= \sum_{|\alpha|=n}\Big(x_1-\frac{\alpha_1}{n}\Big)^p\binom{n}{\alpha_1}x_1^{\alpha_1}(1-x_1)^{n-\alpha_1}\\
&\qquad\times\frac{(n-\alpha_1)!}{\alpha_2!\cdots\alpha_0!}\Big(\frac{x_2}{1-x_1}\Big)^{\alpha_2}\cdots\Big(\frac{x_0}{1-x_1}\Big)^{\alpha_0}\\
&= \sum_{\alpha_1=0}^{n}\Big(x_1-\frac{\alpha_1}{n}\Big)^p\binom{n}{\alpha_1}x_1^{\alpha_1}(1-x_1)^{n-\alpha_1}\\
&\qquad\times\sum_{|(\alpha_2,\dots,\alpha_d,\alpha_0)|=n-\alpha_1}B_{(\alpha_2,\dots,\alpha_d,\alpha_0)}\Big(\frac{x_2}{1-x_1},\cdots,\frac{x_d}{1-x_1}\Big)\\
&= \sum_{\alpha_1=0}^{n}\Big(x_1-\frac{\alpha_1}{n}\Big)^p\binom{n}{\alpha_1}x_1^{\alpha_1}(1-x_1)^{n-\alpha_1}
\end{aligned}
$$

which is the p-th moment of the one-dimensional Bernstein operator. Estimates for these moments are well known an can be found, e.g., in [**10**, Chapter 10, §1]. It follows from Corollary to Theorem 1.1 of this chapter that there is a constant A_p depending only on p such that

$$
\sum_{\alpha_1=0}^{n}\Big(x_1-\frac{\alpha_1}{n}\Big)^p\binom{n}{\alpha_1}x_1^{\alpha_1}(1-x_1)^{n-\alpha_1}\leqslant A_p\,n^{-p/2}.
$$

Consequently,

$$\|I\|_{L^p(\mathbb{S}^d,\rho)} \leqslant \left\{\|\mathbf{B}_n([\varphi_i(x)-\varphi_i(\cdot)]^p)\|_{L^1(\mathbb{S}^d,\rho)}\right\}^{1/p} \leqslant n^{-1/2}A_p^{1/p}\,\left[\rho(\mathbb{S}^d)\right]^{1/p}. \tag{2.5}$$

For an arbitrary $p \geqslant 1$, take $\tilde{p}$ to be the smallest even integer with $p \leqslant \tilde{p}$. The $L^p(\mathbb{S}^d,\rho)$-and $L^{\tilde{p}}(\mathbb{S}^d,\rho)$-norms are connected by the inequality

$$\|\cdot\|_{L^p(\mathbb{S}^d,\rho)} \leqslant \|\cdot\|_{L^{\tilde{p}}(\mathbb{S}^d,\rho)}\,\left[\rho(\mathbb{S}^d)\right]^{\frac{1}{p}-\frac{1}{\tilde{p}}} \tag{2.6}$$

(e.g., [**15**, Chapter IV, §3, Theorem 6]). This estimate and (2.5) yield

$$\|I\|_{L^p(\mathbb{S}^d,\rho)} \leqslant n^{-1/2}A_{\tilde{p}}^{1/\tilde{p}}\left[\rho(\mathbb{S}^d)\right]^{1/p}.$$

Combining this with (2.4), we obtain

$$\begin{aligned}\|\mathbf{M}_{n,\rho}\big(|\varphi_i(x)-\varphi_i(\cdot)|\big)\|_{L^p(\mathbb{S}^d,\rho)} &\leqslant \|\mathbf{B}_n\big(|\varphi_i(x)-\varphi_i(\cdot)|\big)\|_{L^p(\mathbb{S}^d,\rho)} + \|I\|_{L^p(\mathbb{S}^d,\rho)}\\ &\leqslant n^{-1/2}c_p\left[\rho(\mathbb{S}^d)\right]^{1/p}\end{aligned}$$

with $c_p = \frac{1}{2} + A_{\tilde{p}}^{1/\tilde{p}}$, which is (2.2).

It follows from (2.6) that for all $p \leqslant \tilde{p}$ it holds

$$\|\mathbf{M}_{n,\rho}\big(|\varphi_i(x)-\varphi_i(\cdot)|\big)\|_{L^p(\mathbb{S}^d,\rho)} \leqslant n^{-1/2}c_{\tilde{p}}\left[\rho(\mathbb{S}^d)\right]^{1/p}.$$

Thus, $c_p \leqslant c_{\tilde{p}}$ for $p \leqslant \tilde{p}$.

Finally, consider $p = 2$. In this case $A_2 = \frac{1}{4}$ (see (2.3)). Thus, $c_2 = 1$, and we also can take $c_p = 1$ for $1 \leqslant p \leqslant 2$. □

Remark 2.1. Representations of general moments of the multivariate Bernstein operator (1.3) of the form $\mathbf{B}_n\big(\prod_{i=1}^d(\varphi_i(x)-\varphi_i(\cdot))^{p_i}\big)$ with nonnegative integers p_i, $i = 1,\dots,d$, in terms of Stirling numbers are given by Abel and Ivan [**1**]. They also estimated the order of these moments.

Remark 2.2. Alternatively, we could use in Theorem 1.1 the estimate

$$\|\mathbf{M}_{n,\rho}f - f\|_{L^p(\mathbb{S}^d,\rho)} \leqslant \max\{2,d\}\,\mathcal{K}\left(f,\Delta_{n,p}\right)_p$$

instead of (2.1). This inequality leads to the estimate

$$\|\mathbf{M}_{n,\rho}f - f\|_{L^p(\mathbb{S}^d,\rho)} \leqslant \max\{2,d\}\,\mathcal{K}\big(f, n^{-1/2}c_p\left[\rho(\mathbb{S}^d)\right]^{1/p}\big)_p, \qquad 1 \leqslant p < \infty,$$

with a constant c_p like in Lemma 2.1.

Remark 2.3. Our method does not lead to estimates for the rates of convergence of the operator $\mathbf{M}_{n,\rho}$ in the space $L^\infty(\mathbb{S}^d,\rho)$, or to pointwise estimates. These are important and interesting open questions.

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German University of Technology in Oman
Muscat, Oman
elena.berdysheva@gutech.edu.om

Zhejiang University
Hangzhou, China
libingzheng@zju.edu.cn